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Fractional green's function and fractional boundary conditions in diffraction of electromagnetic waves on plane screens

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Abstract. Proposed method to solve difference-integral equation of a special type, arising in problems of diffraction by boundaries is described by fractional boundary condition (FBC). The method is considered on two boundaries – a strip and a half-plane with FBC when the fractional order varies from 0 to 1. The proposed method is based on application of orthogonal polynomials. Gegenbauer polynomials orthogonal on interval $(-1, 1)$ are utilized for a strip, while Lager polynomials orthogonal on interval $(0, \infty)$ are used for a half-plane. One important feature of the considered integral equations is noted: these equations can be solved analytically for one special intermediate value of the fractional order (FO) $\alpha = 0,5$ and it can be done for any value of the frequency.

Key Words and Phrases: fractional operators, fractional boundary conditions, Gegenbauer polynomials, Lager polynomials, fractional Green's theorem
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1. Introduction

Let us assume that an E-polarized plane wave, described by the function $\vec{E}^i = \vec{z}E_z^i\,(x, y) = \vec{z}e^{-ik(x\cos\theta + y\sin\theta)}$, is an incident field scattered by a strip or a half-plane located at the plane $y = 0$ and infinite along the axis$z$. θ is an incidence angle, $k = 2\pi/\lambda$ is a wave number. The time dependence is assumed to be $e^{-i\omega t}$ and argued throughout the paper.

Solution of a diffraction problem on a plane screen S is to be solved under the following conditions:

- The total electric field $\vec{E} = \vec{z}E_z\,(x, y)$ satisfies Helmholtz equation outside the screen

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) E_z\,(x, y) = 0;$$

- The scattered electric field $E_z^s(x, y)$ satisfies the radiation condition at the infinity:

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial E_z^s}{\partial r} - iE_z^s\right) = 0, r = \sqrt{x^2 + y^2};$$

- Meixner's edge conditions;
- The total field $E_z(x, y)$ satisfies suitable boundary conditions on the screen.

Classic electromagnetic theory deals with ordinary derivatives and integrals. Maxwell equations are the main equations that relate electric and magnetic fields via curl operator which is expressed by ordinary derivatives. Classic models and concepts in electromagnetics can be generalized by utilizing fractional operators.

Fractional operators have found many applications in various problems of electromagnetics. These operators are defined as fractionalizations of some commonly used operators. For example, fractional derivatives and integrals [20], [14] are generalizations of derivative and integral. Fractional curl operator defined by N. Engheta [5] is a fractionalized analogue of conventional curl operator. The detailed description of fractional calculus one can find in books [20], [14].

Fractionalization of operators is interesting from theoretical and practical points of view. Fractional Fourier transform and its applications were studied by V. Namias, A. W. Lohmann, D. Mendlovic and H. M. Ozaktas [9],[**?**],[15]. Fractionalization of Hankel transform was considered in [13]. Differential equations of fractional order are discussed in book [16], where was noted that the classical solution to diffraction problem on a wedge can be expressed using the fractional derivative of the order $1/2$.

It was shown by A.Turski [21] that Helmholtz equation has four eigen solutions, each is expressed using the fractional derivative of the order $1/2$. It must be noted that the kernel of Riemann-Liouville integral when the fractional order equal to $1/2$ is similar to the integral kernel frequently used in reflection problems. In some cases application of fractional derivatives allows to reduce the problem to more simple equations. For instance, it can be done for parabolic Schrdinger equation and wave equation in parabolic approximation [12].

It is possible to introduce fractional Green's function G^α by using fractional derivatives that is defined as fractional derivative of the ordinary Green's function of the free space. In two-dimensional case $(0 \le \alpha \le 1)$ G^α can be expressed as [4]:

$$G^\alpha\left(x - x', y\right) = -i/4 D^\alpha_{ky} H_0^{(1)}\left(k\sqrt{\left(x - x'\right)^2 + y^2}\right),$$

where $H_0^{(1)}(x)$ is Hankel function of the first kind of zeroth order.

We use the symbol $D^\alpha_y f$ throughout the paper to denote operator ${}_{-\infty}D^\alpha_y f$ which is defined by the integral of Riemann-Liouville on semi-infinite interval [20]:

$${}_{-\infty}D^\alpha_y f(y) = \frac{1}{(1-\alpha)} \frac{d}{dy} \int_{-\infty}^{y} \frac{f(t)}{(y-t)^\alpha} dt,$$

where $\Gamma(\alpha)$ is Gamma function, $f \in L_1$. Fractional Green's function G^α corresponds to one-dimensional and two-dimensional Green's functions for special cases of fractional order (FO) $\alpha = 0$ and $\alpha = 1$, respectively [4].

Fractional Green's function can be used to generalize Green's theorem. The idea was formulated by E.I.Veliev and N.Engheta [24]. The source-free case of fractional Green's theorem is expressed as [23]:

$$
{}_{-\infty}D_x^{\mu+\nu}\psi(r) = \frac{1}{4\pi}\oint_S \left[{}_{-\infty}D_{x_0}^{\nu}G(r,r_0)\, grad_{0-\infty}D_{x_0}^{\mu}\psi(r_0) - \right.
$$
$$
\left. - {}_{-\infty}D_{x_0}^{\mu}\psi(r_0)\, grad_{0-\infty}D_{x_0}^{\nu}G(r,r_0)\right] ds_0, \qquad (1)
$$

where the point r is inside the surface S. ${}_{-\infty}D_x^{\beta}\psi(r) = 0$ when r is outside S. Special cases of fractional Green's theorem were studied in [23]. Green's theorems are useful tools to present functions which describe scattered fields in scattering problems. Scattered field outside a scattering object, where boundary conditions are introduced, is usually of major interest. The scattered field is presented as integral over the object's surface with an unknown density function.

If one completes the equation (1) with boundary conditions (BC) and conditions on the infinity ($\vec{r} \to \infty$) then the representation for ${}_{-\infty}D_x^{\alpha}\psi(r)$ can be obtained that is true for outer domain. This representation will be used in this paper for representation of a scattered field. Fractional Green's theorem yields to consideration of boundary conditions with fractional derivative.

Fractional Green's theorem will be used to present the scattered field via fractional Green's function. This representation yields to the fractional order difference-integral equation (FODIE). The method to solve this FODIE is proposed for the cases when the FO $\alpha \in [0,1]$. For the limit cases of FO $\alpha = 0$ and 1 the equation is reduced to known integral equations used for perfect electric conductor (PEC) and perfect magnetic conductor (PMC) boundaries, respectively. The proposed method generalizes known method used for PEC and PMC strip and half-plane. As will be shown later the method allows obtaining a solution for one value $\alpha = 0,5$ in an explicit analytical form.

The purpose of this work is to build an effective analytic-numerical method to solve two-dimensional problems of scattering by boundaries described by fractional boundary conditions (FBC) with FO $\alpha \in [0,1]$. The proposed method is applied to model scattering objects – a strip and a half plane.

The method is based on presenting the unknown function as a series of orthogonal polynomials: Gegenbauer polynomials for a strip and Lager polynomials for a half-plane. The degree of polynomials and the weight functions depend on the FOα. These representations result in special type of edge conditions. The FO is chosen so that it allows satisfying edge conditions. FODIE can be reduced to coupled integral equations (CIE) using Fourier transform. Then, using the properties of discontinuous integrals of Weber-Schafheitling (for a strip) and Fourier representation for Lager polynomials (for a half-plane), the CIE are reduced to the infinite system of linear algebraic equations (SLAE) in respect to unknown coefficients in series of orthogonal polynomials. SLAE allows obtaining the solution with any desired accuracy using the reduction method. Physical characteristics of the considered scattering objects can be obtained from the coefficients found by solving SLAE.

2. Strip with fractional boundary conditions

Let us consider a two-dimensional problem of diffraction of E-polarized plane wave on a strip with FBC

$$D_{ky}^{\alpha} E_z(x, y) = 0, y \to \pm 0, \quad -a < x < a. \tag{2}$$

Here, for convenience, the fractional derivative is applied with respect to dimensionless variable ky. An infinitely thin strip of width $2a$ is located in the plane $y = 0$ and infinite along the axis z. The incident field $\vec{E}^i = \vec{z} E_z^i(x, y)$ is described by expression $E_z^i(x, y) = e^{-ik(x\cos\theta + y\sin\theta)}$, where θ is the incidence angle, $k = 2\pi/\lambda$ is the wave number. Time dependence is assumed to be $e^{-i\omega t}$ and deprecated throughout the paper. The scattered field is defined by the function $\vec{E}^s = \vec{z} E_z^s(x, y)$. The total field $\vec{E} = \vec{z} E_z(x, y)$ is a sum of the incident field and scattered field: $E_z = E_z^i + E_z^s$. The solution of the problem, i.e. function $E_z(x, y)$, must satisfy the following conditions:

1) Helmholtz equation outside the strip;

2) Radiation conditions of Zommerfeld at the infinity (for the scattered field only $E_z^s(x, y)$);

3) Edge conditions (for $y = 0$, $x \to \pm a$);

4) Fractional boundary conditions (FBC) (2).

We present the scattered field $E_z^s(x, y)$ via the fractional Green's function

$$E_z^s(x, y) \equiv \int_{-a}^{a} f^{1-\alpha}(x') \, G^{\alpha}(x - x', y) \, dx', \tag{3}$$

where $f^{1-\alpha}(x)$ is an unknown function named density of the fractional potential. The representation (3) comes from application of fractional Green's theorem (1). For special values of FO $\alpha = 0$ and $\alpha = 1$, the representation (3) is reduced to known representations for single and double-layer potentials, commonly used to solve scattering problems for BC of Dirichlet and Neumann types, respectively .

We get the following FODIE after substituting $E_z(x, y)$ into FBC (2):

$$\lim_{y \to 0} D_{ky}^{\alpha} \int_{-a}^{a} f^{1-\alpha}(x') \, G^{\alpha}(x - x', y) \, dx' = -\lim_{y \to 0} D_{ky}^{\alpha} E_z^i(x, y), \tag{4}$$

where the right part of the equation is the known function and $f^{1-\alpha}(x)$ is an unknown function to be found.

It is convenient to use Fourier transform. Let us introduce dimensionless variable $\xi = x/a$ and a new function $\tilde{f}^{1-\alpha}(\xi) \equiv a f^{1-\alpha}(a\xi)$ for $\xi \in [-1, 1]$ and $\tilde{f}^{1-\alpha}(\xi) \equiv 0$ for $|\xi| > 1$. The Fourier transform of the function $f^{1-\alpha}(x)$ is defined as:

$$F^{1-\alpha}(q) \equiv \int_{-1}^{1} \tilde{f}^{1-\alpha}(\xi) \, e^{-ika\xi q} d\xi, \tilde{f}^{1-\alpha}(\xi) = \frac{ka}{2\pi} \int_{-\infty}^{\infty} F^{1-\alpha}(q) \, e^{ika\xi q} dq.$$

Using the spectral representation for the Hankel function $H_0^{(1)}\left(k\sqrt{(x - x')^2 + y^2}\right)$ like [7], [8]:

$$H_0^{(1)}\left(k\sqrt{(x-x')^2+y^2}\right)=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{ik\left[(x-x')q+|y|\sqrt{1-q^2}\right]}\frac{dq}{\sqrt{1-q^2}},$$

where $Im\sqrt{1-q^2}>0$, we obtain for the fractional Green's function $G^{\nu}(x-x',y)$ expression $G^{\alpha}(x-x',y-y')=-i\frac{e^{\pm i\pi\alpha/2}}{4\pi}\int_{-\infty}^{\infty}e^{ik\left[(x-x')q+|y-y'|\sqrt{1-q^2}\right]}\left(1-q^2\right)^{(\alpha-1)/2}dq.$ Taking into account this we get the representation for the scattered field via the Fourier image $F^{1-\alpha}(q)$:

$$E_z^s(x,y)=-i\frac{e^{\pm i\pi\alpha/2}}{4\pi}\int_{-\infty}^{\infty}\frac{F^{1-\alpha}(q)\,e^{ik\left(qx+|y|\sqrt{1-q^2}\right)}}{(1-q^2)^{(1-\alpha)/2}}dq.$$

Now FODIE (4) is reduced to the CIE in respect to the unknown function $F^{1-\alpha}(q)$:

$$\begin{cases}\int_{-\infty}^{\infty}F^{1-\alpha}(q)\,e^{ika\xi\,q}\left(1-q^2\right)^{\alpha-1/2}dq=-4\pi e^{i\pi/2(1-\alpha)}\sin^{\alpha}\theta e^{-ika\xi\cos\theta}\,,\ \xi\in[-1,\,1]\,,\\ \int_{-\infty}^{\infty}F^{1-\alpha}(q)\,e^{ika\xi\,q}dq=0\,,\quad |\xi|>1.\end{cases}\tag{5}$$

It should be noted that CIE (5) for $\alpha=0$ results in CIE for a scattering by E-polarized plane wave on a PEC infinitely thin strip, while for $\alpha=1$ CIE describes the problem of diffraction on a PMC strip. CIE (5) is more general and includes other CIE considered earlier.

Let us analyze one special case $\alpha=0,5$, when the equation (5) has an analytical solution for any value of the frequency parameter ka. Indeed, it is easy to obtain the solution from (5) in the following form

$$\tilde{f}^{0,5}(\xi)=-2ika\sin^{1/2}\theta\,e^{-ika\,\xi\cos\theta+i\pi/4},F^{0,5}(q)=-4i\sin^{1/2}\theta\,e^{i\pi/4}\frac{\sin ka\,(q+\cos\theta)}{q+\cos\theta}.$$

Let us know solution of the CIE (5) for general cases of $0<\alpha<1$. We require $\tilde{f}^{1-\alpha}(\xi)$ to have the following behavior on the edges

$$\tilde{f}^{1-\alpha}(\xi)=O\left(\left(1-\xi^2\right)^{\alpha-1/2}\right)\,,\ \xi\to\pm1,\tag{6}$$

since $\tilde{f}^{1-\alpha}(\xi)$ must satisfy edge conditions for $\xi\to\pm1$. As special cases $\alpha=0$ and $\alpha=1$ these conditions have the form

$$\tilde{f}^{1-\alpha}(\xi)=\begin{cases}O\left(\left(1-\xi^2\right)^{-1/2}\right)\,,\ \alpha=0,\\ O\left(\left(1-\xi^2\right)^{1/2}\right),\ \alpha=1,\end{cases}\quad\xi\to\pm1.\tag{7}$$

The conditions (7) are well-known as Meixner's edge conditions [6].

We will search $\tilde{f}^{1-\alpha}(\xi)$ as a series by Gegenbauer polynomials which are orthogonal on the interval $(-1,\,1)$:

$$\tilde{f}^{1-\alpha}(\xi) = \left(1-\xi^2\right)^{\alpha-1/2} \sum_{n=0}^{\infty} f_n^{\alpha} \frac{1}{\alpha} C_n^{\alpha}(\xi), \tag{8}$$

where f_n^{α} are unknown coefficients, for case when $\tilde{f}^{1-\alpha}(\xi)$ satisfies the edge conditions (6). It can be noted that Gegenbauer polynomials $C_n^{\alpha}(\xi)$ are reduced to Chebyshev polynomials of the first and second kind $T_n(\xi)$, $U_n(\xi)$ [1] when $\alpha = 0$ and $\alpha = 1$, respectively,

$$\lim_{\alpha\to 0} \frac{C_n^{\alpha}(\xi)}{\alpha} = \begin{cases} \frac{2}{n} T_n(\xi)\ ,\ n \neq 0\,, \\ 1\,,\ n = 0\,, \end{cases} \quad \lim_{\alpha\to 1} \frac{C_n^{\alpha}(\xi)}{\alpha} = C_n^1(\xi) = U_n(\xi)\,. \tag{9}$$

From (9) we can say that the Gegenbauer polynomials are intermediate polynomials between Chebyshev polynomials of the first and the second kind.

After application of Fourier transform to the series (8) and utilizing the property [8]:

$$\int_{-1}^{1} \left(1-\xi^2\right)^{\alpha-1/2} C_n^{\alpha}(\xi)\, e^{ib\xi\, q} d\xi = \frac{2\pi\,(-i)^n\,(n+2\alpha)}{(n+1)\,(\alpha)} \frac{J_{n+\alpha}(bq)}{(2bq)^{\alpha}},$$

we obtain the representation for $F^{1-\alpha}$ as a series by Bessel functions:

$$F^{1-\alpha}(q) = \frac{2\pi}{(\alpha+1)} \sum_{n=0}^{\infty} (-i)^n f_n^{\alpha} \frac{(n+2\alpha)}{(n+1)} \frac{J_{n+\alpha}(kaq)}{(2kaq)^{\alpha}}. \tag{10}$$

We modify the equation (5) by multiplying both parts by $e^{-ika\xi\tau}$ and integrating by ξ from -1 to 1. Finally, we obtained

$$\int_{-\infty}^{\infty} F^{1-\alpha}(q) \frac{\sin ka\,(q-\tau)}{q-\tau} \left(1-q^2\right)^{\alpha-1/2} dq = -4\pi e^{i\pi/2(1-\alpha)} \sin^{\alpha}\theta \frac{\sin ka\,(\tau+\cos\theta)}{\tau+\cos\theta}. \tag{11}$$

After substituting expression (10) into integral equation (11) and using the following relation for Bessel functions [8], [6]:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{J_{n+\alpha}(\varepsilon q)}{q^{\alpha}} \frac{\sin\varepsilon\,(q-\beta)}{q-\beta} dq = \frac{J_{n+\alpha}(\varepsilon\beta)}{\beta^{\alpha}},$$

we get SLAE to find the coefficients f_n^{α}:

$$\sum_{n=0}^{\infty} (-i)^n \frac{(n+2\alpha)}{(n+1)} f_n^{\alpha} C_{kn}^{\alpha} = \gamma_k^{\alpha}\,,\ \ k = 0, 1, 2, ..., \tag{12}$$

with the matrix coefficients defined as

$$C_{kn}^{\alpha} \int_{-\infty}^{\infty} J_{n+\alpha}(kaq)\, J_{k+\alpha}(kaq) \left(1-q^2\right)^{\alpha-1/2} \frac{dq}{q^{2\alpha}},$$

$$\gamma_k^{\alpha} = -2\,(\alpha+1)\,(2ka)^{\alpha}\, i^{1-\alpha} \sin^{\alpha}\theta \frac{J_{k+\alpha}(ka\cos\theta)}{\cos\theta}.$$

Let us show that SLAE (12) can be reduced to SLAE of Fredholm type of the second kind. We introduce function $\delta_\alpha(\beta)$ as follows:

$$\delta_\alpha(\beta) = |\beta|^{2\alpha-1} e^{i\pi(\alpha-1/2)} \left[\left(1 - \frac{1}{\beta^2}\right)^{\alpha-1/2} - 1 \right], \tag{13}$$

$$\delta_\alpha(-\beta) = \delta_\alpha(\alpha)\ ,\ \delta_\alpha(\beta) \Big/_{\alpha=0,5} = 0.$$

The function $\delta_\alpha(\beta)$ has the following behavior at the infinity ($\beta \to \infty$):

$$\delta_\alpha(\beta) = O\left((\alpha - 1/2)\,\beta^{2\alpha-3}\right), \beta \to \infty.$$

It results from [4] that

$$\frac{\left(1-\beta^2\right)^{\alpha-1/2}}{\beta^{2\alpha}} = \frac{\delta_\alpha(\beta)}{\beta^{2\alpha}} + \frac{e^{i\pi(\alpha-1/2)}}{|\beta|}. \tag{14}$$

Taking into account the expression (14) matrix coefficients C_{kn}^{α} can be presented as a sum

$$C_{kn}^{\alpha} = C_{kn}^{1\alpha} + C_{kn}^{2\alpha},$$

$$C_{kn}^{1\alpha} = e^{i\pi(\alpha-1/2)} \left[1 + (-1)^{k+n}\right] \int_0^\infty J_{k+\alpha}(kaq)\, J_{n+\alpha}(kaq)\, \frac{dq}{q},$$

$$C_{kn}^{2\alpha} = \left[1 + (-1)^{k+n}\right] \int_0^\infty J_{k+\alpha}(kaq)\, J_{n+\alpha}(kaq)\, \delta_\alpha(q)\, \frac{dq}{q^{2\alpha}}.$$

The integral in $C_{kn}^{1\alpha}$ is expressed analytically [8, 211]: $C_{kn}^{1\alpha} = e^{i\pi(\alpha-1/2)} \frac{1}{k+\alpha} \delta_{kn}$, where δ_{kn} is Kronecker symbol. Finally, we get modified SLAE

$$f_n^{\alpha} + \sum_{n=0}^{\infty} f_n^{\alpha} \tilde{C}_{kn}^{\alpha} = \tilde{\gamma}_k^{\alpha}\ ,\quad k = 0, 1, 2, ..., \tag{15}$$

where

$$\tilde{C}_{kn}^{2\alpha} = (-1)^{n-k} \left[1 + (-1)^{k+n}\right] \frac{(n+2\alpha)}{(n+1)} \frac{(k+1)}{(k+2\alpha)} (k+\alpha)\, e^{-i\pi(\alpha-1/2)} C_{kn}^{2\alpha},$$

$$\tilde{\gamma}_k^{\alpha} = i^k (k+\alpha)\, e^{-i\pi(\alpha-1/2)} \frac{(k+1)}{(k+2\alpha)} \gamma_k^{\alpha}.$$

It can be shown that the coefficients [7], [11] satisfy equations

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \left|\tilde{C}_{kn}^{2\alpha}\right|^2 < \infty\, ,\ \sum_{n=0}^{\infty} \left|\tilde{\gamma}_k^{\alpha}\right|^2 < \infty.$$

It means that SLAE (15) is of Fredholm type of the second kind and the unknown coefficients f_k^α can be found with any given accuracy using the reduction method of solving infinite SLAE.

For the special case of the FO $\alpha = 0,5$ SLAE (15) is reduced to more simple system. Indeed, we have $\tilde{C}_{kn}^{2\alpha} \big/_{\alpha=0,5} = 0$. SLAE can be solved analytically

$$f_k^{0,5} = \tilde{\gamma}_k^{0,5} = i^k\,(k+0,5)\,\gamma_k^{0,5} = -i\sqrt{\frac{\pi\, k\, a}{2\cos\theta}}e^{i\pi/4}\,(-i)^k\,(2k+1)\sin^{1/2}\theta\, J_{k+0,5}\,(ka\cos\theta)\,.$$

Substituting $f_n^{0,5}$ into (8) and (10) we get the representations

$$\tilde{f}^{1-\alpha}\,(\xi)\Big/_{\alpha=0,5} = 2\sum_{n=0}^{\infty} f_n^{0,5} C_n^{0,5}\,(\xi),\;\; F^{1-\alpha}\,(q) = \frac{2\pi}{(3/2)}\sum_{n=0}^{\infty}(-i)^n\, f_n^{0,5}\frac{J_{n+0,5}\,(kaq)}{\sqrt{2kaq}}.$$

Using the formulas [1, 185]:

$$2kae^{-ika\cos\theta\xi} = \sqrt{\frac{2\pi ka}{\cos\theta}}\sum_{n=0}^{\infty}(-i)^n\,(2n+1)\,J_{n+0,5}\,(ka\cos\theta)\,C_n^{0,5}\,(\xi)\,,$$

and [8, 667]:

$$\sum_{n=0}^{\infty}(-1)^n\,(n+1/2)\,J_{n+0,5}\,(\varepsilon q_0)\,J_{n+0,5}\,(\varepsilon\, q) = \frac{\sqrt{qq_0}}{\pi\,(q+q_0)}\sin\varepsilon\,(q+q_0)\,,$$

we get the representations for $\tilde{f}^{0,5}\,(\xi)$ and $F^{0,5}\,(\alpha)$ which is confirmed with the expressions obtained above. After the coefficients f_n^α are found, the function of potential density $\tilde{f}^{1-\alpha}\,(\xi)$ and its Fourier transform $F^{1-\alpha}\,(q)$ can be obtained from equations (8) and (10), respectively.

Then we can obtain such electrodynamic characteristics of the scattered field as radiation pattern (RP), monostatic radar cross-section (MRCS), total scattering cross-section (TCS) and surface current densities. The scattered field $E_z^s\,(x,y)$ in the far-zone $kr\to\infty$ in the cylindrical coordinate system (r,φ), $x = r\cos\varphi$, $x = r\sin\varphi$, is expressed as

$$E_z^s\,(r,\varphi) = \frac{i}{4\pi}\,(\pm i)^\alpha\int_{-\infty}^{+\infty} F^{1-\alpha}\,(\cos\beta)\,e^{ikr\cos(\varphi\pm\beta)}\sin^\alpha\beta\, d\beta,$$

where the upper sign is chosen for $\varphi\in[0,\,\pi]$, and the bottom one when $\varphi\in[\pi,\,2\pi]$. Using the stationary phase method for $kr\to\infty$ we present $E_z^s\,(x,y)$ as

$$E_z^s\,(x,y)\approx A\,(kr)\,\Phi^\alpha\,(\varphi)\,, kr\to\infty,$$

where

$$A\,(kr) = \sqrt{\frac{2}{\pi kr}}e^{ikr-i\frac{\pi}{4}}, \Phi^\alpha\,(\varphi) = -\frac{i}{4}\,(\pm i)^\alpha\, F^{1-\alpha}\,(\cos\varphi)\sin^\alpha\varphi.$$

The function $\Phi^{\alpha}(\varphi)$ describes RP and can be expressed via the coefficients f_n^{α} as

$$\Phi^{\alpha}(\varphi) = \frac{\pi i (\pm i)^{\alpha}}{2(\alpha+1)} \tan^{\alpha}\varphi \sum_{n=0}^{\infty} (-i)^n f_n^{\alpha} \frac{(n+2\alpha)}{(n+1)} \frac{J_{n+\alpha}(ka\cos\varphi)}{(2ka)^{\alpha}}.$$

In physical optics (PO) approximation $\Phi^{\alpha}(\varphi)$ ($k << 1$) has more simple form. Using the following formula

$$\lim_{ka\to\infty} \frac{\sin ka(\alpha-\beta)}{\alpha-\beta} = \pi\delta(\alpha-\beta),$$

in IE (11) we get the expressions for $F^{\alpha}(q)$ and $\Phi^{\alpha}(\varphi)$:

$$F^{1-\alpha}(q) \approx -4i^{\alpha} \frac{\sin^{1-\alpha}\theta}{(1-q^2)^{(1-2\alpha)/2}} \frac{\sin ka(q-\cos\theta)}{q-\cos\theta},$$

$$\Phi^{\alpha}(\varphi) \approx (\mp 1)^{\alpha} \sin\varphi \left(\frac{\sin\theta}{\sin\varphi}\right)^{\alpha} \frac{\sin ka(\cos\varphi+\cos\theta)}{\cos\varphi+\cos\theta}.$$

For the special case $\alpha = 0,5$ and any value of ka we get an analytical expression for the RP

$$\Phi^{0,5}(\varphi) = (\mp 1)^{1/2} \sqrt{\sin\varphi\sin\theta} \frac{\sin ka(\cos\varphi+\cos\theta)}{\cos\varphi+\cos\theta}.$$

Cross section (CS) $\frac{\sigma_{2d}}{\lambda}$ is expressed from RP $\Phi(\varphi)$ as $\frac{\sigma_{2d}}{\lambda}(\varphi) = \frac{2}{\pi}|\Phi(\varphi)|^2$. MRCS σ_{2D}^{mono} is defined as $\sigma_{2D}^{mono} = \frac{\sigma_{2d}}{\lambda}(\theta) = \frac{2}{\pi}|\Phi(\theta)|^2$. We have in PO approximation the following representations

$$\frac{\sigma_{2d}}{\lambda} = \frac{2}{\pi}\sin^2\varphi \left(\frac{\sin\theta}{\sin\varphi}\right)^{2\alpha} \left\{\frac{\sin ka(\cos\varphi+\cos\theta)}{\cos\varphi+\cos\theta}\right\}^2, \; k << 1,$$

$$\sigma_{2D}^{mono} = \frac{2}{\pi}\sin^2\theta \left\{\frac{\sin ka(2\cos\theta)}{2\cos\theta}\right\}^2, k << 1.$$

Surface currents are defined as the discontinuity of field components on the strip boundary

$$j_z^{\alpha(e)} = -(H_x(x,+0) - H_x(x,-0)), j_x^{\alpha(m)} = -(E_z(x,+0) - E_z(x,-0)).$$

In E-polarization case the electrical current has only one non-zero component, z-component, $\vec{j}^{\alpha(e)} = \vec{z} j_z^{\alpha(e)}$, while the magnetic current has only x-component, $\vec{j}^{\alpha(m)} = \vec{x} j_x^{\alpha(m)}$. Physical current can be expressed via the fractional potential density $\tilde{f}^{1-\alpha}(\xi)$ using the integrals

$$j_z^{\alpha(e)} = -2i\cos\left(\frac{\pi\alpha}{2}\right) \frac{i}{4\pi} \int_{-\infty}^{\infty} F^{1-\alpha}(q) e^{ikqx} (1-q^2)^{\alpha/2} dq, \tag{16}$$

$$j_x^{\alpha(m)} = -2\sin\left(\frac{\pi\alpha}{2}\right)\frac{i}{4\pi}\int_{-\infty}^{\infty} F^{1-\alpha}(q)\, e^{ik\beta x}\left(1-q^2\right)^{\alpha/2-1/2} dq. \tag{17}$$

It is evident from equations (16), (17) that there exists only one current – electrical ($\alpha = 0$) or magnetic ($\alpha = 1$) for the limit values of the FO. There are two currents on the strip for intermediate values $0 < \alpha < 1$. This fact yields from fractional Green's theorem.

One can find detailed numerical results for physical characteristics of the strip with FBC in [20, 21].

In case when we have H-polarized incident plane wave $\vec{H}^i\left(0, 0, H_z^i\right)$, where $H_z^i(x, y) = e^{-ik(x\cos\theta + y\sin\theta)}$, the method proposed above can be applied. We define FBC as

$$D_{ky}^{1-\alpha} H_z(x, y) \Big/_{y\to\pm 0} = D_{ky}^{1-\alpha}\left[H_z^i(x, y) + H_z^s(x, y)\right] \Big/_{y\to\pm 0} = 0\ , x \in (-a, a)\,. \tag{18}$$

Case $\alpha = 0$ corresponds to diffraction of H-polarized plane wave on a PEC strip, while case $\alpha = 1$ describes diffraction of H-polarized plane wave on a PMC strip.

As before we present the scattered field via fractional Green's function

$$H_z^s(x, y) \equiv \int_{-a}^{a} f^{\alpha}\left(x'\right) G^{1-\alpha}\left(x - x', y\right) dx'. \tag{19}$$

After substituting (18) into FBC (19) we get the equation

$$\lim_{y\to 0} D_{ky}^{1-\alpha}\int_{-a}^{a} f^{\alpha}\left(x'\right) G^{1-\alpha}\left(x - x', y\right) dx' = -\lim_{y\to 0} D_{ky}^{1-\alpha} H_z^i(x, y)\,.$$

This equation can be solved by repeating all steps as for E-polarization case just changing α to $1 - \alpha$.

3. Half-plane with fractional boundary conditions

Here we will consider a problem of scattering of plane waves by a half-plane . The method introduced to solve integral equation (IE) for a finite object (a strip) will be modified to solve IE for semi-infinite objects such as half-plane. There are many papers devoted to classical problem of diffraction by a half-plane. The method to solve the scattering problem for a perfectly conducting half-plane is presented in [7]. Usually, it is solved using Wiener-Hopf method. First application of the method to a PEC half-plane can be referred to the papers of Copson [3] in 1946 and independently to papers of Carlson and Heins in 1947 [2]. In 1952 Senior first applied Wiener-Hopf method to a diffraction by an impedance half-plane [17] and later oblique incidence was considered [18]. Diffraction problems by a resistive and conductive half-plane and also by different types of junctions are analyzed in details in [19].

We propose a new approach for a rigorous analysis of the considered problem which generalized results [22] for PEC boundaries and include them as special cases. The proposed method allowed to reduce the problem to a SLAE to find unknown coefficients of series used in representation of the scattered field.

Let an E-polarized plane wave $E_z^i(x,y) = e^{-ik(x\cos\theta + y\sin\theta)}$, where θ is an angle of incidence, to be scattered by a half-plane ($y = 0,\ x > 0$). The total field $E_z = E_z^i + E_z^s$ satisfies FBC

$$D_{ky}^{\alpha} E_z(x,y) = 0, y \to \pm 0,\ x > 0. \tag{20}$$

Also Meixner's edge conditions must be satisfied for $x \to 0$ [7].

We present the scattered field using the fractional potential

$$E_z^s(x,y) \equiv \int_0^{\infty} f^{1-\alpha}(x')\, G^{\alpha}(x - x',\, y)\, dx', \tag{21}$$

where $f^{1-\alpha}(x)$ is an unknown function, G^{α} is fractional Green's function defined earlier. After substituting the total field into FBC (20) we get FODIE:

$$\frac{-i}{4}\lim_{y\to 0} D_{ky}^{2\alpha} \int_0^{\infty} f^{1-\alpha}(x')\, H_0^{(1)}\left(k\sqrt{(x-x')^2 + y^2}\right) dx' = -\lim_{y\to 0} D_{ky}^{\alpha} E_z^i(x,y),\ x > 0. \tag{22}$$

The Fourier transform of $f^{1-\alpha}(x)$ is defined as

$$F^{1-\alpha}(q) = \int_{-\infty}^{\infty} \tilde{f}^{1-\alpha}(\xi)\, e^{-ikq\,\xi} d\xi = \int_0^{\infty} f^{1-\alpha}(x)\, e^{-ik\beta x} dx,$$

$$\tilde{f}^{1-\alpha}(\xi) = \frac{k}{2\pi}\int_{-\infty}^{\infty} F^{1-\alpha}(\beta)\, e^{ik\beta\,\xi} d\beta,$$

where $\tilde{f}^{1-\alpha}(\xi) \equiv f^{1-\alpha}(\xi)(\xi > 0)$, $\tilde{f}^{1-\alpha}(\xi) \equiv 0\ (\xi < 0)$. Using the technique proposed for a strip with FBC, FODIE (22) is reduced to CIE

$$\begin{cases} \int_{-\infty}^{\infty} F^{1-\alpha}(q)\, e^{ik\xi\, q}\left(1 - q^2\right)^{\alpha - 1/2} dq = -4\pi e^{i\pi/2(1-\alpha)} \sin^{\alpha}\theta e^{-ik\xi\cos\theta}\ ,\ \xi > 0\,, \\ \int_{-\infty}^{\infty} F^{1-\alpha}(q)\, e^{ik\xi\, q} dq = 0\,, \qquad \xi < 0. \end{cases} \tag{23}$$

For one special case $\alpha = 0,5$ CIE (23) has an analytical solution

$$f^{1/2}(x) = -2\sin^{1/2}\theta\, e^{i\pi/4} e^{-ikx\cos\theta}, F^{1/2}(q) = -2\sin^{1/2}\theta\, e^{i\pi/4}\frac{\pi}{k}\delta(q + \cos\theta)\,.$$

In this case the scattered field can be obtained in the following form

$$E_z^s(x,y) = \frac{i}{2k} e^{\pm i\pi\alpha/2} e^{i\pi/4} \sin^{\alpha - 1/2}\theta\, e^{ik(-\cos\theta\, x + |y|\sin\theta)}.$$

Let us go back to a general case $0 < \alpha < 1$. The function $\tilde{f}^{1-\alpha}(\xi)$ must satisfy edge conditions for $\xi \to 0$. Let $\tilde{f}^{1-\alpha}(\xi)$ satisfies the following condition

$$\tilde{f}^{1-\alpha}(\xi) = O\left(\xi^{\alpha - 1/2}\right), \xi \to 0. \tag{24}$$

For special cases $\alpha = 0$ and $\alpha = 1$ edge conditions reduce to well-known equations [23], [6] for perfectly conducting boundaries. We present the function $\tilde{f}^{1-\alpha}$ as a series by Lagger polynomials with unknown coefficients f_n^α [10]:

$$\tilde{f}^{1-\alpha}(x) = e^{-x} x^{\alpha-1/2} \sum_{n=0}^{\infty} f_n^\alpha L_n^{\alpha-1/2}(2x) . \tag{25}$$

The representation (25) guarantees that $\tilde{f}^{1-\alpha}$ satisfies the edge conditions (24). After substituting (25) into the first equation of (23) we get IE

$$\sum_{n=0}^{\infty} f_n^\alpha \int_{-\infty}^{\infty} \left[\int_0^\infty e^{-t} t^{\alpha-1/2} L_n^{\alpha-1/2}(2t) e^{-ikqt} dt \right] \times e^{ik\xi q} \left(1-q^2\right)^{\alpha-1/2} dq = R(\xi), \tag{26}$$

where $R(\xi) = -4\pi e^{i\pi/2(1-\alpha)} \sin^\alpha \theta \, e^{-ik\xi\cos\theta}$ is known. Using the representation for Fourier transform of Lagger polynomials [8, 462] and making some transforms we reduce IE (26) to

$$\sum_{n=0}^{\infty} f_n^\alpha \frac{(n+\alpha+1/2)}{(n+1)} \times \int_{-\infty}^{\infty} \frac{(ikq-1)^n}{(ikq+1)^{n+\alpha+1/2}} \left(1-q^2\right)^{\alpha-1/2} e^{ik\xi q} = R(\xi), \xi > 0. \tag{27}$$

To discretize the equation (27) we integrate both sides $\int_0^\infty (\cdot) \, e^{-\xi} \xi^{\alpha-1/2} L_m^{\alpha-1/2}(2\xi) \, d\xi$ and get SLAE

$$\sum_{n=0}^{\infty} f_n^\alpha C_{mn}^\alpha = B_m^\alpha, m = 0, 1, 2, ..., \infty,$$

where the matrix elements are defined as

$$C_{mn}^\alpha = \frac{(n+\alpha+1/2)}{(n+1)} \int_{-\infty}^{\infty} \frac{(ikq+1)^{m-n-\alpha-1/2}}{(ikq-1)^{n-m-\alpha-1/2}} \left(1-q^2\right)^{\alpha-1/2} dq,$$

$$B_m^\alpha = -4\pi \, e^{i\pi/2(1-\alpha)} (-1)^{\alpha+1/2} \frac{\sin^\alpha \theta \, (1-ik\cos\theta)^m}{(1+ik\cos\theta)^{\alpha+m+1/2}}.$$

It can be shown that the unknown coefficients f_n^α can be found with any desired accuracy by solving SLAE. $\tilde{f}^{1-\alpha}(x)$ is expressed as the series (25) that allows to obtain the scattered field (21).

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