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Simultaneous statistical approximation of analytic functions and their derivatives by k-positive linear operators

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Abstract. In this paper we obtain the theorems on simultaneous statistical approximation of analytic functions and their derivatives by the sequences of k-positive linear operators and its derivatives in the unite disk of complex plane.

Key Words and Phrases: The space of analytical functions, linear k-positive operators, Simultaneous approximation, Korovkin type theorem.

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1. Introduction

Let D be the open unit disk of $|z| < 1$, $D = \{z : |z| < 1\}$ and $A(D)$ denote the space of all analytic functions in D. For each function $f \in A(D)$ the Taylor expansion is given by

$$f(z) = \sum_{k=0}^{\infty} f_k z^k,$$

where f_k is the Taylor coefficients of $f(z)$, and $\limsup\limits_{k\to\infty} | f_k |^{\frac{1}{k}} = 1$. This condition imply that for any fixed $r < 1$ the series in right hand-side is uniformly convergent if $|z| \le r$. Denoting for any $r < 1$

$$\|f\|_{A(D),r} = \max_{|z|\le r} |f(z)|, \tag{1}$$

we see that $A(D)$ is a Fréchet space with the family of norms $\| f \|_{A(D),r}$ depending on the number r.

It is easy to see that any linear operator acting from $A(D)$ to $A(D)$ can be represented in the form

$$Tf(z) = \sum_{k=0}^{\infty} z^k \sum_{m=0}^{\infty} f_m T_{k,m}, \tag{2}$$

with some matrix of $T_{k,m}$ such that

$$\limsup_{k\to\infty}\left|\sum_{m=0}^{\infty} f_m T_{k,m}\right|^{\frac{1}{k}} = 1.$$

We will study the sequence of linear operators

$$T_n f(z) = \sum_{k=0}^{\infty} z^k \sum_{m=0}^{\infty} f_m T_{k,m}^{(n)},$$

acting on functions $f \in A(D)$ and having the properties of "k-positivity" in the sense of our work [6]. Recall, that by the definition (see [6]) a linear operator T, acting from $A(D)$ to $A(D)$, is called k-positive if it preserves the class of functions with non-negative Taylor coefficients. As were shown in [6] linear operator T_n, given by the formula (2), is k-positive if and only if $T_{k,m}^{(n)} \geq 0$ for all n, k, m. Note that the different approximation properties of linear k-positive operators were studied in the papers [1], [2], [3], [7], [8], [9], [10], [12], [13]. The papers [2],[3],[8] and [10] is devoted to statistical approximations of analytic functions by the sequences of k-positive linear operators. We recall the concept of "statistical convergence" (see [5]). A sequence α_n is said to be statistically convergent to a number α if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{|\{k \leq n : |\alpha_k - \alpha| > \varepsilon\}|}{n} = 0,$$

where $|\{k \leq n : |\alpha_k - \alpha| > \varepsilon\}|$ be the number of all $k \leq n$, for which $|\alpha_k - \alpha| > \varepsilon$. In this case we write $st - \lim_{n\to\infty} \alpha_n = \alpha$.

Note that in the paper [11] have been proved the first Korovkin type theorems on the statistical approximation by positive operators and given the definition of order of statistical approximation by positive linear operators.

2. Main results

Let $f_n(z)$ be a sequence of analytic functions in D. Then we can write

$$f_n(z) = \sum_{k=0}^{\infty} f_{n,k} z^k, \text{ where } \limsup_{k\to\infty} |\, f_{n,k} \,|^{\frac{1}{k}} = 1, \text{ for any fixed natural } n.$$

The following lemma is a statistical analogue of the lemma on the uniform convergence of the sequence of analytic functions in $A(D)$, proved in the paper [4].

Lemma 1. *To the sequence $f_n(z)$ statistically tends to zero in $A(D)$ necessary and sufficient to satisfy the condition*

$$|f_{n,k}| \leq \varepsilon_n \left(1 + \delta_n\right)^k, \qquad k = 0, 1, 2, ..., \tag{3}$$

where δ_n tends to zero and ε_n-statistically tends to zero as $n \to \infty$.

Proof. Let (3) holds. Then, by (1), for any $r < 1$:

$$\|f_n\|_{A(D),r} \leq \varepsilon_n \sum_{k=0}^{\infty} (1+\delta_n)^k r^k,$$

and we can write

$$\|f_n\|_{A(D),r} \leq \frac{\varepsilon_n}{1 - r(1+\delta_n)} \ ,$$

because by the condition $\lim_{n\to\infty} \delta_n = 0$ we can choose $\delta_n < \frac{1}{r} - 1$.

Since $\lim_{n\to\infty} \dfrac{1}{1 - r(1+\delta_n)} = \dfrac{1}{1-r}$ is finite, then there exists a positive constant $K(r)$ such that for all n

$$\frac{1}{1 - r(1+\delta_n)} \leq K(r).$$

Therefore, for all $n = 1, 2, ...$

$$\|f_n\|_{A(D),r} \leq \varepsilon_n K(r).$$

This inequality allows us to write the following embedding

$$\{n : \|f_n\|_{A(D),r} > \varepsilon\} \subset \{n : \varepsilon_n K(r) > \varepsilon\}.$$

Therefore,

$$\left|\left\{n : \|f_n\|_{A(D),r} > \varepsilon\right\}\right| \leq |\{n : \varepsilon_n K(r) > \varepsilon\}|.$$

From this, since ε_n statistically tends to zero as $n \to \infty$, we can write by the definition of statistical convergence

$$st - \lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \frac{|\{n : \varepsilon_n K(r) > \varepsilon\}|}{n} = 0.$$

It follows that

$$st - \lim_{n\to\infty} \|f_n\|_{A(D),r} = 0,$$

and the sufficiency is proved. To prove a necessity, we choose δ_n so slowly tending to zero that

$$\varepsilon_n = \max_{|z|=\frac{1}{1+\delta_n}} |f_n(z)|,$$

statistically tends to zero as $n \to \infty$.

Then we can write

$$f_{n,k} = \frac{1}{2\pi i} \int\limits_{|z|=\frac{1}{1+\delta_n}} \frac{f_n(z)}{z^{k+1}} dz,$$

and

$$|f_{n,k}| \leq \varepsilon_n \int\limits_{|z|=\frac{1}{1+\delta_n}} \frac{|dz|}{|z|^{k+1}} = \varepsilon_n (1+\delta_n)^k. \ ◀$$

Corollary 1. *The following two statements are equivalent:*

a) $f_n^{(m)}(z)$ *statistically tends to zero in* $A(D)$ *for any* $m = 0, 1, 2, 3, ...$;

b) $|f_{n,k+m}| \le \dfrac{k!}{(k+m)!}\varepsilon_n(1+\delta_n)^{k+m}, \ \ m = 0, 1, 2, ...,$

where ε_n *and* δ_n *as in Lemma 1.*

Proof. From the Taylor expansion of $f_n(z)$, we can write the following obvious representation

$$f_n^{(m)}(z) = \sum_{k=0}^{\infty}(k+1)...(k+m)f_{n,k+m}z^k. \tag{4}$$

By Lemma 1, $st-\lim_{n\to\infty}\|f_n^{(m)}\|_{A(D)} = 0$ if and only if

$$(k+1)...(k+m)|f_{n,k+m}| \le \varepsilon_n(1+\delta_n)^{k+m}.$$

Therefore, a) holds iff

$$|f_{n,k+m}| \le \frac{k!}{(k+m)!}\varepsilon_n(1+\delta_n)^{k+m},$$

which gives b).

Inversely, if b) holds then

$$\|f_n^{(m)}(z)\|_{A(D),r} \le \sum_{k=0}^{\infty}(k+1)...(k+m)|f_{n,k+m}|r^k \le$$

$$\le \varepsilon_n(1+\delta_n)^m\sum_{k=0}^{\infty}r^k(1+\delta_n)^k = \varepsilon_n\frac{(1+\delta_n)^m}{1-r(1+\delta_n)} \ ,$$

which gives a).

Lemma 2. *To*

$$st-\lim_{n\to\infty}\|f_n^{(m)}\|_{A(D),r} = 0, \quad m = 0, 1, 2, ..., \tag{5}$$

it necessary and sufficient to satisfy the condition (5) for $m = 0$.

Proof. Let the condition (5) holds for $m = 0$. Then by Lemma 1 there exist sequences ε_n and δ_n such that $\lim_{n\to\infty}\delta_n = 0$, $st-\lim_{n\to\infty}\varepsilon_n = 0$ and the inequality (3) holds. From the equality (4) we have for any natural m

$$\|f_n^{(m)}\|_{A(D),r} \le \varepsilon_n(1+\delta_n)^m\sum_{k=0}^{\infty}(k+1)...(k+m)(1+\delta_n)^k r^k,$$

since

$$\frac{d^m}{dz^m}\frac{1}{1-z} = \sum_{k=0}^{\infty}(k+1)...(k+m)z^k,$$

we can write

$$\frac{m!}{(1-z)^{m+1}} = \sum_{k=0}^{\infty}(k+1)...(k+m)z^k.$$

Therefore,

$$\|f_n^{(m)}\|_{A(D),r} \le \varepsilon_n(1+\delta_n)^m \frac{m!}{(1-r(1+\delta_n))^{m+1}}.$$

Since

$$\lim_{n\to\infty}(1+\delta_n)^m \frac{m!}{(1-r(1+\delta_n))^{m+1}} = \frac{m!}{(1-r)^{m+1}},$$

there exist a constant $K(m,r)$ such that for any $n = 1, 2, ...$

$$(1+\delta_n)^m \frac{m!}{(1-r(1+\delta_n))^{m+1}} \le K(m,r).$$

So we have the inequality

$$\|f_n^{(m)}\|_{A(D),r} \le \varepsilon_n K(m,r),$$

which implies

$$\left|\left\{k \le n : \|f_k^{(m)}\|_{A(D),r} > \varepsilon\right\}\right| \le \left|\left\{k \le n : \varepsilon_n > \frac{\varepsilon}{K(m,r)}\right\}\right|.$$

The last inequality gives the proof as in the proof of Lemma 1.◄

Consider now a sequence of linear k-positive operator T_n given by the formula (2). Obviously, for any natural p:

$$\frac{d^p}{dz^p}T_n f(z) = \sum_{k=0}^{\infty} z^k (k+1)(k+2)...(k+p) \sum_{m=0}^{\infty} T_{k+p,m}^{(n)} f_m.$$

Since T_n is a sequence of k-positive operators then, by the definition, $T_{k+p,m}^{(n)} \ge 0$ for any k, p and m. For positive coefficients f_m we have

$$(k+1)(k+2)...(k+p) \sum_{m=0}^{\infty} T_{k+p,m}^{(n)} f_m \ge 0.$$

By the definition, T_n is k-positive operator if for any $f \in A(D)$, having non-negative Taylor coefficients, $T_n f(z) \in A(D)$ and also has a non-negative Taylor coefficients. Therefore we have a

Proposition 1. *If $T_n f(z)$ is a k-positive operators then for any natural p, $\frac{d^p}{dz^p}T_n f(z)$ is also k-positive operator.*

Lemma 2 allows us to formulate any theorem on statistical convergence $T_n f(z)$ to $f(z)$ as $n \to \infty$ in $A(D)$ as a Theorem on simultaneous convergence of $\frac{d^p}{dz^p}T_n f(z)$ to $f^{(p)}(z)$, $p = 0, 1, 2, ...$.

For example, using the general result, proven in [10](see, p.399, Theorem 2.1) we can formulate the following result.

Theorem 1. *Let $g_k \geq 1$ be an increasing sequence of real numbers, $\limsup\limits_{k\to\infty} g_k^{\frac{1}{k}} = 1$ and*

$$g_\upsilon(z) = \sum_{k=0}^{\infty} g_k^{\frac{\upsilon}{2}} z^k.$$

Then, for linear k-positive operators T_n, given by (2) and acting from $A(D)$ into itself the following are equivalent:

a) st-$\lim\limits_{n\to\infty} \|\frac{d^p}{dz^p} T_n f(z) - f^{(p)}(z)\|_{A(D),r} = 0$, $p = 0, 1, 2, ...$ for any function f with Taylor coefficient satisfying the inequality

$$|f_k| \leq M g_k, \ \ k = 0, 1, 2, ...; \tag{6}$$

b) st-$\lim\limits_{n\to\infty} \|T_n g_\upsilon - g_\upsilon\|_{A(D),r} = 0$, $\upsilon = 0, 1, 2$.

Proof. In [10] have proved that under the conditions of Theorem 1

$$st - \lim_{n\to\infty} \|T_n f - f\|_{A(D),r} = 0,$$

for any function $f \in A(D)$ with Taylor coefficients satisfying (6). Therefore, applying Lemma 2 we obtain the desired result.◄

Theorem 2. *Let $g_k \geq 1$, $k = 0, 1, 2, ...,$ $\limsup\limits_{k\to\infty} g_k^{\frac{1}{k}} = 1$ and $g(z) = \sum\limits_{k=0}^{\infty} g_k z^k$ be an analytic function in $D = \{z : |z| < 1\}$. Let T_n be a sequence of linear k-positive operators from $A(D)$ into itself. If for any $r < 1$*

$$st - \lim_{n\to\infty} \|T_n g(z) - g(z)\|_{A(D),r} = 0, \tag{7}$$

$$st - \lim_{n\to\infty} \|T_n(zg'(z)) - zg'(z)\|_{A(D),r} = 0, \tag{8}$$

$$st - \lim_{n\to\infty} \|T_n(z^2 g''(z)) - z^2 g''(z)\|_{A(D),r} = 0, \tag{9}$$

then for any function f with Taylor coefficients satisfying (6)

$$st - \lim_{n\to\infty} \|\frac{d^p T_n f(z)}{dz^p} - f^{(p)}(z)\|_{A(D),r} = 0, \quad p = 0, 1, 2, ... \ . \tag{10}$$

Proof. Obviously, we have

$$zg'(z) = \sum_{k=0}^{\infty} k g_k z^k,$$

$$z^2 g''(z) = \sum_{k=0}^{\infty} k(k-1) g_k z^k,$$

and therefore using (2) the conditions (7)-(9) gives as $n \to \infty$:

$$\left\| \sum_{k=0}^{\infty} z^k \sum_{m=0}^{\infty} T_{k,m}^{(n)} g_m - \sum_{k=0}^{\infty} z^k g_k \right\|_{A(D),r} \longrightarrow^{(st)} 0,$$

$$\left\| \sum_{k=0}^{\infty} z^k \sum_{m=0}^{\infty} T_{k,m}^{(n)} m g_m - \sum_{k=0}^{\infty} z^k k g_k \right\|_{A(D),r} \longrightarrow^{(st)} 0,$$

$$\left\| \sum_{k=0}^{\infty} z^k \sum_{m=0}^{\infty} T_{k,m}^{(n)} m(m-1) g_m - \sum_{k=0}^{\infty} z^k k(k-1) g_k \right\|_{A(D),r} \longrightarrow^{(st)} 0,$$

where $\longrightarrow^{(st)} 0$ denote that statistical limit tends to zero. By Lemma 1 this means that there exist a sequences δ_n and ε_n such that $st - \lim_{n\to\infty} \varepsilon_n = 0$ and $\lim_{n\to\infty} \delta_n = 0$ and the following inequalities hold

$$\left| \sum_{m=0}^{\infty} T_{k,m}^{(n)} g_m - g_k \right| < \varepsilon_n (1+\delta_n)^k,$$

$$\left| \sum_{m=0}^{\infty} T_{k,m}^{(n)} m g_m - k g_k \right| < \varepsilon_n (1+\delta_n)^k,$$

$$\left| \sum_{m=0}^{\infty} T_{k,m}^{(n)} m(m-1) g_m - k(k-1) g_k \right| < \varepsilon_n (1+\delta_n)^k.$$

From these inequalities follows that

$$\sum_{k=0}^{\infty} (m-k)^2 g_m T_{k,m}^{(n)} < \varepsilon_n (1+\delta_n)^k (1+k)^2. \tag{11}$$

Let now $f(z)$ be any function in $A(D)$ with Taylor coefficients satisfying (6). Then

$$\| T_n f - f \|_{A(D),r} \leq 2M \left\{ \sum_{k=0}^{\infty} r^k g_k \sum_{m=0}^{\infty} (m-k)^2 g_m T_{k,m}^{(n)} + \sum_{k=0}^{\infty} r^k g_k \left| \sum_{k=0}^{\infty} T_{k,m}^{(n)} - 1 \right| \right\} =$$

$$= 2M \left\{ S_n^{'} + S_n^{''} \right\}. \tag{12}$$

Using (11), we have

$$S_n^{'} \leq \varepsilon_n \sum_{k=0}^{\infty} r^k g_k (1+k)^2 (1+\delta_n)^k,$$

and since the series converges, the right-hand side tends to zero as $n \to \infty$ statistically.

Now we estimate $S_n^{''}$. Using (11)

$$\sum_{k\neq m} T_{k,m}^{(n)} \leq \sum_{k=0}^{\infty}(k-m)^2 T_{k,m}^{(n)} \leq \varepsilon_n(1+\delta_n)^k(1+k)^2.$$

Further, by the condition (7)

$$st-\lim_{n\to\infty}\left\|\sum_{k=0}^{\infty} z^k \sum_{m=0}^{\infty} T_{k,m}^{(n)} g_m - \sum_{k=0}^{\infty} z^k g_k\right\|_{A(D),r} = 0,$$

and therefore

$$\left|\sum_{m=0}^{\infty} T_{k,m}^{(n)} g_m - g_k\right| < \varepsilon_n(1+\delta_n)^k,$$

or

$$\left|g_k\left(T_{k,k}^{(n)}-1\right) + \sum_{k\neq m} T_{k,m}^{(n)} g_m\right| < \varepsilon_n(1+\delta_n)^k.$$

This inequality gives

$$g_k|T_{k,k}^{(n)}-1| \leq \varepsilon_n(1+\delta_n)^k + \sum_{k\neq m} T_{k,m}^{(n)} g_m (m-k)^2 \leq 2\varepsilon_n(1+\delta_n)^k(1+k)^2, \tag{13}$$

by (11). On the other side, (11) gives

$$\sum_{k\neq m} T_{k,m}^{(n)} \leq \varepsilon_n(1+\delta_n)^k(1+k)^2. \tag{14}$$

From the inequalities (13) and (14) we obtain

$$S_n^{''} \leq \varepsilon_n \sum_{k=0}^{\infty} r^k g_k (1+k)^2(1+\delta_n)^k + 2\varepsilon_n \sum_{k=0}^{\infty} r^k (1+k)^2 (1+\delta_n)^k.$$

Since both series in right-hand side converge, then $S_n^{''}$ statistically tends to zero as $n\to\infty$ because statistically tends to zero ε_n.

Therefore, both $S_n^{'}$ and $S_n^{''}$ statistically tend to zero as $n\to\infty$ and using (12), we see that $\|T_n f - f\|_{A(D),r}$ statistically convergent to zero.

Using Lemma 2 we get (10) and the proof of Theorem 2 is completed.◄

Corollary 2. *Let T_n be a sequence of linear k-positive operators from $A(D)$ into itself. If*

$$st-\lim_{n\to\infty}\|T_n\frac{z^\upsilon}{(1-z)^{\upsilon+1}} - \frac{z^\upsilon}{(1-z)^{\upsilon+1}}\|_{A(D),r} = 0, \ \ \upsilon = 0,1,2,$$

then for any function $f\in A(D)$, having bounded Taylor coefficients

$$st-\lim_{n\to\infty}\|\frac{d^m T_n f(z)}{dz^m} - f^{(m)}(z)\|_{A(D),r} = 0, m = 0,1,2,\dots\ .$$

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