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TOPICAL REVIEW

Inverse scattering problem on the axis for the triangular 2×2 matrix potential with or without a virtual level

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Abstract. A survey of authors' works. The characteristic properties of scattering data for the Schrödinger operator on the axis with a triangular 2×2 matrix potential are obtained in the case when a simple or multiple virtual levels is present, as well as in the case of absent virtual level. Under a multiple virtual level, a pole for the reflection coefficient at $k = 0$ is possible. For this case, the modified Parseval equality is constructed.

Key Words and Phrases: scattering on the axis, inverse problem, triangular matrix potential, discrete level, virtual level, Parseval equality

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1. Introduction

Initially, a complete solution of the inverse scattering problem (ISP) on the axis for the Schrödinger equation

$$-Y'' + V(x)Y = k^2Y, \qquad -\infty < x < \infty, \tag{1.1}$$

with a real scalar potential having the first moment, which allows presence of a virtual level (VL), i.e., (1.1) for $k = 0$ admits existence of bounded on the whole axis non-trivial solutions, has been given in the monograph by V. A. Marchenko [18, Ch. 3] (see also [19]). In the case of the potential having the second moment, a solution for the ISP is considered in [11], [16, ch. VI].

In [27], [29], [30], [31] the authors solve the ISP on the axis for the equation (1.1) with a triangular 2×2 matrix potential. In this case, necessary and sufficient conditions are obtained for a given collection of values to be the scattering data (SD) for a problem of this form. A solution of such problem is reduced to the Marchenko equation under the assumption that the upper triangular 2×2 matrix potential $V(x)$ has the second moment on the axis and real diagonal elements:

$$(1 + x^2)|V(x)| \in L^1(-\infty, +\infty), \qquad \text{Im } v_{ll}(x) = 0, \quad l = 1, 2; \quad v_{21}(x) \equiv 0. \tag{1.2}$$

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Here $|V(x)|$ denotes the (operator) norm of the matrix $V(x)$. Besides, in the case of the matrix problem (1.1) in question we introduce a method of addition of a discrete spectrum (cf. in the scalar self-adjoint case [16, Ch. VI], [3]) and the method of elimination of eigenvalues.

The case when a VL exists (simple or multiple) for $k = 0$ appears to be essentially more difficult. Theorems 2 and 3 of this survey are devoted to the solution of ISP with a VL being present. For the case of no VL, see Theorem 1.

It should be noted that important results in scattering theory in the case of the Schrödinger operator both in 1- and 3-dimensional space, along with the works cited above [3], [11], [16], [18], [19] are contained in the monographs [1], [2], [5], [6], [10], [21], [22], [23], [25], [26], [28]. See also the bibliography therein. The eigenvalue expansion in 3-dimensional scattering problem for the Schrödinger equation was initially established in [24].

2. Basic definitions

Index '0' will be used to mark values which are either related to a problem with no discrete spectrum or those derived from SD of a problem with discrete spectrum whose existence is in no way used in the construction. Matrices are denoted by capital letters, while the matrix elements by the corresponding small letters.

$$I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad J \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad J^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We also use the matrix Wronski determinant defined as

$$W\{G(x), H(x)\} \equiv G(x)H'(x) - G'(x)H(x).$$

In addition to (1.1), we consider also the tilde- ($\sim$)-equation:

$$\widetilde{l}[\widetilde{Z}] \equiv -\widetilde{Z}'' + \widetilde{Z}V(x) = k^2\widetilde{Z}, \qquad -\infty < x < \infty. \tag{2.1}$$

The solutions $E_\pm(x,k)$, $\widetilde{E}_\pm(x,k)$ of (1.1), (2.1) with asymptotics

$$E_\pm(x,k) \sim e^{\pm ikx}I, \qquad \widetilde{E}_\pm(x,k) \sim e^{\pm ikx}I, \qquad x \to \pm\infty, \quad \operatorname{Im} k \geq 0, \tag{2.2}$$

are called the Jost solutions. These can be written in the form [15] (see also [18], [16], [1])

$$\begin{aligned} E_\pm(x,k) &= Ie^{\pm ikx} \pm \int_x^{\pm\infty} K_\pm(x,t)e^{\pm ikt}dt, \\ \widetilde{E}_\pm(x,k) &= Ie^{\pm ikx} \pm \int_x^{\pm\infty} \widetilde{K}_\pm(x,t)e^{\pm ikt}dt, \qquad \operatorname{Im} k \geq 0, \end{aligned} \tag{2.3}$$

in terms of transformation operators, where

$$V(x) = \mp 2dK_{\pm}(x,x)/dx = \mp 2d\widetilde{K}_{\pm}(x,x)/dx. \tag{2.4}$$

In addition to the Jost solutions (2.2), we will need the solutions

$$E^{\wedge}_{\pm}(x,k) \sim e^{\mp ikx} I, \qquad \widetilde{E}^{\wedge}_{\pm}(x,k) \sim e^{\mp ikx} I, \qquad x \to \pm\infty, \quad \mathrm{Im}\, k \geq 0, \quad k \neq 0, \tag{2.5}$$

which form fundamental systems together with $E_{\pm}(x,k)$ and, respectively, with $\widetilde{E}_{\pm}(x,k)$. Matrix solutions $E^{\wedge}_{\pm}(x,k)$ have been constructed and investigated in [1], and the solutions $\widetilde{E}^{\wedge}_{\pm}(x,k)$ can be constructed in a similar way. However, unlike the Jost solutions, the solutions (2.5) are not determined by their asymptotics unambiguously for Im $k > 0$. On the other hand, once one of the solutions (2.5) is fixed, let it be $E^{\wedge}_{+}(x,k)$, then the corresponding solution $\widetilde{E}^{\wedge}_{+}(x,k)$ is determined uniquely under the additional assumption

$$W\left\{\widetilde{E}^{\wedge}_{+}(x,k), E^{\wedge}_{+}(x,k)\right\} \equiv \widetilde{E}^{\wedge}_{+}(x,k)\frac{d}{dx}E^{\wedge}_{+}(x,k) - \frac{d}{dx}\widetilde{E}^{\wedge}_{+}(x,k)E^{\wedge}_{+}(x,k) = 0, \qquad \mathrm{Im}\, k \geq 0, \qquad k \neq 0. \tag{2.6}$$

Given an arbitrary $\varepsilon > 0$, the solutions (2.5) can be chosen analytic in k for $|k| > \varepsilon$, Im $k > 0$, and we will assume they are chosen exactly this way. (See also [21] for the scalar case.)

With real $k \neq 0$, the pairs of functions $E_{+}(x,\pm k)$ or $E_{-}(x,\pm k)$, together with $\widetilde{E}_{+}(x,\pm k)$ or $\widetilde{E}_{-}(x,\pm k)$, form fundamental systems of solutions for (1.1) or (2.1), respectively. Their Wronski determinants are independent of x, and (see, e.g., [4])

$$\begin{aligned} E_{+}(x,k) &= E_{-}(x,-k)A(k) + E_{-}(x,k)B(k),\\ E_{-}(x,k) &= E_{+}(x,-k)C(k) + E_{+}(x,k)D(k),\\ \widetilde{E}_{+}(x,k) &= C(k)\widetilde{E}_{-}(x,-k) - D(-k)\widetilde{E}_{-}(x,k),\\ \widetilde{E}_{-}(x,k) &= A(k)\widetilde{E}_{+}(x,-k) - B(-k)\widetilde{E}_{+}(x,k), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} A(k) &= \frac{1}{2ik}W\left\{\widetilde{E}_{-}(x,k), E_{+}(x,k)\right\};\\ C(k) &= -\frac{1}{2ik}W\left\{\widetilde{E}_{+}(x,k), E_{-}(x,k)\right\};\\ B(k) &= -\frac{1}{2ik}W\left\{\widetilde{E}_{-}(x,-k), E_{+}(x,k)\right\};\\ D(k) &= \frac{1}{2ik}W\left\{\widetilde{E}_{+}(x,-k), E_{-}(x,k)\right\}, \qquad k \in \mathbb{R}\backslash\{0\}. \end{aligned} \tag{2.8}$$

The values

$$\begin{aligned} R^{+}(k) &\equiv D(k)C(k)^{-1} = -A(k)^{-1}B(-k);\\ R^{-}(k) &\equiv B(k)A(k)^{-1} = -C(k)^{-1}D(-k), \end{aligned} \tag{2.9}$$

are called right (respectively, left) reflection coefficients. A relation between them is given by

$$R^-(k) = -A(-k)R^+(-k)A(k)^{-1} = -C(k)^{-1}R^+(-k)C(-k), \qquad k \in \mathbb{R}. \tag{2.10}$$

As a consequence of the well-known relations

$$\begin{gathered} A(-k)C(k) = I - B(k)D(k); \qquad C(-k)A(k) = I - D(k)B(k); \\ B(-k)C(k) + A(k)D(k) = D(-k)A(k) + C(k)B(k) = 0, \end{gathered} \tag{2.11}$$

which are themselves due to (2.7), (2.8), one has (see [18], [4]):

$$\begin{gathered} (I - R^-(-k)R^-(k))^{-1} = A(k)C(-k); \\ (I - R^+(-k)R^+(k))^{-1} = C(k)A(-k). \end{gathered} \tag{2.12}$$

The eigenvalues k_j^2 of the problem (1.1), $j = \overline{1,p}$, coincide with the collection of eigenvalues for scalar scattering problems with real potentials $v_{ll}(x)$, which are just the diagonal elements of the matrix potential $V(x)$. Therefore, k_j^2 are roots of the equation $\det A(k) = a_{11}(k)a_{22}(k) = 0$, $\operatorname{Im} k > 0$. Hence there are only finitely many eigenvalues, and $k_j^2 < 0$, $\operatorname{Im} k_j > 0$. Note that $a_{ll}(k) = c_{ll}(k)$ and $\det A(k) = \det C(k)$.

We call the polynomials

$$\begin{aligned} Z_j^+(t) &= -ie^{-ik_jt}\operatorname{Res}_{k_j}\left\{W^+(k)C(k)^{-1}e^{ikt}\right\}, \\ Z_j^-(t) &= -ie^{ik_jt}\operatorname{Res}_{k_j}\left\{W^-(k)A(k)^{-1}e^{-ikt}\right\}, \quad j = \overline{1,p}, \quad t \in \mathbb{R}, \\ \widetilde{Z}_j^+(t) &= -ie^{-ik_jt}\operatorname{Res}_{k_j}\left\{A^{-1}(k)\widetilde{W}^+(k)e^{ikt}\right\}. \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} W^\pm(k) &= \pm\frac{1}{2ik}W\left\{\widetilde{E}_\pm^\wedge(x,k), E_\mp(x,k)\right\}, \\ \widetilde{W}^+(k) &= -\frac{1}{2ik}W\left\{\widetilde{E}_-(x,k), E_+^\wedge(x,k)\right\}, \end{aligned} \tag{2.14}$$

respectively, the right and the left normalizing polynomials (compare to the scalar case [21], [17], [7]). Normalizing polynomials do not depend on the choice of $\widetilde{E}_\pm^\wedge$ in the expression (2.17) for $W^\pm(k)$.

In the upper half-plane, one has the following representation

$$E_-(x,k) = E_+(x,k)W^+(k) + E_+^\wedge(x,k)C(k), \qquad \operatorname{Im} k > 0. \tag{2.15}$$

Definition 1. *The problem* (1.1) *of the considered form will be said to have a multiplicity two (respectively, simple, i.e., multiplicity one) VL for* $k = 0$ *if* $\det\{kA(k)\}$ *has a multiplicity two (respectively, simple) root at* $k = 0$.

Note that $ka_{ll}(k)$ can have root of multiplicity at most one at $k = 0$ because $|a_{ll}(k)| \geq 1$ for $k \in \mathbb{R}$ and $d\{ka_{ll}(k)\}/dk$ are continuous.

This can be rephrased by saying that the problem (1.1) possesses a multiplicity two VL for $k = 0$ if every scalar equation of the form (1.1) with potentials $v_{11}(x)$ and $v_{22}(x)$ has a VL. The problem (1.1) has a simple VL if just one of the potentials $v_{ll}(x)$ has a VL.

Also note that def $\{kA(k)\}_{k=0} = 1$ in the case of simple VL, and it is 1 or 2 in the case of multiplicity two VL.

Definition 2. *A scattering data is a collection of values*

$$\left\{R^+(k),\ k \in \mathbb{R};\ k_j^2 < 0,\ Z_j^+(t),\ j = 1, \ldots, p < \infty\right\}, \tag{2.16}$$

where $R^+(k)$ is a matrix reflection coefficient, $Z_j^+(t)$ matrix normalizing polynomials, $k_j^2 < 0$ the discrete spectrum of the problem (1.1). $R^+(k)$, $Z_j^+(t)$ are upper triangular 2×2 matrices, as well as the potential $V(x)$. These are right SD; the left SD will be indexed by '-'. If $R^+(k)$ is the right matrix reflection coefficient of a problem of the form (1.1), then its diagonal elements $r_{ll}^+(k)$ $(l = 1, 2)$ are right reflection coefficients for scalar problems of the form (1.1) with potentials $v_{ll}(x)$ $(l = 1, 2)$, respectively. Therefore, $r_{11}^+(k)$ and $r_{22}^+(k)$ possess all the properties of scalar reflection coefficients (see [18]). In particular, they are continuous on the axis (for $k = 0$ see also [14]), and the potential $v_{ll}(x)$ $(l = 1$ or $2)$ determines a problem with no VL if and only if $r_{ll}^+(0) = -1$, $l = 1, 2$. On the contrary, the potential $v_{ll}(x)$ makes sure a VL is present if and only if $-1 < r_{ll}^+(0) < 1$.

Remark 1. *No special symbol will be reserved for the integrals in the sense of Cauchi principal value. Fourier integrals of functions from $L^2(-\infty, \infty)$, $L^2(a, \infty)$, or $L^2(-\infty, a)$ are treated implicitly in the sense of convergence in a corresponding L^2 space. In particular,*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{k} dk = \frac{i}{2}\operatorname{sign} x.$$

The values

$$T^+(k) = C(k)^{-1}; \qquad T^-(k) = A(k)^{-1},$$

are called the right and the left transmission coefficients, respectively. Note that, in opposition to the scalar case, in our problem generically

$$T^+(k) \neq T^-(k).$$

Let us introduce the notation

$$g[-1] \equiv \lim_{k\to 0}\{kg(k)\}, \qquad \eta(x) \equiv \begin{cases} 1, & 0 < x < \infty, \\ 0, & -\infty < x < 0; \end{cases} \text{(the Heaviside function)} \tag{2.17}$$

3. Auxiliary propositions

Lemma 1. *a) One has the following inequalities that involve degrees of normalizing polynomials:*

$$\deg Z_j^{\pm}(t) \leq \sum_{l=1}^{2} \operatorname{sign} z_{ll}^{[j]\pm} - 1 \leq 1, \qquad j = \overline{1,p}, \tag{3.1}$$

with the diagonal elements $z_{ll}^{[j]\pm}$ *being non-negative and independent of* t. *(The degree of the identically zero polynomial is assumed negative.)*
b) Ranks of normalizing polynomials satisfy the following relations:

$$\operatorname{rg} Z_j^{\pm}(t) = \operatorname{rg} \operatorname{diag} Z_j^{\pm}(t) = \operatorname{rg} \operatorname{diag} Z_j^{\pm}(0), \qquad j = \overline{1,p}. \tag{3.2}$$

c) Matrix elements of normalizing polynomials and those of matrices $C(k)$, $A(k)$ *are related as follows:*

$$\begin{aligned} &a_{11}(k_j) z_{12}^{[j]-}(0) + c_{12}(k_j) z_{22}^{[j]-} + i\dot{a}_{11}(k_j)(z_{12}^{[j]-})'(0) = 0; \\ &z_{11}^{[j]-} a_{12}(k_j) + z_{12}^{[j]-}(0) a_{22}(k_j) + i(z_{12}^{[j]-})'(0)\dot{a}_{22}(k_j) = 0; \end{aligned} \tag{3.3}$$

$$\begin{aligned} &z_{11}^{[j]+} c_{12}(k_j) + z_{12}^{[j]+}(0) a_{22}(k_j) - i(z_{12}^{[j]+})'(0)\dot{a}_{22}(k_j) = 0; \\ &a_{11}(k_j) z_{12}^{[j]+}(0) + a_{12}(k_j) z_{22}^{[j]+} - i\dot{a}_{11}(k_j)(z_{12}^{[j]+})'(0) = 0; \end{aligned} \tag{3.4}$$

$$\begin{aligned} &a_{11}(k_j)(z_{12}^{[j]\pm})'(0) = a_{22}(k_j)(z_{12}^{[j]\pm})'(0) = 0; \\ &a_{ll}(k_j) z_{ll}^{[j]\pm} = 0, \qquad l = 1, 2; \quad j = \overline{1,p}. \end{aligned} \tag{3.5}$$

Proof. The claims a) and b) can be proved just as in [9]. Prove (3.4), (3.5) for $Z_j^+(t)$. Use (3.1) to deduce from the definition (2.13) that

$$Z_j^+(0) = -i\frac{d}{dk}(W^+(k)C(k)^{-1}(k-k_j)^2)_{k_j}; \qquad (Z_j^+)'(0) = (W^+(k)C(k)^{-1}(k-k_j)^2)_{k_j},$$

that is

$$(Z_j^+)'(0)C(k_j) = (W^+(k)C(k)^{-1}C(k)(k-k_j)^2)_{k_j} = (W^+(k)(k-k_j)^2)_{k_j} = 0;$$

$$\begin{aligned} &Z_j^+(0)C(k_j) - i(Z_j^+)'(0)\dot{C}(k_j) = \\ &\quad = -i\frac{d}{dk}\left(W^+(k)C(k)^{-1}(k-k_j)^2\right)_{k_j} C(k_j) - i(W^+(k)C(k)^{-1}(k-k_j)^2)_{k_j}\dot{C}(k_j) = \\ &\qquad = -i\frac{d}{dk}(W^+(k)C(k)^{-1}C(k)(k-k_j)^2)_{k_j} = -i\frac{d}{dk}\left(W^+(k)(k-k_j)^2\right)_{k_j} = 0. \end{aligned}$$

Now write down the latter relations separately for the matrix elements to get (3.4) and (3.5) for $Z_j^+(t)$, $j = \overline{1,p}$. The validity of (3.3), (3.5) for $Z_j^-(t)$ can be proved in a similar way. Lemma 1 is proved. ◄

Lemma 2. *The associated right SD* (2.16) *and tilde-SD for the problems* (1.1) *and* (2.1) *of the above form coincide. Similarly, the left data for the problems* (1.1) *and* (2.1) *coincide.*

Proof. We present a proof for the case of right SD. Define the reflection coefficient for the problem (2.1) by

$$\widetilde{R}^+(k) = -A^{-1}(k)B(-k), \qquad k \in \mathbb{R}, \tag{3.6}$$

with $A(k)$ and $B(k)$ being determined by (2.8). The third equality in (2.11), together with (3.6), implies

$$R^+(k) = D(k)C^{-1}(k) = -A^{-1}(k)B(-k) = \widetilde{R}^+(k), \qquad k \in \mathbb{R}. \tag{3.7}$$

Coincidence of the eigenvalues k_j^2, $\operatorname{Im} k_j > 0$, for the problems (1.1) and (2.1) follows from the upper-triangular form of the problems and the fact that the diagonal elements of the matrix potential $V(x)$ are real, that is $\det A(k) = \det C(k) = a_{11}(k)a_{22}(k)$, $\operatorname{Im} k \geq 0$. By the definition of the normalizing polynomial for the problem (2.1) one has

$$\widetilde{Z}_j^+(t) = -ie^{-ik_jt}\operatorname{Res}_{k_j}\left\{A^{-1}(k)\widetilde{W}^+(k)e^{ikt}\right\}, \tag{3.8}$$

with

$$\widetilde{W}^+(k) = -\frac{1}{2ik}W\left\{\widetilde{E}_-(x,k); E_+^\wedge(x,k)\right\}. \tag{3.9}$$

In the upper half-plane, similarly to (2.7), one has the following representations:

$$\begin{aligned} E_+(x,k) &= E_-(x,k)W^-(k) + E_-^\wedge(x,k)A(k), \\ E_+^\wedge(x,k) &= E_-(x,k)W^\wedge(k) - E_-^\wedge(x,k)\widetilde{W}^+(k), \\ E_-(x,k) &= E_+(x,k)W^+(k) + E_+^\wedge(x,k)C(k), \qquad \operatorname{Im} k > 0, \end{aligned} \tag{3.10}$$

with $W^\pm(k)$ and $\widetilde{W}^+(k)$ being defined by (2.14), $W^\wedge(k) = -\frac{1}{2ik}W\left\{\widetilde{E}_-^\wedge(x,k),\ E_+^\wedge(x,k)\right\}$. Now substitute the initial two relations of (3.10) into the third one to obtain:

$$\begin{aligned} E_-(x,k) &= E_-(x,k)W^-(k)W^+(k)+ \\ &\quad + E_-^\wedge(x,k)A(k)W^+(k) + E_-(x,k)W^\wedge(k)C(k) - E_-^\wedge(x,k)\widetilde{W}^+(k)C(k). \end{aligned}$$

Grouping the summands we get with $\operatorname{Im} k > 0$

$$E_-(x,k)(I - W^-(k)W^+(k) - W^\wedge(k)C(k)) = E_-^\wedge(x,k)(A(k)W^+(k) - \widetilde{W}^+(k)C(k)).$$

Since the solutions $E_-(x,k)$ and $E_-^\wedge(x,k)$ form a fundamental system with $\operatorname{Im} k > 0$, the following relations are valid:

$$A(k)W^+(k) = \widetilde{W}^+(k)C(k), \qquad I = W^-(k)W^+(k) + W^\wedge(k)C(k), \qquad \operatorname{Im} k > 0.$$

So, $A(k)^{-1}\widetilde{W}^+(k) = W^+(k)C(k)^{-1}$, that is $Z_j^+(t) = \widetilde{Z}_j^+(t)$, which was to be proved. Lemma 2 is proved. ◄

Remark 2. *Similarly to* (3.7)*, one has for the left reflection coefficient*

$$R^-(k) := B(k)A(k)^{-1} = -C(k)^{-1}D(-k) =: \widetilde{R}^-(k), \qquad k\in\mathbb{R}, \tag{3.11}$$

and for the left normalizing polynomial

$$Z_j^-(t) = -ie^{ik_jt}\mathrm{Res}_{k_j}\{W^-(k)A(k)^{-1}e^{-ikt}\} = -ie^{ik_jt}\mathrm{Res}_{k_j}\left\{C(k)^{-1}\widetilde{W}^-(k)e^{-ikt}\right\},$$

$j=\overline{1,p}$, *with* $\widetilde{W}^-(k) = \frac{1}{2ik}W\left\{\widetilde{E}_+(x,k), E_-^{\wedge}(x,k)\right\}$.

The formulas (3.11) and (3.7) allow one to write down a relationship between right and left reflection coefficients:

$$R^-(k) = -A(-k)R^+(-k)A(k)^{-1} = -C(k)^{-1}R^+(-k)C(-k), \qquad k\in\mathbb{R}. \tag{3.12}$$

Lemma 3. *One has the relations as follows between right and left normalizing polynomials for each* $\tau\in\mathbb{R}$*:*

$$\begin{aligned} Z_j^-(t) &= -C_j(t-\tau)[Z_j^+(\tau)+Q_j]^{-1}A_{-1}^{<k_j>},\\ Z_j^+(t) &= -A_j(t-\tau)[Z_j^-(\tau)+Q_j]^{-1}C_{-1}^{<k_j>}; \end{aligned} \tag{3.13}$$

$$\begin{aligned} Z_j^-(t) &= -C_{-1}^{<k_j>}[Z_j^+(\tau)+Q_j]^{-1}A_j(\tau-t),\\ Z_j^+(t) &= -A_{-1}^{<k_j>}[Z_j^-(\tau)+Q_j]^{-1}C_j(\tau-t), \end{aligned} \tag{3.14}$$

with

$$\begin{aligned} C_j(t) &= e^{ik_jt}\mathrm{Res}_{k_j}\{C(k)^{-1}e^{-ikt}\} = C_{-1}^{<k_j>} + (-it)C_{-2}^{<k_j>},\\ A_j(t) &= e^{-ik_jt}\mathrm{Res}_{k_j}\{A(k)^{-1}e^{ikt}\} = A_{-1}^{<k_j>} + itA_{-2}^{<k_j>}, \end{aligned}$$

Q_j *being an arbitrary upper triangular matrices with the property* $q_{ll}^{[j]}=0$ *if* $z_{ll}^{[j]\pm}\neq 0$ *and* $q_{ll}^{[j]}\neq 0$ *if* $z_{ll}^{[j]\pm}=0$ *(*$l=1,2$*),* $\mathrm{rg}\, Q_j = 2-\mathrm{rg}\, Z_j^{\pm}(t)$*.*

Proof. It is easy to demonstrate that at k_j, the eigenvalues of problems (1.1) and (2.1), one has the following relations:

$$\begin{cases} E_+(x,k_j)A_{-2}^{<k_j>} = i^2E_-(x,k_j)(Z_j^-)'(0);\\ E_+(x,k_j)A_{-1}^{<k_j>} + \dot{E}_+(x,k_j)A_{-2}^{<k_j>} = iE_-(x,k_j)Z_j^-(0) + i^2\dot{E}_-(x,k_j)(Z_j^-)'(0); \end{cases} \tag{3.15}$$

$$\begin{cases} E_-(x,k_j)C_{-2}^{<k_j>} = -i^2E_+(x,k_j)(Z_j^+)'(0);\\ E_-(x,k_j)C_{-1}^{<k_j>} + \dot{E}_-(x,k_j)C_{-2}^{<k_j>} = iE_+(x,k_j)Z_j^+(0) - i^2\dot{E}_+(x,k_j)(Z_j^+)'(0); \end{cases} \tag{3.16}$$

$$\begin{cases} C_{-2}^{<k_j>}\widetilde{E}_+(x,k_j) = i^2(Z_j^-)'(0)\widetilde{E}_-(x,k_j);\\ C_{-1}^{<k_j>}\widetilde{E}_+(x,k_j) + C_{-2}^{<k_j>}\dot{\widetilde{E}}_+(x,k_j) = iZ_j^-(0)\widetilde{E}_-(x,k_j) + i^2(Z_j^-)'(0)\dot{\widetilde{E}}_-(x,k_j); \end{cases} \tag{3.17}$$

$$\begin{cases} A_{-2}^{<k_j>}\widetilde{E}_-(x,k_j) = -i^2(Z_j^+)'(0)\widetilde{E}_+(x,k_j); \\ A_{-1}^{<k_j>}\widetilde{E}_-(x,k_j) + A_{-2}^{<k_j>}\dot{\widetilde{E}}_-(x,k_j) = iZ_j^+(0)\widetilde{E}_+(x,k_j) - i^2(Z_j^+)'(0)\dot{\widetilde{E}}_+(x,k_j). \end{cases} \tag{3.18}$$

Prove the first relation in (3.13). The second one of (3.13) and (3.14) can be proved in a similar way. Transform the system (3.15) as follows:

$$\begin{cases} -iE_+(x,k_j)A_j'(-t) = i^2E_-(x,k_j)(Z_j^-)'(t); \\ E_+(x,k_j)A_j(-t) - i\dot{E}_+(x,k_j)A_j'(-t) = iE_-(x,k_j)Z_j^-(t) + i^2\dot{E}_-(x,k_j)(Z_j^-)'(t); \end{cases}$$

to be rewritten in the block matricial form:

$$\begin{pmatrix} E_+(x,k_j) & \dot{E}_+(x,k_j) \\ 0 & E_+(x,k_j) \end{pmatrix} \begin{pmatrix} A_j(-t) \\ -iA_j'(-t) \end{pmatrix} = \begin{pmatrix} E_-(x,k_j) & \dot{E}_-(x,k_j) \\ 0 & E_-(x,k_j) \end{pmatrix} \begin{pmatrix} iZ_j^-(t) \\ i^2(Z_j^-)'(t) \end{pmatrix}.$$

Furthermore, it follows from (3.16) that

$$\begin{pmatrix} E_-(x,k_j) & \dot{E}_-(x,k_j) \\ 0 & E_-(x,k_j) \end{pmatrix} \begin{pmatrix} C_{-1}^{<k_j>} \\ C_{-2}^{<k_j>} \end{pmatrix} (Z_j^+(t)+Q_j)^{-1}A_{-1}^{<k_j>} =$$

$$= \begin{pmatrix} E_+(x,k_j) & \dot{E}_+(x,k_j) \\ 0 & E_+(x,k_j) \end{pmatrix} \begin{pmatrix} iZ_j^+(0) \\ -i^2(Z_j^+)'(0) \end{pmatrix} (Z_j^+(t)+Q_j)^{-1}A_{-1}^{<k_j>}.$$

Apply the relation

$$Z_j^+(\tau)[Z_j^+(t)+Q_j]^{-1}A_{-1}^{<k_j>} = A_j(\tau - t), \tag{3.19}$$

to be proved below, to deduce that

$$\begin{pmatrix} E_-(x,k_j) & \dot{E}_-(x,k_j) \\ 0 & E_-(x,k_j) \end{pmatrix} \begin{pmatrix} C_{-1}^{<k_j>} \\ C_{-2}^{<k_j>} \end{pmatrix} (Z_j^+(t)+Q_j)^{-1}A_{-1}^{<k_j>} =$$

$$= \begin{pmatrix} E_+(x,k_j) & \dot{E}_+(x,k_j) \\ 0 & E_+(x,k_j) \end{pmatrix} \begin{pmatrix} iA_j(-t) \\ -i^2(A_j)'(-t) \end{pmatrix} = i\begin{pmatrix} E_-(x,k_j) & \dot{E}_-(x,k_j) \\ 0 & E_-(x,k_j) \end{pmatrix} \begin{pmatrix} iZ_j^-(t) \\ i^2(Z_j^-)'(t) \end{pmatrix}.$$

Compare left and right hand sides to deduce

$$Z_j^-(t) = -C_{-1}^{<k_j>}(Z_j^+(t)+Q_j)^{-1}A_{-1}^{<k_j>};$$
$$(Z_j^-)'(t) = iC_{-2}^{<k_j>}(Z_j^+(t)+Q_j)^{-1}A_{-1}^{<k_j>}.$$

Multiply the second relation by $(\tau - t)$ and add to the first one to get $Z_j^-(\tau) = -C_j(\tau - t)(Z_j^+(t)+Q_j)^{-1}A_{-1}^{<k_j>}$, or equivalently $Z_j^-(t) = -C_j(t-\tau)(Z_j^+(\tau)+Q_j)^{-1}A_{-1}^{<k_j>}$.

Now prove (3.19). Consider the cases:

1) $a_{11}(k_j) = a_{22}(k_j) = 0$, then $z_{ll}^{[j]\pm} \neq 0$, hence $Q_j = 0$, and so by a virtue of the second relation in (3.4):

$$Z_j^+(\tau)Z_j^+(t)^{-1}A_{-1}^{<k_j>}=\begin{pmatrix}1 & \frac{z_{12}^{[j]+}(\tau)-z_{12}^{[j]+}(t)}{z_{22}^{[j]+}}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\dot a_{11}(k_j)} & -\frac{d}{dk}\left(\frac{a_{12}(k)(k-k_j)^2}{a_{11}(k)a_{22}(k)}\right)_{k_j}\\ 0 & \frac{1}{\dot a_{22}(k_j)}\end{pmatrix}=$$
$$=A_{-1}^{<k_j>}+i(\tau-t)A_{-2}^{<k_j>}=A_j(\tau-t).$$

2) $a_{11}(k_j)=0$; $a_{22}(k_j)\neq 0$, hence $q_{11}^{[j]}=0$; $q_{22}^{[j]}\neq 0$ and $Z_j^+(t)=Z_j^+(\tau)=Z_j^+$; $A_j(\tau-t)=A_{-1}$. Thus (3.19) is equivalent to $Q_j(Z_j^++Q_j)^{-1}A_{-1}^{<k_j>}=0$. So, in our case

$$Q_j(Z_j^++Q_j)^{-1}A_{-1}^{<k_j>}=\begin{pmatrix}0 & q_{12}^{[j]}\\ 0 & q_{22}^{[j]}\end{pmatrix}\begin{pmatrix}\frac{1}{z_{11}^{[j]+}} & -\frac{z_{12}^{[j]+}+q_{12}^{[j]}}{z_{11}^{[j]+}q_{22}^{[j]}}\\ 0 & \frac{1}{q_{22}^{[j]}}\end{pmatrix}\begin{pmatrix}\frac{1}{\dot a_{11}(k_j)} & -\frac{a_{12}(k_j)}{\dot a_{11}(k_j)a_{22}(k_j)}\\ 0 & 0\end{pmatrix}=0.$$

3) $a_{11}(k_j)\neq 0$; $a_{22}(k_j)=0$, hence $q_{11}^{[j]}\neq 0$; $q_{22}^{[j]}=0$ and $Z_j^+(t)=Z_j^+(\tau)=Z_j^+$; $A_j(\tau-t)=A_{-1}^{<k_j>}$. Thus (3.19) is equivalent to $Q_j(Z_j^++Q_j)^{-1}A_{-1}^{<k_j>}=0$. By virtue of the second relation in (3.4) one has

$$Q_j(Z_j^++Q_j)^{-1}A_{-1}^{<k_j>}=\begin{pmatrix}q_{11}^{[j]} & q_{12}^{[j]}\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{q_{11}^{[j]}} & -\frac{z_{12}^{[j]+}+q_{12}^{[j]}}{q_{11}^{[j]}z_{22}^{[j]+}}\\ 0 & \frac{1}{z_{22}^{[j]+}}\end{pmatrix}\begin{pmatrix}0 & -\frac{a_{12}(k_j)}{a_{11}(k_j)\dot a_{22}(k_j)}\\ 0 & \frac{1}{\dot a_{22}(k_j)}\end{pmatrix}=0.$$

Lemma 3 is proved. ◄

Note that Lemma 3 and (3.12) indicate that, to determine the right scattering data given the left SD or conversely, it suffices to retrieve simultaneously the matrices $A(k)$ and $C(k)$.

Lemma 4. *In the case of without VL for the problem* (1.1), (1.2) *under consideration one has the following decomposition of the Dirac δ-function:*

$$\delta(x-t)I=$$
$$=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}E_+(x,k)A(k)^{-1}\widetilde{E}_-(t,k)\,dk+\sum_{j=1}^{p}\sum_{l=0}^{1}\frac{d^l}{i^l dk^l}\{E_+(x,k)(Z_j^+)^{(l)}(0)\widetilde{E}_+(t,k)\}_{k_j},\tag{3.20}$$

or equivalently

$$\delta(x-t)I=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\Big\{E_+(x,k)\widetilde{E}_+(t,-k)+E_+(x,k)R^+(k)\widetilde{E}_+(t,k)\Big\}\,dk+$$
$$+\sum_{j=1}^{p}\sum_{l=0}^{1}\frac{d^l}{i^l dk^l}\Big\{E_+(x,k)(Z_j^+)^{(l)}(0)\widetilde{E}_+(t,k)\Big\}_{k_j},\tag{3.21}$$

which is known to be equivalent to the Parseval equality.

Proof. (cf. [8]) uses the method of contour integration, to be combined with a passage to weak limit for the resolvent $zR_z(L) \to E$ as $z \to \infty$, with E being the identity operator generated by the kernel $\delta(x-t)$ as an integral operator, along with (2.7), (2.9). ◀

Lemma 5. [The following assumptions can be valid only in the case without VL.] *Suppose that an upper triangular 2×2 matrix $R^+(k)$ is continuous in $k \in \mathbb{R}$ and has a continuous derivative which is bounded on the entire axis and is such that $\frac{d}{dk}R^+(k) = o(k^{-1})$ as $k \to \pm\infty$; $R^+(0) = -I$, $R^+(k) = O(k^{-1})$ as $k \to \pm\infty$, and $I - R^+(-k)R^+(k) = O(k^2)$ as $k \to 0$. Assume that its diagonal elements are such that $\overline{r^+_{ll}(k)} = r^+_{ll}(-k)$, $|r^+_{ll}(k)| \le 1 - \frac{C_l k^2}{1+k^2}$, $l = 1, 2$, and the associated functions*

$$za^0_{ll}(z) \equiv zc^0_{ll}(z) = z\exp\left\{-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\ln(1-|r^+_{ll}(k)|^2)}{k-z}dk\right\}, \qquad l=1,2, \quad \operatorname{Im} z>0, \tag{3.22}$$

are continuous in the closed upper half-plane (see [18], [11]*), and such that the function*

$$h^0(k) \equiv ka^0_{11}(-k)a^0_{22}(k)\left\{r^+_{11}(-k)r^+_{12}(k) + r^+_{12}(-k)r^+_{22}(k)\right\}, \qquad h^0(0)=0, \tag{3.23}$$

satisfies the Hölder condition on the real axis, that is, there exist constants α and μ, $0<\mu\le 1$, such that $|h^0(k_1)-h^0(k_2)| \le \alpha|k_1-k_2|^\mu$ for all $-\infty<k_1<k_2<\infty$, and moreover,

$$h^0(k) = O(k^{-1}), \qquad k \to \pm\infty. \tag{3.24}$$

Then the following Riemann-Hilbert problem is solvable uniquely with respect to $c^0_{12}(k)$ and $a^0_{12}(-k)$ (cf. (2.12)*), which are regular and bounded, respectively, in upper and lower half-planes:*

$$\frac{kc^0_{12}(k)}{a^0_{11}(k)} = \frac{-ka^0_{12}(-k)}{a^0_{22}(-k)} + h^0(k), \qquad k \in \mathbb{R}. \tag{3.25}$$

It turns out that this solution satisfies the assumption

$$R^-(0) = -I, \tag{3.26}$$

with $R^-(k)$ being produced as in (3.12) *via the matrices $A(k)$ and $C(k)$, determined from* (3.22), (3.25)*. The above solution admits a representation in the form*

$$\begin{aligned} c^0_{12}(z) &= \frac{\psi^+_0(z)-\psi^+_0(0)}{z}a^0_{11}(z); \\ a^0_{12}(z) &= \frac{\psi^-_0(-z)-\psi^+_0(0)}{z}a^0_{22}(z), \qquad \operatorname{Im} z>0, \end{aligned} \tag{3.27}$$

with

$$\begin{aligned} \psi^\pm_0(z) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{h^0(k)}{k-z}dk, \qquad \pm\operatorname{Im} z>0, \\ \psi^\pm_0(0) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}k^{-1}h^0(k)dk. \end{aligned} \tag{3.28}$$

Proof. The Riemann problem (3.25) under the assumptions of the Lemma has index $\nu = -1$, hence its solution exists and is unique (see [12], 14.7). Unlike [12], we do not require the Hölder condition for $h^0(k)$ at the neighborhood of the infinity. This is because the related formulas get simpler since the coefficient of the equation (3.25) is already written in the factorized form $\frac{a^0_{11}(k)}{a^0_{22}(-k)}$, so we obtain

$$\begin{aligned} zc^0_{12}(z) &= \{a_0 + \psi^+_0(z)\}a^0_{11}(z), \qquad \operatorname{Im} z > 0, \\ -za^0_{12}(-z) &= \{a_0 + \psi^-_0(z)\}a^0_{22}(-z), \qquad \operatorname{Im} z < 0, \end{aligned} \tag{3.29}$$

with a_0 being a constant determined by the requirement for the right hand sides to be continuous at $z = 0$, i.e.,

$$a_0 = -\psi^+_0(0) = -\psi^-_0(0), \tag{3.30}$$

and the functions $\psi^\pm_0(z)$ being determined by (3.28) where one should note that $a^0_{ll}(-k) = \overline{a^0_{ll}(k)}$, $l = 1, 2$, by a virtue of (3.22). The second equality in (3.30) follows from formula (3.28) which has already been established, and the assumptions of the Lemma. The solution (3.29) is derived from (3.25), after one takes into account (3.28) and the Sokhotski-Plemelj formulas. This all together results in

$$\frac{kc^0_{12}(k)}{a^0_{11}(k)} - \psi^+_0(k) = -\frac{ka^0_{12}(-k)}{a^0_{22}(-k)} - \psi^-_0(k) = a_0, \tag{3.31}$$

where the second equality (with a constant as a right hand side) follows from the Liouville theorem, since the left and the central parts of (3.31) appear to be analytic continuations of each other to the entire complex plane. Now an application of (3.30) allows one to deduce (3.27) from (3.31).

Now use (3.12) to construct a function

$$r^-_{12}(k) = -\frac{r^+_{11}(-k)c^0_{12}(-k)}{a^0_{11}(k)} - \frac{a^0_{22}(-k)}{a^0_{11}(k)}r^+_{12}(-k) - \frac{r^-_{22}(-k)c^0_{12}(k)}{a^0_{11}(k)}.$$

An application of the Sokhotski-Plemelj formulas for $\psi^\pm_0(z)$ (3.28) allows one to deduce from (3.29) with $k \neq 0$ on the real axis

$$r^-_{12}(k) = -\frac{a^0_{22}(-k)}{a^0_{11}(k)}r^+_{12}(-k) - \frac{r^-_{11}(k)}{k}\left\{a_0 + \psi^+_0(-k)\right\} - \frac{r^-_{22}(k)}{k}\left\{a_0 + \psi^+_0(k)\right\}.$$

It follows that as $k \to 0$ by (3.31), (3.30) one has $r^-_{12}(0) = 0$ since $r^-_{ll}(0) = -1$ due to the properties of the scalar inverse scattering problem, which yields (3.26).

Lemma 5 is proved. ◀

Remark 3. *Suppose that in Lemma 5, besides the matrix $R^+(k)$ that satisfies the assumptions of the Lemma, we are given numbers $k_j^2 < 0$ (with $\operatorname{Im} k_j > 0$) and polynomials $Z^+_j(t)$, $j = \overline{1,p}$, $t \in \mathbb{R}$, which satisfy the assumptions a), b), and (3.5) of Lemma 1. Set*

$$zc_{ll}(z) \equiv za_{ll}(z) := za^0_{ll}(z)\prod_{j=1}^{p}\left(\frac{z-k_j}{z+k_j}\right)^{s^l_j}, \qquad \operatorname{Im} z > 0, \tag{3.32}$$

with $s_j^l = \operatorname{sign} z_{ll}^{[j]+} \geq 0$, $l = 1, 2$. *In this case the Riemann-Hilbert problem*

$$\frac{kc_{12}(k)}{a_{11}(k)} = \frac{-ka_{12}(-k)}{a_{22}(-k)} + h(k), \tag{3.33}$$

where

$$h(k) = h^0(k) \prod_{j=1}^{p} \left(\frac{k - k_j}{k + k_j} \right)^{s_j^2 - s_j^1}, \qquad h(0) = 0, \tag{3.34}$$

is also solvable uniquely under the assumptions (3.4) *so that*

$$zc_{12}(z) = \\ = \left\{ \frac{-\psi^+(0) \prod\limits_{j=1}^{p} k_j^{\kappa_j} + a_1 z + \ldots + a_\kappa z^\kappa}{\prod\limits_{j=1}^{p} (z + k_j)^{\kappa_j}} + \psi^+(z) \right\} a_{11}(z) \prod_{j=1}^{p} \left(\frac{z + k_j}{z - k_j} \right)^{s_j^1}, \quad \operatorname{Im} z > 0, \tag{3.35}$$

$$- za_{12}(-z) = \\ = \left\{ \frac{-\psi^+(0) \prod\limits_{j=1}^{p} k_j^{\kappa_j} + a_1 z + \ldots + a_\kappa z^\kappa}{\prod\limits_{j=1}^{p} (z + k_j)^{\kappa_j}} + \psi^-(z) \right\} a_{22}(-z) \prod_{j=1}^{p} \left(\frac{z + k_j}{z - k_j} \right)^{s_j^1}, \\ \operatorname{Im} z < 0,$$

with $a_1, \ldots, a_\kappa$ *being retrievable uniquely using the systems* (3.4), (3.5), *and given normalizing polynomials,* $\kappa = \sum\limits_{j=1}^{p} \kappa_j$, *where* $\kappa_j = s_j^1 + s_j^2 = \operatorname{sign} z_{11}^{[j]+} + \operatorname{sign} z_{22}^{[j]+}$, *and the functions* $\psi^\pm(z)$ *and* $\psi^+(0)$ *being determined by*

$$\psi^\pm(z) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{h(k)}{k - z} \prod_{j=1}^{p} \left(\frac{k - k_j}{k + k_j} \right)^{s_j^1} dk, \qquad \pm \operatorname{Im} z > 0, \tag{3.36}$$

$$\psi^+(0) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} k^{-1} h(k) \prod_{j=1}^{p} \left(\frac{k - k_j}{k + k_j} \right)^{s_j^1} dk. \tag{3.37}$$

The solution we get this way appears to be such that (3.26) *is valid if* $R^-(k)$ *is constructed as in* (3.12) *which involves matrices* $A(k)$ *and* $C(k)$ *determined by* (3.32), (3.35), (3.33), (3.34). *Note that the problem* (3.33), (3.34) *in our case has index* $\nu = \kappa - 1$.

Explanations to Remark 3. Note that the determinant of the linear system that defines $a_1, \dots, a_\kappa$ is non-zero in the cases under consideration, as one can see from (3.5).

Observe also that, unlike [12, 14.7], we use the denominator $\prod_{j=1}^{p}(z+k_j)^{\kappa_j}$ instead of $(z+i)^\kappa$. This replacement turns out to be more suitable in our case (e.g., in (3.35), etc.). Since Im $k_j > 0$, our product $\prod_{j=1}^{p}(z+k_j)^{\kappa_j}$ as well as $(z+i)^\kappa$ from [12, 14.7], have the same index as z varies from $-\infty$ to $+\infty$, namely $(-\kappa/2)$.

Lemma 6. *The coefficients $A(k)$ and $B(k)$ given by* (2.8)*, admit representations as follows:*

$$
\begin{aligned}
A(k) &= I - \frac{1}{2ik}\left\{\int_{-\infty}^{\infty} V(x)dx + \int_{-\infty}^{0} A_1(t)e^{-ikt}dt\right\} = I + O\left(\frac{1}{k}\right),\\
B(k) &= \frac{1}{2ik}\int_{-\infty}^{\infty} B_1(t)e^{-ikt}dt = o\left(\frac{1}{k}\right), \qquad k \to \pm\infty,
\end{aligned}
\tag{3.38}
$$

with $A_1(t)$ being a summable matrix function whose first moment exists on $(-\infty; 0]$ *under condition* (1.2)*;*
$B_1(t)$ is a summable matrix function whose first moment exists on $(-\infty; \infty)$ *under condition* (1.2)*.*

Proof of Lemma 6 coincides to that of a Lemma by V. A. Marchenko [18, Lemma 3.5.1.], if one takes into account that under condition (1.2) the kernel $K(x,t)$ of the transformation operator is a summable function, which has the first moment with respect to $t \in [x; \infty)$. ◄

Lemma 7. *In the case without VL, suppose that a matrix potential $V(x)$ has the second moment on the axis as in* (1.2)*. Then the matrix reflection coefficient $R^+(k)$ for the problem* (1.1)*, is a bounded function of k on the whole axis with a continuous derivative and such that $\frac{d}{dk}R^+(k) = o(\frac{1}{k})$ as $k \to \pm\infty$.*

Proof. It follows from Lemma 6 that it is possible to differentiate $kA(k)$ and $kB(k)$ in k in the integrands (3.38). Thus we have

$$
d\{kA(k)\}/dk = I + \frac{1}{2}\int_{-\infty}^{0} tA_1(t)e^{-ikt}dt = I + o(1),
$$

$$
d\{kB(k)\}/dk = o(1), \qquad k \to \pm\infty.
$$

This, together with the definition $R^+(k)$ (2.9), implies the claim of Lemma 7, after one observes that under absence of VL

$$
\begin{aligned}
&\lim_{k\to 0} kA(k) = C_1; \qquad && \lim_{k\to 0} kC(k) = C_2;\\
&\det C_1 \neq 0; && \det C_2 \neq 0.
\end{aligned}
\tag{3.39}
$$
◄

Lemma 8. ([13, Chapter I, § 2, Example 5]) *Let*

$$F(x)=\frac{P(x)}{\prod\limits_{j=1}^{n}(x-ia_j)^{k_j}\prod\limits_{l=1}^{m}(x+ib_l)^{r_l}}, \qquad Re\ a_j>0, \quad Re\ b_l>0, \tag{3.40}$$

where $P(x)$ is a polynomial whose degree is lower than degree of the denominator. Then

$$\int\limits_{-\infty}^{\infty}F(x)e^{-ixt}dx=\begin{cases}\sum\limits_{l=1}^{m}p_l(t)e^{-b_lt}, & t>0,\\ \sum\limits_{j=1}^{n}q_j(t)e^{a_jt}, & t<0,\end{cases} \qquad -\infty<t<\infty. \tag{3.41}$$

Here $p_l(t)$, $q_j(t)$ are polynomials of degrees r_l-1 and k_j-1, respectively.

Proof. Decompose $F(x)$ (3.40) as a sum of simple fractions and find the Fourier transforms of each of those fractions. The subsequent summing up the results gives (3.41). ◀

4. The case of absent VL

4.1. A formulation of Theorem 1. The necessity in the version 4)

In the formulation of the next theorem we do not restrict ourselves with citing the numbers of the necessary formulas written earlier in the text, but reproduce them here after appropriate references for the reader's convenience.

Theorem 1. (See [31, Theorem 2]). *In order to have the set of values* (2.16)*:*

$$\left\{R^+(k),\ k\in\mathbb{R};\ k_j^2<0,\ Z_j^+(t),\ j=1,\dots,p<\infty\right\},$$

the right SD of the problem (1.1), (1.2) *without VL, it is necessary and sufficient that the following conditions 1) – 6) are satisfied. Here $R^+(k)$ and the polynomials $Z_j^+(t)$, $j=\overline{1,p}$, are upper triangular 2×2 matrix functions. This theorem is valid in the two versions: either condition 4) or condition 4a) is involved.*

1) *$R^+(k)$ is continuous in $k\in\mathbb{R}$ and has a continuous derivative which is bounded on the entire axis, so that $dR^+(k)/dk=o(k^{-1})$ as $k\to\pm\infty$. Moreover, $\overline{r_{ll}^+(k)}=r_{ll}^+(-k)$, $\left|r_{ll}^+(k)\right|\le 1-\frac{C_lk^2}{1+k^2}$, with $C_l>0$, $l=1,2$, $R^+(0)=-I$; $I-R^+(-k)R^+(k)=O(k^2)$ as $k\to 0$ and $R^+(k)=O(k^{-1})$ as $k\to\pm\infty$ (the two latter conditions here can be enhanced as necessary ones to requiring continuity for the function $\{I-R^+(-k)R^+(k)\}k^{-2}$ for $k\in\mathbb{R}$ together with the estimate $R^+(k)=o(k^{-1})$ as $k\to\pm\infty$).*

2) *The function*

$$F_R^+(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} R^+(k)e^{ikx}dk, \tag{4.1}$$

is absolutely continuous, and for every $a > -\infty$ *one has*

$$(1+x^2)\left|\frac{d}{dx}F_R^+(x)\right| \in L^1(a,+\infty). \tag{4.2}$$

3) *The functions* $zc_{ll}^0(z) \equiv za_{ll}^0(z)$, $l = 1,2$, *given by* (3.22)*:*

$$zc_{ll}^0(z) \equiv za_{ll}^0(z) := z\exp\left\{-\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\ln(1-|r_{ll}^+(k)|^2)}{k-z}dk\right\}, \qquad \operatorname{Im} z > 0, \tag{4.3}$$

are continuously differentiable in the closed upper half-plane after being defined on the real axis by continuity. Here one has $\lim_{z\to 0}(za_{ll}^0(z)) \neq 0$ *due to the absence of VL.*

4) *The function*

$$F_R^-(x) \equiv -\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} C(k)^{-1}R^+(-k)C(-k)e^{-ikx}dk, \tag{4.4}$$

is absolutely continuous, and for every $a < +\infty$ *one has*

$$\left(1+x^2\right)\left|dF_R^-(x)/dx\right| \in L^1(-\infty,a). \tag{4.5}$$

Here the matrix $C(k)$ *for* $k \in \mathbb{R}$ *is defined as* $C(k+i0)$*. For* $l = 1,2$ *its elements* $c_{ll}(k)$ *are given by* (3.32)*:*

$$zc_{ll}(z) \equiv za_{ll}(z) := zc_{ll}^0(z)\prod_{j=1}^{p}\left(\frac{z-k_j}{z+k_j}\right)^{s_j^l}, \qquad \operatorname{Im} z > 0, \tag{4.6}$$

where $\operatorname{Im} k_j > 0$, $s_j^l = \operatorname{sign} z_{ll}^{[j]+} \geq 0$*. Furthermore,* $c_{21}(k) \equiv 0$, $c_{12}(k) \equiv c_{12}(k+i0)$ *with* $zc_{12}(z)$ *being given by* (3.35)*, which can be rewritten as*

$$zc_{12}(z) = \left\{\frac{-\psi^+(0)\prod\limits_{j=1}^{p}k_j^{\kappa_j} + a_1 z + \ldots + a_\kappa z^\kappa}{\prod\limits_{j=1}^{p}(z+k_j)^{\kappa_j}} + \psi^+(z)\right\} a_{11}^0(z), \quad \operatorname{Im} z > 0. \tag{4.7}$$

The constants $a_1,\ldots,a_\kappa$ *are uniquely derivable from the given polynomials* $Z_j^+(t)$*, together with* $a_{ll}(z)$ *determined by* (3.32), (3.22)*,* $\kappa = \sum_{j=1}^{p}\kappa_j$, $\kappa_j = \operatorname{sign} z_{11}^{[j]+} +$

sign $z_{22}^{[j]+}$. *Besides* $zc_{12}(z)$ *is bounded and continuous in the closed upper half-plane,*

$$\psi^{\pm}(z) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{h(k)}{k-z} \prod_{j=1}^{p} \left(\frac{k-k_j}{k+k_j}\right)^{s_j^1} dk, \qquad \pm\mathrm{Im}\, z > 0, \tag{4.8}$$

where

$$h(k) = ka_{11}(-k)a_{22}(k)\{r_{11}^{+}(-k)r_{12}^{+}(k) + r_{12}^{+}(-k)r_{22}^{+}(k)\}, \qquad h(0) = 0, \tag{4.9}$$

or equivalently, $h(k)$ *given by* (3.34),

$$\psi^{+}(0) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} k^{-1}h(k) \prod_{j=1}^{p} \left(\frac{k-k_j}{k+k_j}\right)^{s_j^1} dk. \tag{4.10}$$

4a) (4.5) *is still valid if one substitutes in* (4.4) $C(k)$ *by the matrix* $C^0(k)$, *where* $c_{ll}^0(z) \equiv a_{ll}^0(z)$, $l = 1, 2$, *are given by* (3.22), *and* $c_{12}^0(z)$ *is determined as follows*

$$zc_{12}^0(z) = [\psi_0^{+}(z) - \psi_0^{+}(0)]a_{11}^0(z), \quad \mathrm{Im}\, z > 0, \tag{4.11}$$

$$\psi_0^{+}(z) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{h^0(k)}{k-z} dk, \tag{4.12}$$

$$h^0(k) = ka_{11}^0(-k)a_{22}^0(k)\{r_{11}^{+}(-k)r_{12}^{+}(k) + r_{12}^{+}(-k)r_{22}^{+}(k)\}, \quad h^0(0) = 0, \tag{4.13}$$

$$\psi_0^{+}(0) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} k^{-1}h^0(k)dk. \tag{4.14}$$

5) $\deg Z_j^{+}(t) \leq \sum\limits_{l=1}^{2} \mathrm{sign}\, z_{ll}^{[j]+} - 1$, $j = \overline{1,p}$, *with* $z_{ll}^{[j]+}$ *being non-negative and constant.*

6) $\mathrm{rg}\, Z_j^{+}(t) = \mathrm{rg}\,\mathrm{diag}\, Z_j^{+}(t) = \mathrm{rg}\,\mathrm{diag}\, Z_j^{+}(0)$, $j = \overline{1,p}$.

Remark 4. *The conditions of the theorem related to the diagonal matrix elements only, are direct consequences of* [18, ch. 3], [11], [16, ch. VI].

Remark 5. *In the case when the discrete spectrum is absent, the conditions 5) and 6) of the Theorem become inapplicable and should be discarded. The conditions 4) and 4a) become the same.*

Proof of Theorem 1. The necessity.

Similarly to [18], under the condition (1.2) one has that $R^{+}(k)$ is a continuous function of $k \in \mathbb{R}$. In this context, since the upper triangular potential of the scattering problem

(1.1) has its principal diagonal formed by real functions, the following relations hold: $\overline{r_{ll}^{+}(k)} = r_{ll}^{+}(-k)$ and $\left|r_{ll}^{+}(k)\right| \le 1 - \frac{C_l k^2}{1+k^2}$, $l = 1, 2$ (see [18]).

Furthermore, (2.7) and (2.9) imply with $k \in \mathbb{R}$

$$kE_{-}(x,k) = \{E_{+}(x,k)[R^{+}(k) + I] + E_{+}(x,-k) - E_{+}(x,k)\}kC(k). \tag{4.15}$$

Since the scattering problem (1.1), (1.2) in question is assumed to have no virtual level, the definition (2.8) implies the existence of the limits

$$\begin{aligned} \lim_{k\to 0} kA(k) = C_1; \qquad & \lim_{k\to 0} kC(k) = C_2; \\ \det C_1 \neq 0; \qquad & \det C_2 \neq 0. \end{aligned} \tag{4.16}$$

Thus, passage to a limit as $k \to 0$ in (4.15) yields $0 = \lim_{k\to 0}\{E_{+}(x,k)[R^{+}(k)+I]\}kC(k) = E_{+}(x,0)\lim_{k\to 0}[R^{+}(k)+I]C_2$. By a continuity of $R^{+}(k)$ one has $R^{+}(0) = -I$. Also by (4.16), we deduce from (2.12) that $I - R^{+}(-k)R^{+}(k) = O(k^2)$ as $k \to 0$.

Lemma 6 and the definition of reflection coefficient (2.9), (3.7) imply $R^{+}(k) = o(k^{-1})$ as $k \to \pm\infty$. The rest of the properties of $R^{+}(k)$ listed in condition 1) of Theorem follow from Lemma 7. Condition 1) of Theorem 1 is proved completely.

Condition 2) of Theorem 1 follows by arguments, with the help of which the Marchenko equation is derived for the given right SD (see [18], [1], [9]).

The fact that $za_{ll}(z)$ (3.32), (3.22) (hence identically the same function $zc_{ll}(z)$) is continuous in the closed upper half-plane, is proved in [18] by an application of Lemma 3.5.1 from [18], and the continuous differentiability of these functions in the closed upper half-plane is an obvious consequence of Lemma 6. Condition 3) of Theorem 1 is proved.

Using Lemma 5 and Remark 3, the relation (3.12) between left and right reflection coefficients, and the argument that derives the Marchenko equation by a contour integration for the given left SD, (see [18], the text that starts at (3.5.14) and ends at (3.5.19′)) we thus prove that condition 4) holds. An additional observation to be used here is that the uniqueness for constants $a_1, \ldots, a_\kappa$ in the expression (4.7) for $zc_{12}(z)$ follows from the system of linear equations (3.4) after substituting therein the expressions for $c_{12}(k_j)$ (4.7) and for $a_{12}(k_j)$ (3.35), as well as those for $a_{ll}(k_j)$, $\dot{a}_{ll}(k_j)$ (3.32) (cf. Remark 3). Note that the determinant of the above system that defines $a_1, \ldots, a_\kappa$ is non-zero in the cases under consideration, as one can see from (3.5).

Necessity of conditions 5) and 6) of Theorem 1 follows from claims a) and b) of Lemma 1.

Necessity of assumptions of Theorem 1 is proved in the version with condition 4).

Prove sufficiency of assumptions of Theorem 1.

4.2. The case of absence of discrete spectrum

First, reconstruct the problem (1.1), (1.2) given $R^{+}(k)$, without eigenvalues and normalizing polynomials. In this case, use (3.12) and the formulas (3.22) – (3.25) of Lemma

5, to construct the function $R_0^-(k)$ as follows

$$R_0^-(k) = -C_0(k)^{-1}R^+(-k)C_0(-k), \qquad C_0(k) = \begin{pmatrix} a_{11}^0(k) & c_{12}^0(k) \\ 0 & a_{22}^0(k) \end{pmatrix}, \tag{4.17}$$

where zero indices indicate absence of eigenvalues.

Prove that $R^+(k)$ and $R_0^-(k)$ are right and left reflection coefficients of the same differential equation (1.1), whose potential is triangular, summable, and has the second moment on the real axis. Since $R^+(k)$ and $R_0^-(k)$ are upper triangular and the diagonal elements $r_{ll}^+(k)$ and $r_{ll}^{[0]-}(k)$ $(l = 1, 2)$ satisfy the assumptions of the Marchenko Lemma [18, Lemma 3.5.3], one deduces that the Marchenko equations associated to $R^+(k)$ and $R_0^-(k)$ respectively, have unique solutions $K_+^0(x, y)$ and $K_-^0(x, y)$, and similarly $\widetilde{K}_+^0(x, y)$ and $\widetilde{K}_-^0(x, y)$. (In fact, the equations for diagonal elements are solvable unambiguously by [18], and the equations for k_{12+}^0 and k_{12-}^0 differ from those for the diagonal elements only by a free term). By the same Lemma 3.5.3 of [18], the functions $E_\pm^0(x, k) = e^{\pm ikx}I \pm \int_x^{\pm\infty} K_\pm^0(x, t)e^{\pm ikt}dt$ are the Jost solutions of Schrödinger equations on the entire axis, in which the potentials $V_0^\pm(x)$ possess the property (1.2), and similarly $\widetilde{E}_\pm^0(x, k) = e^{\pm ikx}I \pm \int_x^{\pm\infty} \widetilde{K}_\pm^0(x, t)e^{\pm ikt}dt$ are the Jost tilde-solutions.

To prove that $R^+(k)$ and $R_0^-(k)$ are the right and the left reflection coefficients of the same equation, it suffices to demonstrate that

$$\begin{aligned} E_-^0(x, k)C_0(k)^{-1} &= E_+^0(x, -k) + E_+^0(x, k)R^+(k); \\ E_+^0(x, k)A_0(k)^{-1} &= E_-^0(x, -k) + E_-^0(x, k)R_0^-(k), \qquad k \in \mathbb{R}. \end{aligned} \tag{4.18}$$

We follow the ideas of [16], [18] in proving (4.18).

Define a function

$$\Phi_+(x, y) := F_R^+(x + y) + \int_x^\infty K_+^0(x, t)F_R^+(t + y)dt,$$

with F_R^+ being given by (4.1). It follows from the above that at every fixed x, the function $\Phi_+(x, y)$ is in $L^1(-\infty, \infty)$ since $F_R^+(y) \in L^1(-\infty, \infty)$. Furthermore, by virtue of the specific expression (4.1) of F_R^+ one has

$$\int_{-\infty}^\infty \Phi_+(x, y)e^{-iky}dy = E_+^0(x, k)R^+(k).$$

By the Marchenko equation $\Phi_+(x, y) = -K_+^0(x, y)$ with $x < y < \infty$.

Since $\int_x^\infty K_+^0(x, y)e^{-iky}dy = E_+^0(x, -k) - e^{-ikx}I$, one has $\int_{-\infty}^\infty \Phi_+(x, y)e^{-iky}dy = \int_{-\infty}^x \Phi_+(x, y)e^{-iky}dy + e^{-ikx}I - E_+^0(x, -k)$. Thus

$$E_+^0(x, k)R^+(k) + E_+^0(x, -k) = H_-(x, k)C_0(k)^{-1}, \tag{4.19}$$

where

$$H_-(k) = e^{-ikx}\left\{ I + \lim_{N\to\infty}\int\limits_{-N}^{x} \Phi_+(x,y)e^{-ik(y-x)}dy \right\} C_0(k).$$

It suffices to demonstrate that

$$H_-(x,k) = E_-^0(x,k), \tag{4.20}$$

and this will prove (4.18). In fact, consider the system

$$E_+^0(x,k)R^+(k) + E_+^0(x,-k) = H_-(x,k)C_0(k)^{-1},$$

$$E_+^0(x,k) + E_+^0(x,-k)R^+(-k) = H_-(x,-k)C_0(-k)^{-1},$$

with respect to $E_+^0(x,\pm k)$ to deduce from (2.12) that

$$H_-(x,k)R_0^-(k) + H_-(x,-k) = E_+^0(x,k)A_0(k)^{-1}, \tag{4.21}$$

which, by a virtue of (4.20) yields (4.18).

We follow the proof of Theorem 6.5.1 of [16] to establish the following three properties of the function $H_-(x,k)$:

1. $H_-(x,k)$ admits an analytic continuation into the upper half-plane, and for large z one has the estimate $|H_-(x,z) - e^{-ixz}I| = O\left(\frac{e^{x\,\mathrm{Im}\,z}}{|z|}I\right)$.

2. $zH_-(x,z)$ is continuous in the closed upper half-plane, and $zH_-(x,z) = o(I)$ as $z \to 0$ (uniformly in x).

3. $H_-(x,k) - e^{-ikx}I \in L_2(-\infty,\infty)$ in k.

Use these properties of $H_-(x,k)$ to prove (4.20). Consider for $x < y$ an analytic in the upper half-plane function $[H_-(x,z) - e^{-ixz}I]e^{iyz}$. Use the method of contour integration to obtain, in view of the properties 1-3,

$$\lim_{R\to\infty}\int\limits_{-R}^{R}\left[H_-(x,k) - e^{-ixk}I\right]e^{iyk}dk = 0, \qquad (x<y).$$

Hence

$$H_-(x,k) = e^{-ikx}I + \int\limits_{-\infty}^{x} G_-(x,y)e^{-iky}dy, \tag{4.22}$$

for some $G_-(x,y) \in L_2(-\infty,x)$.

One has from (4.21) and (4.22)

$$E_+^0(x,k)A_0(k)^{-1} - e^{ikx}I = \int\limits_{-\infty}^{x} G_-(x,y)e^{iky}dy + e^{-ikx}R_0^-(k) + \int\limits_{-\infty}^{x} G_-(x,y)e^{-iky}dyR_0^-(k).$$

By a construction, $A_0(z)$ and $A_0^{-1}(z)$ are regular in the open upper half-plane, hence with $t < x$ one has

$$0 = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left(E_+^0(x,k)A_0(k)^{-1} - e^{ikx}I\right)e^{-ikt}dt =$$

$$= G_-(x,t) + F_{R_0}^-(x+t) + \int\limits_{-\infty}^{x} G_-(x,y)F_{R_0}^-(t+y)dt.$$

That is, $G_-(x,y)$ satisfies the Marchenko equation. It follows from the unambiguous solvability of the Marchenko equation that $K_-^0(x,t) \equiv G_-(x,t)$, whence one deduces (4.20) in view of (4.22).

Thus $R^+(k)$ and $R_0^-(k)$ are the right and the left reflection coefficients for the problem (1.1), (1.2) under the absence of discrete spectrum, so in this special case the Theorem 1 is proved.

4.3. The addition of discrete spectrum

Now consider the general case when the problem might have a finite number p of different eigenvalues (for the 2×2 triangular potential under consideration those can be either simple or multiplicity two, and respectively the ranks of normalizing polynomials $Z_j^+(t)$ should be either 1 or 2). We proceed by induction (cf., for example, [16] in the scalar case). Suppose that for the data

$$\{R^+(k);\ k_1^2,\dots,k_p^2;\ Z_1^+(t),\dots,Z_p^+(t)\}, \tag{4.23}$$

the inverse problem on the axis is solved, that is those values form the right SD for a problem of the form (1.1), (1.2) and a potential $V(x) = V_p(x)$. We are about to demonstrate in this case how to obtain a solution of the inverse problem with $p+1$ different eigenvalues and normalizing polynomials, that is, with the right SD of the form

$$\{R^+(k);\ k_1^2,\dots,k_p^2,k_{p+1}^2;\ Z_1^+(t),\dots,Z_p^+(t),Z_{p+1}^+(t)\}. \tag{4.24}$$

Denote by $E_+^p(x,k)$ and $\widetilde{E}_+^p(x,k)$ the Jost solutions for the equations, respectively,

$$-Y'' + V_p(x)Y = k^2Y, \qquad -\infty < x < +\infty, \tag{4.25}$$

$$-\widetilde{Z}'' + \widetilde{Z}V_p(x) = k^2\widetilde{Z}, \qquad -\infty < x < +\infty, \tag{4.26}$$

with asymptotics $E_+^p(x,k) \sim e^{ikx}I$, $\widetilde{E}_+^p(x,k) \sim e^{ikx}I$ as $x \to +\infty$, $\operatorname{Im} k \ge 0$. Since k_{p+1}^2 is not an eigenvalue of the equations (4.25) and (4.26), one has that $E_+^p(x,k_{p+1})$ and $\widetilde{E}_+^p(x,k_{p+1})$ decays exponentially as $x \to +\infty$, and increase exponentially as $x \to -\infty$.

We generalize the procedure of attaching the discrete spectrum expounded in [16] (Chapter VI, § 6).

Set $F(x,y) = E_+^p(x,k_{p+1})Z_{p+1}^+(0)\widetilde{E}_+^p(y,k_{p+1}) - i\frac{d}{dk}\left\{E_+^p(x,k)(Z_{p+1}^+)'(0)\widetilde{E}_+^p(y,k)\right\}_{k_{p+1}}$, and consider the degenerate integral equation

$$B(x,y) + F(x,y) + \int_x^\infty B(x,t)F(t,y)dt = 0, \qquad (x < y). \tag{4.27}$$

Solve it to obtain

$$B(x,y) = -E_+^p(x,k_{p+1})Z_{p+1}^+(0)\left[I + \int_x^\infty \widetilde{E}_+^p(t,k_{p+1})E_+^p(t,k_{p+1})dtZ_{p+1}^+(0)\right]^{-1} \cdot \\ \cdot \widetilde{E}_+^p(y,k_{p+1}) + ib(x,y)(Z_{p+1}^+)'(0), \tag{4.28}$$

with

$$b(x,y) = \frac{\dot{e}_{11+}^p(x,k_{p+1})e_{22+}^p(y,k_{p+1})}{1 + z_{22}^{[p+1]+}\int_x^\infty e_{22+}^p(t,k_{p+1})^2dt} + \frac{e_{11+}^p(x,k_{p+1})\dot{e}_{22+}^p(y,k_{p+1})}{1 + z_{11}^{[p+1]+}\int_x^\infty e_{11+}^p(t,k_{p+1})^2dt} - \\ - \frac{e_{11+}^p(x,k_{p+1})e_{22+}^p(y,k_{p+1})}{\left(1 + z_{11}^{[p+1]+}\int_x^\infty e_{11+}^p(t,k_{p+1})^2dt\right)\left(1 + z_{22}^{[p+1]+}\int_x^\infty e_{22+}^p(t,k_{p+1})^2dt\right)} \cdot \\ \cdot \left[z_{11}^{[p+1]+}\int_x^\infty \dot{e}_{11+}^p(t,k_{p+1})e_{11+}^p(t,k_{p+1})dt + z_{22}^{[p+1]+}\int_x^\infty \dot{e}_{22+}^p(t,k_{p+1})e_{22+}^p(t,k_{p+1})dt\right]. \tag{4.29}$$

Set

$$\triangle V(x) = -2\frac{d}{dx}B(x,x), \qquad V_{p+1}(x) = V_p(x) + \triangle V(x). \tag{4.30}$$

Let us prove the following Lemma.

Lemma 9. *The matrix function $\triangle V(x)$ given by* (4.30) *possesses the property*

$$\int_{-\infty}^{+\infty}(1 + |x|^2)|\triangle V(x)|dx < \infty, \tag{4.31}$$

if $V_p(x)$ satisfy the condition (1.2).

Proof. Since $E_+^p(x,k_{p+1})$ and $\widetilde{E}_+^p(x,k_{p+1})$ decay exponentially as $x \to +\infty$, it follows from (4.28) – (4.30) that $\triangle V(x)$ also decays exponentially as $x \to +\infty$.

Thus it remains to demonstrate that $\triangle V(x)$ has the second moment at $-\infty$ if $V_p(x)$ satisfies (1.2). For this, we introduce the notation

$$\begin{aligned}
\Phi(x,k_{p+1}) &= e^{-ik_{p+1}x}E_+^p(x,k_{p+1});\\
\widetilde{\Phi}(x,k_{p+1}) &= e^{-ik_{p+1}x}\widetilde{E}_+^p(x,k_{p+1});\\
\Gamma(x,k_{p+1}) &= e^{-2ik_{p+1}x}\int\limits_x^{\infty}\widetilde{E}_+^p(t,k_{p+1})E_+^p(t,k_{p+1})dt,
\end{aligned} \tag{4.32}$$

and generalize the techniques of [16] to obtain following statements which are similar to Lemmas 6.6.1 – 6.6.3 of [16]:

Lemma 10. *The matrix functions* $\Phi(x,k_{p+1})$, $\widetilde{\Phi}(x,k_{p+1})$ *given by* (4.32), *and* $\dot{\varphi}_{ll}(x,k_{p+1}) = \frac{d}{dk}\left(e^{-ikx}e^p_{ll+}(x,k)\right)_{k_{p+1}}$, *where* φ_{ll} *are the diagonal elements of* Φ, $\widetilde{\Phi}$, $l = 1,2$, *are bounded on* $-\infty < x < N$ *for each* $N < +\infty$.

A *proof* results immediately from the representations (2.3) for the Jost solutions and the inequalities for transformation operators (see [1]):

$$\left|K_+^p(x,t)\right| \le C\int\limits_{\frac{x+t}{2}}^{\infty}|V_p(s)|ds; \qquad \left|\widetilde{K}_+^p(x,t)\right| \le C\int\limits_{\frac{x+t}{2}}^{\infty}|V_p(s)|ds, \tag{4.33}$$

with some constant C and also with the use of exponentially rising solutions as $x \to -\infty$ with asymptotics of (2.5). ◀

Lemma 11. *One has following inequalities for the functions* $\Phi(x,k_{p+1})$, $\widetilde{\Phi}(x,k_{p+1})$ *from* (4.32) *and* $\dot{\varphi}_{ll}(x,k_{p+1})$*:*

$$\begin{gathered}
\int\limits_{-\infty}^{0}(1+|t|^2)\left|\frac{d}{dt}\Phi(t,k_{p+1})\right|dt < \infty; \qquad \int\limits_{-\infty}^{0}(1+|t|^2)\left|\frac{d}{dt}\widetilde{\Phi}(t,k_{p+1})\right|dt < \infty;\\
\int\limits_{-\infty}^{0}(1+|t|^2)\left|\frac{d}{dt}\dot{\varphi}_{ll}(t,k_{p+1})\right|dt < \infty, \qquad (l=1,2).
\end{gathered} \tag{4.34}$$

A *proof* is similar to that of Lemma 6.6.2 [16] in view of the fact that $V_p(x)$ satisfies (1.2).

Lemma 12. *The following statements are valid for the matrix function* $\Gamma(x,k_{p+1})$ *from* (4.32)*:*

a) $|\Gamma(x,k_{p+1})|$ *and* $|\Gamma^{-1}(x,k_{p+1})|$ *are bounded as* $x \to -\infty$*;*

b)

$$\int\limits_{-\infty}^{0}(1+|t|^2)\left|\frac{d}{dt}\Gamma(t,k_{p+1})\right|dt < \infty; \tag{4.35}$$

c) $|\dot\gamma_{ll}(x,k_{p+1})|$ *are bounded as* $x\to-\infty$, $(l=1,2)$, *and*

$$\int\limits_{-\infty}^{0}(1+|t|^2)\left|\frac{d}{dt}\dot\gamma_{ll}(t,k_{p+1})\right|dt<\infty, \qquad (l=1,2), \tag{4.36}$$

with γ_{ll}, $(l=1,2)$, *being the diagonal elements of the matrix function* $\Gamma(x,k_{p+1})$.

Proofs of propositions a) and b) of Lemma 12 are similar to that of Lemma 6.6.3 [16] in view of the fact that $V_p(x)$ satisfies 1.2. Also note that the boundedness of $|\Gamma(x,k_{p+1})|$ as $x\to-\infty$ follows from the fact that $|\Gamma^{-1}(x,k_{p+1})|$ is bounded and $|\gamma_{ll}(x,k_{p+1})|\ge a_l>0$, $(l=1,2)$, as $x\to-\infty$.

Prove the proposition c) of Lemma 12.

One has from (4.32):

$$\dot\gamma_{ll}(x,k_{p+1})=2\int\limits_{x}^{\infty}\dot\varphi_{ll}(t,k_{p+1})\varphi_{ll}(t,k_{p+1})e^{-2ik_{p+1}(x-t)}dt-$$
$$-2i\int\limits_{x}^{\infty}\varphi_{ll}^2(t,k_{p+1})(x-t)e^{-2ik_{p+1}(x-t)}dt=$$
$$=2\int\limits_{-\infty}^{0}\dot\varphi_{ll}(x-z,k_{p+1})\varphi_{ll}(x-z,k_{p+1})e^{-2ik_{p+1}z}dz-$$
$$-2i\int\limits_{-\infty}^{0}\varphi_{ll}^2(x-z,k_{p+1})ze^{-2ik_{p+1}z}dz. \tag{4.37}$$

Thus we obtain in view of Lemma 10 that

$$|\dot\gamma_{ll}(x,k_{p+1})|\le C_{ll}\left(\int\limits_{-\infty}^{0}e^{-2ik_{p+1}z}dz+\int\limits_{-\infty}^{0}|z|e^{-2ik_{p+1}z}dz\right)<\infty, \qquad l=1,2,$$

as $x\to-\infty$.

Since $\int\limits_0^{+\infty}e^p_{ll+}(x,k_{p+1})dx<\infty$, with the notation

$$\gamma^p_{ll}(x,k_{p+1})=e^{-2ik_{p+1}x}\int\limits_{x}^{0}e^p_{ll+}(t,k_{p+1})^2dt, \qquad \alpha_{ll}(k_{p+1})=\int\limits_{0}^{\infty}e^p_{ll+}(t,k_{p+1})^2dt,$$

we obtain the following representation:

$$\gamma_{ll}(x,k_{p+1})=\alpha_{ll}(k_{p+1})e^{-2ik_{p+1}x}+\gamma^p_{ll}(x,k_{p+1}),$$

and hence

$$\dot{\gamma}_{ll}(x,k_{p+1}) = -2ixe^{-2ik_{p+1}x}\alpha_{ll}(k_{p+1}) + e^{-2ik_{p+1}x}\dot{\alpha}_{ll}(k_{p+1}) + \dot{\gamma}^p_{ll}(x,k_{p+1}).$$

Thus it suffices to prove (4.36) for the function $\frac{d}{dt}\dot{\gamma}^p_{ll}(t,k_{p+1})$.
So

$$\begin{aligned}\dot{\gamma}^p_{ll}(x,k_{p+1}) = -2ix\int_x^0 \varphi_{ll}(t,k_{p+1})^2 e^{-2ik_{p+1}(x-t)}dt + 2\int_x^0 (\dot{\varphi}_{ll}(t,k_{p+1}) + \\ + it\varphi_{ll}(t,k_{p+1}))\varphi_{ll}(t,k_{p+1})e^{-2ik_{p+1}(x-t)}dt = \\ = 2\int_x^0 \dot{\varphi}_{ll}(x-z,k_{p+1})\varphi_{ll}(x-z,k_{p+1})e^{-2ik_{p+1}z}dz - \\ - 2i\int_x^0 \varphi_{ll}(x-z,k_{p+1})^2 z e^{-2ik_{p+1}z}dz.\end{aligned}$$

Hence

$$\begin{aligned}\frac{d}{dt}\dot{\gamma}^p_{ll}(t,k_{p+1}) = -2\dot{\varphi}_{ll}(0,k_{p+1})\varphi_{ll}(0,k_{p+1})e^{-2ik_{p+1}t} + 2i\varphi_{ll}(0,k_{p+1})^2 t e^{-2ik_{p+1}t} + \\ + 2\int_t^0 \left[\frac{d}{dt}\dot{\varphi}_{ll}(t-z,k_{p+1})\varphi_{ll}(t-z,k_{p+1}) + \dot{\varphi}_{ll}(t-z,k_{p+1})\frac{d}{dt}\varphi_{ll}(t-z,k_{p+1})\right]e^{-2ik_{p+1}z}dz - \\ - 4i\int_t^0 \frac{d}{dt}\varphi_{ll}(t-z,k_{p+1})\varphi_{ll}(t-z,k_{p+1})z e^{-2ik_{p+1}z}dz.\end{aligned}$$

Thus, in view of Lemmas 10, 11, one has

$$\begin{aligned}\int_{-\infty}^0 (1+|t|^2)\left|\frac{d}{dt}\dot{\gamma}^p_{ll}(t,k_{p+1})\right|dt \le C_0 + C_1\int_{-\infty}^0\left(\int_t^0\left|\frac{d}{dt}\dot{\varphi}_{ll}(t-z,k_{p+1})\right|e^{-2ik_{p+1}z}dz + \right. \\ \left. + \int_t^0\left|\frac{d}{dt}\varphi_{ll}(t-z,k_{p+1})\right|e^{-2ik_{p+1}z}dz\right)(1+|t|^2)dt + \\ + C_2\int_{-\infty}^0 (1+|t|^2)\int_t^0\left|\frac{d}{dt}\dot{\varphi}_{ll}(t-z,k_{p+1})\right| z e^{-2ik_{p+1}z}dzdt = \\ = C_0 + C_1\int_{-\infty}^0 e^{-2ik_{p+1}z}dz\int_{-\infty}^z (1+|t|^2)\left|\frac{d}{dt}\dot{\varphi}_{ll}(t-z,k_{p+1})\right|dt +\end{aligned}$$

$$+ C_2 \int\limits_{-\infty}^{0} z e^{-2ik_{p+1}z} dz \int\limits_{-\infty}^{z} (1+|t|^2) \left| \frac{d}{dt} \varphi_{ll}(t-z, k_{p+1}) \right| dt < \infty,$$

and Lemma 12 is proved. ◀

We turn back to the proof of Lemma 9. Use the notation (4.32) to rewrite $\triangle V(x)$:

$$\triangle\, V(x) = -2\frac{d}{dx} B(x,x) =$$
$$= -2\frac{d}{dx}\Big[-\Phi(x,k_{p+1}) Z^+_{p+1}(0) \{ I e^{-2ik_{p+1}x} + \Gamma(x,k_{p+1}) Z^+_{p+1}(0)\}^{-1} \widetilde{\Phi}(x,k_{p+1}) +$$
$$+ i \left\{ \frac{e^{-2ik_{p+1}x} \frac{d}{dk}(\varphi_{11}(x,k)\varphi_{22}(x,k))_{k_{p+1}} + 2ix\varphi_{11}(x,k_{p+1})\varphi_{22}(x,k_{p+1}) e^{-2ik_{p+1}x}}{\left(e^{-2ik_{p+1}x} + z_{11}^{[p+1]+}\gamma_{11}(x,k_{p+1})\right)\left(e^{-2ik_{p+1}x} + z_{22}^{[p+1]+}\gamma_{22}(x,k_{p+1})\right)} + \right.$$
$$+ \frac{\dot\varphi_{11}(x,k_{p+1})\varphi_{22}(x,k_{p+1}) z_{11}^{[p+1]+}\gamma_{11}(x,k_{p+1})}{\left(e^{-2ik_{p+1}x} + z_{11}^{[p+1]+}\gamma_{11}(x,k_{p+1})\right)\left(e^{-2ik_{p+1}x} + z_{22}^{[p+1]+}\gamma_{22}(x,k_{p+1})\right)} +$$
$$+ \frac{\varphi_{11}(x,k_{p+1})\dot\varphi_{22}(x,k_{p+1}) z_{22}^{[p+1]+}\gamma_{22}(x,k_{p+1})}{\left(e^{-2ik_{p+1}x} + z_{11}^{[p+1]+}\gamma_{11}(x,k_{p+1})\right)\left(e^{-2ik_{p+1}x} + z_{22}^{[p+1]+}\gamma_{22}(x,k_{p+1})\right)} -$$
$$\left. - \frac{\varphi_{11}(x,k_{p+1})\varphi_{22}(x,k_{p+1})\left(z_{11}^{[p+1]+}\dot\gamma_{11}(x,k_{p+1}) + z_{22}^{[p+1]+}\dot\gamma_{22}(x,k_{p+1})\right)}{2\left(e^{-2ik_{p+1}x} + z_{11}^{[p+1]+}\gamma_{11}(x,k_{p+1})\right)\left(e^{-2ik_{p+1}x} + z_{22}^{[p+1]+}\gamma_{22}(x,k_{p+1})\right)} \right\} (Z^+_{p+1})'(0) \Big].$$

Now consider possible (with the assumption 6) of Theorem 1 being taken into account) Cases I – III:

I) $z_{22}^{[p+1]+} = 0$. In view of the assumption 5) of the Theorem one has $(Z^+_{p+1})'(t) \equiv 0$, $Z^+_{p+1}(t) \equiv Z^+_{p+1}$, that is

$$\triangle\, V(x) =$$
$$= 2\frac{1}{e^{-2ik_{p+1}x} + z_{11}^{[p+1]+}\gamma_{11}} \left\{ \Phi'(x,k_{p+1}) Z^+_{p+1} \widetilde{\Phi}(x,k_{p+1}) + \Phi(x,k_{p+1}) Z^+_{p+1} \widetilde{\Phi}'(x,k_{p+1}) \right\} -$$
$$- 2\frac{-2ik_{p+1}e^{-2ik_{p+1}x} + z_{11}^{[p+1]+}\gamma'_{11}}{\left(e^{-2ik_{p+1}x} + z_{11}^{[p+1]+}\gamma_{11}\right)^2} \Phi(x,k_{p+1}) Z^+_{p+1} \widetilde{\Phi}(x,k_{p+1}). \quad (4.38)$$

II) $z_{11}^{[p+1]+} = 0$. In view of the assumption 5) of the Theorem one has $(Z^+_{p+1})'(t) \equiv 0$, $Z^+_{p+1}(t) \equiv Z^+_{p+1}$, that is

$$\triangle\, V(x) =$$
$$= 2\frac{1}{e^{-2ik_{p+1}x} + z_{22}^{[p+1]+}\gamma_{22}} \left\{ \Phi'(x,k_{p+1}) Z^+_{p+1} \widetilde{\Phi}(x,k_{p+1}) + \Phi(x,k_{p+1}) Z^+_{p+1} \widetilde{\Phi}'(x,k_{p+1}) \right\}$$

$$
- 2\frac{-2ik_{p+1}e^{-2ik_{p+1}x}+z_{22}^{[p+1]+}\gamma_{22}'}{\left(e^{-2ik_{p+1}x}+z_{22}^{[p+1]+}\gamma_{22}\right)^2}\Phi(x,k_{p+1})Z_{p+1}^{+}\widetilde{\Phi}(x,k_{p+1}). \quad (4.39)
$$

III) $z_{ll}^{[p+1]+}\neq 0,\ \ (l=1,2)$, then

$$
\begin{aligned}
\triangle V(x) = {} & 2\Phi'(x,k_{p+1})\left\{e^{-2ik_{p+1}x}Z_{p+1}^{+}(0)^{-1}+\Gamma(x,k_{p+1})\right\}^{-1}\widetilde{\Phi}(x,k_{p+1})+\\
& + 2\Phi(x,k_{p+1})\left\{e^{-2ik_{p+1}x}Z_{p+1}^{+}(0)^{-1}+\Gamma(x,k_{p+1})\right\}^{-1}\widetilde{\Phi}'(x,k_{p+1})-\\
& - 2\Phi(x,k_{p+1})\left\{e^{-2ik_{p+1}x}Z_{p+1}^{+}(0)^{-1}+\Gamma(x,k_{p+1})\right\}^{-1}[-2ik_{p+1}e^{-2ik_{p+1}x}Z_{p+1}^{+}(0)^{-1}+\\
& + \Gamma'(x,k_{p+1})]\left\{e^{-2ik_{p+1}x}Z_{p+1}^{+}(0)^{-1}+\Gamma(x,k_{p+1})\right\}^{-1}\widetilde{\Phi}(x,k_{p+1})-\\
& - 2i\left[\frac{-2ik_{p+1}e^{-2ik_{p+1}x}\left[\frac{d}{dk}(\varphi_{11}\varphi_{22})+2ix\varphi_{11}\varphi_{22}\right]}{\left(e^{-2ik_{p+1}x}+z_{11}^{[p+1]+}\gamma_{11}\right)\left(e^{-2ik_{p+1}x}+z_{22}^{[p+1]+}\gamma_{22}\right)}+\right.\\
& + \frac{e^{-2ik_{p+1}x}\left[\frac{d^2}{dkdx}(\varphi_{11}\varphi_{22})+2i\varphi_{11}\varphi_{22}+2ix(\varphi_{11}\varphi_{22})'\right]}{\left(e^{-2ik_{p+1}x}+z_{11}^{[p+1]+}\gamma_{11}\right)\left(e^{-2ik_{p+1}x}+z_{22}^{[p+1]+}\gamma_{22}\right)}+\\
& + \frac{z_{11}^{[p+1]+}\left[(\dot\varphi_{11}\varphi_{22})'\gamma_{11}+\dot\varphi_{11}\varphi_{22}\gamma_{11}'-\frac{(\varphi_{11}\varphi_{22})'}{2}\dot\gamma_{11}-\frac{\varphi_{11}\varphi_{22}}{2}\dot\gamma_{11}'\right]}{\left(e^{-2ik_{p+1}x}+z_{11}^{[p+1]+}\gamma_{11}\right)\left(e^{-2ik_{p+1}x}+z_{22}^{[p+1]+}\gamma_{22}\right)}+\\
& + \frac{z_{22}^{[p+1]+}\left[(\varphi_{11}\dot\varphi_{22})'\gamma_{22}+\varphi_{11}\dot\varphi_{22}\gamma_{22}'-\frac{(\varphi_{11}\varphi_{22})'}{2}\dot\gamma_{22}-\frac{\varphi_{11}\varphi_{22}}{2}\dot\gamma_{22}'\right]}{\left(e^{-2ik_{p+1}x}+z_{11}^{[p+1]+}\gamma_{11}\right)\left(e^{-2ik_{p+1}x}+z_{22}^{[p+1]+}\gamma_{22}\right)}-\\
& - \left[\frac{-4ik_{p+1}e^{-4ik_{p+1}x}-2ik_{p+1}e^{-2ik_{p+1}x}\left(z_{11}^{[p+1]+}\gamma_{11}+z_{22}^{[p+1]+}\gamma_{22}\right)}{\left(e^{-2ik_{p+1}x}+z_{11}^{[p+1]+}\gamma_{11}\right)^2\left(e^{-2ik_{p+1}x}+z_{22}^{[p+1]+}\gamma_{22}\right)^2}+\right.\\
& \left. + \frac{e^{-2ik_{p+1}x}\left(z_{11}^{[p+1]+}\gamma_{11}'+z_{22}^{[p+1]+}\gamma_{22}'\right)+z_{11}^{[p+1]+}z_{22}^{[p+1]+}(\gamma_{11}\gamma_{22})'}{\left(e^{-2ik_{p+1}x}+z_{11}^{[p+1]+}\gamma_{11}\right)^2\left(e^{-2ik_{p+1}x}+z_{22}^{[p+1]+}\gamma_{22}\right)^2}\right]\cdot\\
& \cdot\left\{e^{-2ik_{p+1}x}\left[\frac{d}{dk}(\varphi_{11}\varphi_{22})+2ix\varphi_{11}\varphi_{22}\right]+\dot\varphi_{11}\varphi_{22}z_{11}^{[p+1]+}\gamma_{11}+\right.\\
& \left.\left. +\varphi_{11}\dot\varphi_{22}z_{22}^{[p+1]+}\gamma_{22}-\frac{\varphi_{11}\varphi_{22}}{2}\left(z_{11}^{[p+1]+}\dot\gamma_{11}+z_{22}^{[p+1]+}\dot\gamma_{22}\right)\right\}\right](Z_{p+1}^{+})'(0).
\end{aligned}
$$

It follows from Lemmas 10 – 12 that in each of the three cases the matrix function $\triangle V(x)$ satisfies (4.31) if $V_p(x)$ satisfies (1.2). Lemma 9 is proved. ◄

It is possible to deduce from (4.27) the following differential equation for $B(x,y)$ (cf., for instance, [16]):

$$\frac{\partial^2 B(x,y)}{\partial x^2} - V_{p+1}(x)B(x,y) = \frac{\partial^2 B(x,y)}{\partial y^2} - B(x,y)V_p(y), \tag{4.40}$$

with $B(x,x) = \frac{1}{2}\int\limits_x^\infty \triangle V(t)dt$.

It follows from (4.40), in view of the fact that $B(x,y)$ tends to zero as $y \to +\infty$ (see (4.28)), that the function

$$E_+(x,k) = E_+^p(x,k) + \int\limits_x^\infty B(x,y)E_+^p(y,k)dy, \qquad \operatorname{Im} k \geq 0, \tag{4.41}$$

is the Jost solution for the equation

$$-Y'' + V_{p+1}(x)Y = k^2 Y, \qquad -\infty < x < +\infty, \tag{4.42}$$

with asymptotics $E_+(x,k) \sim e^{ikx}I$ as $x \to +\infty$.

Lemma 13. *The right SD of the problems* (4.42), (1.2) *(with* $V(x) = V_{p+1}(x)$*) coincide to the values given in* (4.24)*.*

A *proof* of the Lemma 13 is based on the computation of the coefficients $A(k)$ and $B(k)$ (2.8) for the equation (4.42).

In view of the assumptions 5) and 6) of Theorem 1, one has three possible cases again:

Case I). $Z_{p+1}^+(t) \equiv Z_{p+1}^+ = \begin{pmatrix} 0 & z_{12}^{[p+1]+} \\ 0 & z_{22}^{[p+1]+} \end{pmatrix}$: $z_{22}^{[p+1]+} > 0$, then we obtain from (4.28) and (4.41), in view of (2.7) and (3.10) as $x \to -\infty$,

$$\begin{aligned} E_+(x,k) &= E_+^p(x,k) - \\ &- E_+^p(x,k_{p+1})Z_{p+1}^+\left[I + \int\limits_x^\infty \widetilde{E}_+^p(t,k_{p+1})E_+^p(t,k_{p+1})dtZ_{p+1}^+\right]^{-1}\int\limits_x^\infty \widetilde{E}_+^p(y,k_{p+1})E_+^p(y,k)dy \sim \\ &\sim e^{ikx}\left\{I - \frac{2k_{p+1}}{(k+k_{p+1})z_{22}^{[p+1]+}a_{22}^p(k_{p+1})}A_p(k_{p+1})Z_{p+1}^+\right\}A_p(k) + \\ &+ e^{-ikx}\left\{I + \frac{2k_{p+1}}{(k-k_{p+1})z_{22}^{[p+1]+}a_{22}^p(k_{p+1})}A_p(k_{p+1})Z_{p+1}^+\right\}B_p(k), \end{aligned}$$

hence

$$A(k) = \alpha(k)A_p(k), \qquad B(k) = \alpha(-k)B_p(k), \qquad k \in \mathbb{R}, \tag{4.43}$$

with

$$\alpha(k) = I - \frac{2k_{p+1}}{(k+k_{p+1})z_{22}^{[p+1]+}a_{22}^{p}(k_{p+1})}A_p(k_{p+1})Z_{p+1}^{+} =$$
$$= \begin{pmatrix} 1 & -\frac{2k_{p+1}\left(a_{11}^{p}(k_{p+1})z_{12}^{[p+1]+}+a_{12}^{p}(k_{p+1})z_{22}^{[p+1]+}\right)}{(k+k_{p+1})z_{22}^{[p+1]+}a_{22}^{p}(k_{p+1})} \\ 0 & \frac{k-k_{p+1}}{k+k_{p+1}} \end{pmatrix}. \quad (4.44)$$

It is clear from the first relation in (4.43) that the equation (4.42) has the same eigenvalues $k_1^2, k_2^2, \ldots, k_p^2$ as the initial equation (4.25), together with one more eigenvalue k_{p+1}^2.

To compute the right reflection coefficient $R_{p+1}^{+}(k)$ and $C(k)$ that correspond to the constructed equation (4.42), we use (3.7) and (2.12):

$$R_{p+1}^{+}(k) = -A(k)^{-1}B(-k) = -A_p(k)^{-1}\alpha^{-1}(k)\alpha(k)B_p(-k) = R^{+}(k), \qquad k \in \mathbb{R}, \quad (4.45)$$

and

$$C(k) = (I - R_{p+1}^{+}(-k)R_{p+1}^{+}(k))^{-1}A(-k)^{-1} =$$
$$= (I - R^{+}(-k)R^{+}(k))^{-1}A_p(-k)^{-1}\alpha^{-1}(-k) = C_p(k)\alpha^{-1}(-k), \qquad k \in \mathbb{R}. \quad (4.46)$$

It is clear from (4.44) that $\alpha^{-1}(-k) = \alpha(k)$, hence $C(k) = C_p(k)\alpha(k)$.

Now prove that initial p normalizing polynomials of the problem (4.42), (1.2) with $V(x) = V_{p+1}(x)$, coincide with the normalizing polynomials of the problem (4.25), (1.2) with $V(x) = V_p(x)$.

It follows from the definition (2.13) of the normalizing polynomial $Z_{j<p+1>}^{-}(t)$ for the equation (4.42) and the relation (4.43) that

$$Z_{j<p+1>}^{-}(t) =$$
$$= -i\left\{\dot{W}^{-}(k_j)A_{-2}^{p<k_j>}\alpha(-k_j) - W^{-}(k_j)A_{-2}^{p<k_j>}\dot{\alpha}(-k_j) + W^{-}(k_j)A_{-1}^{p<k_j>}\alpha(-k_j)\right\} -$$
$$- tW^{-}(k_j)A_{-2}^{p<k_j>}\alpha(-k_j), \qquad j = \overline{1,p}, \quad (4.47)$$

with

$$A_{-2}^{p<k_j>} = (A_p(k)^{-1}(k-k_j)^2)_{k_j}; \qquad A_{-1}^{p<k_j>} = \frac{d}{dk}(A_p(k)^{-1}(k-k_j)^2)_{k_j},$$

being the Laurent coefficients.

One can show, using (3.10) and (3.13) with $x \to +\infty$ and noting that k_j is an eigenvalue of the equation (4.25) with $k_j < k_{p+1}$, $(j = \overline{1,p})$, that $W^{-}(k_j)A_{-2}^{p<k_j>} = -\alpha(-k_j)(Z_j^{-})'(0)$; $\dot{W}^{-}(k_j)A_{-2}^{p<k_j>} + W^{-}(k_j)A_{-1}^{p<k_j>} = i\alpha(-k_j)(Z_j^{-})(0) + \dot{\alpha}(-k_j)(Z_j^{-})'(0)$, hence (4.47) can be rewritten as follows: $Z_{j<p+1>}^{-}(t) = \alpha(-k_j)Z_j^{-}(t)\alpha(-k_j) - i\frac{d}{dk}(\alpha(k)(Z_j^{-})'(0)\alpha(k))_{k=-k_j}$.

Next, use the relation (3.11) with $\tau = t$:

$$Z_{j<p+1>}^{+}(t) = -A_{-1}^{<k_j>}(Z_{j<p+1>}^{-}(t) + Q_j)^{-1}C_{-1}^{<k_j>} =$$
$$= -\left[A_{-1}^{p<k_j>}\alpha(-k_j) - A_{-2}^{p<k_j>}\dot{\alpha}(-k_j)\right]\alpha(k_j)\cdot$$

$$\cdot\left\{Z_j^-(t)-i\alpha(k_j)\frac{d}{dk}(\alpha(k)(Z_j^-)'(0)\alpha(k))_{k=-k_j}\alpha(k_j)+\alpha(k_j)Q_j\alpha(k_j)\right\}^{-1}\cdot$$
$$\cdot\,\alpha(k_j)\left[\alpha(-k_j)C_{-1}^{p<k_j>}-\dot\alpha(-k_j)C_{-2}^{p<k_j>}\right],$$

with $C_{-1}^{p<k_j>}=\frac{d}{dk}(C_p(k)^{-1}(k-k_j)^2)_{k_j}$; $C_{-2}^{p<k_j>}=(C_p(k)^{-1}(k-k_j)^2)_{k_j}$ being the Laurent coefficients.

In view of (4.44) one has

$$\left\{Z_j^-(t)+\alpha(k_j)Q_j\alpha(k_j)-i\alpha(k_j)\frac{d}{dk}(\alpha(k)(Z_j^-)'(0)\alpha(k))_{k=-k_j}\alpha(k_j)\right\}^{-1}=$$
$$=(Z_j^-(t)+\alpha(k_j)Q_j\alpha(k_j))^{-1}+\frac{2ik_{p+1}(Z_j^-)'(0)}{\left(k_j^2-k_{p+1}^2\right)\left(z_{11}^{[j]-}+q_{11}^{[j]}\right)\left(z_{22}^{[j]-}+\left(\frac{k_j-k_{p+1}}{k_j+k_{p+1}}\right)^2q_{22}^{[j]}\right)},$$

hence

$$Z_{j<p+1>}^+(t)=-A_{-1}^{p<k_j>}(Z_j^-(t)+\alpha(k_j)Q_j\alpha(k_j))^{-1}C_{-1}^{p<k_j>}+$$
$$+A_{-1}^{p<k_j>}(Z_j^-(t)+\alpha(k_j)Q_j\alpha(k_j))^{-1}\alpha(k_j)\dot\alpha(-k_j)C_{-2}^{p<k_j>}+$$
$$+A_{-2}^{p<k_j>}\dot\alpha(-k_j)\alpha(k_j)(Z_j^-(t)+\alpha(k_j)Q_j\alpha(k_j))^{-1}C_{-1}^{p<k_j>}-$$
$$-\frac{2ik_{p+1}}{\left(k_j^2-k_{p+1}^2\right)\left(z_{11}^{[j]-}+q_{11}^{[j]}\right)\left(z_{22}^{[j]-}+\left(\frac{k_j-k_{p+1}}{k_j+k_{p+1}}\right)^2q_{22}^{[j]}\right)}A_{-1}^{p<k_j>}(Z_j^-)'(0)C_{-1}^{p<k_j>}.$$

If $z_{11}^{[j]-}>0$ and $z_{22}^{[j]-}>0$, then $q_{11}^{[j]}=q_{22}^{[j]}=0$, with the definition of Q_j of Lemma 3 being taken into account; $A_{-2}^{p<k_j>}=\begin{pmatrix}0&\frac{-a_{12}^p(k_j)}{\dot a_{11}^p(k_j)\dot a_{22}^p(k_j)}\\0&0\end{pmatrix}$, hence

$$Z_{j<p+1>}^+(t)=Z_j^+(t)+\frac{2k_{p+1}}{\left(k_j^2-k_{p+1}^2\right)z_{22}^{[j]-}\dot a_{22}^p(k_j)\dot a_{11}^p(k_j)}\begin{pmatrix}0&\frac{-a_{12}^p(k_j)}{\dot a_{22}^p(k_j)}-i\frac{(z_{12}^{[j]-})'(0)}{z_{11}^{[j]-}}\\0&0\end{pmatrix}=$$
$$=Z_j^+(t),$$

for $j=\overline{1,p}$ in view of (3.3) and (3.5).

On the other hand, if either $z_{11}^{[j]-}=0$ or $z_{22}^{[j]-}=0$, then by the assumption 5) of Theorem 1 $(Z_j^-)'(0)=0=A_{-2}^{p<k_j>}$, hence $Z_{j<p+1>}^+(t)=Z_j^+(t)$, $(j=\overline{1,p})$.

At any case, the initial p normalizing polynomials of the equation (4.42) turn out to be the same as those of the equation (4.25).

Now prove that Z_{p+1}^+ is a normalizing polynomial of (4.42).

Similarly to (4.47), use (4.43), (4.44) to obtain

$$Z^-_{p+1<p+1>}(t) =$$

$$= -i\frac{d}{dk}(W^-(k)A_p^{-1}(k)\alpha(-k)(k-k_{p+1})^2)_{k_{p+1}} - t(W^-(k)A_p^{-1}(k)\alpha(-k)(k-k_{p+1})^2)_{k_{p+1}} =$$

$$= -iW^-(k_{p+1})A_p^{-1}(k_{p+1})\left[\frac{2k_{p+1}}{z_{22}^{[p+1]+}a_{22}^p(k_{p+1})}A_p(k_{p+1})Z^+_{p+1}\right] =$$

$$= -iW^-(k_{p+1})Z^+_{p+1}\frac{2k_{p+1}}{z_{22}^{[p+1]+}a_{22}^p(k_{p+1})} =$$

$$= W\left\{\widetilde{E}^\wedge_-(x,k_{p+1});\ E^p_+(x,k_{p+1}) - \frac{\int\limits_x^\infty e_{22}^{p+}(t,k_{p+1})^2dt}{1+z_{22}^{[p+1]+}\int\limits_x^\infty e_{22}^{p+}(t,k_{p+1})^2dt}E^p_+(x,k_{p+1})Z^+_{p+1}\right\}\cdot$$

$$\cdot\frac{1}{z_{22}^{[p+1]+}a_{22}^p(k_{p+1})}Z^+_{p+1} =$$

$$= \frac{1}{a_{22}^p(k_{p+1})}W\left\{\widetilde{E}^\wedge_-(x,k_{p+1});\ E^p_+(x,k_{p+1})\left[I - \frac{\int\limits_x^\infty e_{22}^{p+}(t,k_{p+1})^2dt}{1+z_{22}^{[p+1]+}\int\limits_x^\infty e_{22}^{p+}(t,k_{p+1})^2dt}Z^+_{p+1}\right]\right\}.$$

Assume $x\to-\infty$ and apply the asymptotics $E^p_+(x,k_{p+1})\sim e^{ik_{p+1}x}A_p(k_{p+1})$, that is, as $x\to-\infty$

$$Z^-_{p+1<p+1>}(t) \equiv Z^-_{p+1<p+1>} =$$

$$= \frac{1}{a_{22}^p(k_{p+1})}W\left\{e^{ik_{p+1}x}I;\ e^{ik_{p+1}x}A_p(k_{p+1})\begin{pmatrix}1 & -\frac{z_{12}^{[p+1]+}}{z_{22}^{[p+1]+}}\\ 0 & -\frac{2ik_{p+1}e^{-2ik_{p+1}x}}{z_{22}^{[p+1]+}a_{22}^p(k_{p+1})^2}\end{pmatrix}\right\} =$$

$$= \frac{1}{a_{22}^p(k_{p+1})}A_p(k_{p+1})\begin{pmatrix}0 & 0\\ 0 & \frac{(2ik_{p+1})^2}{z_{22}^{[p+1]+}a_{22}^p(k_{p+1})^2}\end{pmatrix} =$$

$$= \frac{(2ik_{p+1})^2}{z_{22}^{[p+1]+}a_{22}^p(k_{p+1})^3}A_p(k_{p+1})\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \frac{-4k_{p+1}^2}{z_{22}^{[p+1]+}a_{22}^p(k_{p+1})^3}\begin{pmatrix}0 & a_{12}^p(k_{p+1})\\ 0 & a_{22}^p(k_{p+1})\end{pmatrix}.$$

It follows from (3.13) with $\tau=t$ and (4.46) that

$$Z^+_{p+1<p+1>} = -A_{-1}^{<k_{p+1}>}(Z^-_{p+1<p+1>}+Q_{p+1})^{-1}C_{-1}^{<k_{p+1}>} =$$

$$= -A_p^{-1}(k_{p+1})\left\{\frac{2k_{p+1}}{z_{22}^{[p+1]+}a_{22}^p(k_{p+1})}A_p(k_{p+1})Z^+_{p+1}\right\}\frac{z_{22}^{[p+1]+}a_{22}^p(k_{p+1})^3}{-4k_{p+1}^2a_{22}^p(k_{p+1})}\cdot$$

$$\cdot\left\{\frac{2k_{p+1}}{z_{22}^{[p+1]+}a_{22}^p(k_{p+1})}A_p(k_{p+1})Z^+_{p+1}\right\}C_p^{-1}(k_{p+1}) =$$

$$= Z^+_{p+1}\frac{a^p_{22}(k_{p+1})}{z^{[p+1]+}_{22}}z^{[p+1]+}_{22}\frac{1}{a^p_{22}(k_{p+1})} = Z^+_{p+1},$$

and Lemma 13 in the Case I) is proved.

Case II). $Z^+_{p+1}(t) \equiv Z^+_{p+1} = \begin{pmatrix} z^{[p+1]+}_{11} & z^{[p+1]+}_{12} \\ 0 & 0 \end{pmatrix}$, where $z^{[p+1]+}_{11} > 0$, then similarly to the Case I) we obtain

$$A(k) = \beta(k)A_p(k), \qquad B(k) = \beta(-k)B_p(k), \qquad k \in \mathbb{R}, \tag{4.48}$$

with

$$\beta(k) = I - \frac{2ik_{p+1}}{(k+k_{p+1})z^{[p+1]+}_{11}a^p_{11}(k_{p+1})}Z^+_{p+1}C_p(k_{p+1}) =$$
$$= \begin{pmatrix} \frac{k-k_{p+1}}{k+k_{p+1}} & -\frac{2ik_{p+1}(z^{[p+1]+}_{11}c^p_{12}(k_{p+1})+z^{[p+1]+}_{12}a^p_{22}(k_{p+1}))}{(k+k_{p+1})z^{[p+1]+}_{11}a^p_{11}(k_{p+1})} \\ 0 & 1 \end{pmatrix}. \tag{4.49}$$

It is clear from the first relation in (4.48) that the equation (4.42) has the same eigenvalues $k^2_1, k^2_2, \dots, k^2_p$ as the original equation (4.25), and one more eigenvalue k^2_{p+1}.

Just as in (4.46), we deduce from (3.7) and (2.12) that

$$C(k) = C_p(k)\beta^{-1}(-k) = C_p(k)\beta(k), \tag{4.50}$$

and the reflection coefficient of the equation (4.42) coincides to that of (4.25): $R^+_{p+1}(k) = R^+(k)$.

The coincidence of p normalizing polynomials of the equation (4.42) to those of (4.25) can be proved just as in the case I) with the substitution of $\beta(k)$ (4.49) for $\alpha(k)$ (4.44).

Now prove that Z^+_{p+1} is a normalizing polynomial of (4.42). Similarly to (4.47), use (4.48) and (4.49) to obtain

$$Z^-_{p+1<p+1>}(t) \equiv Z^-_{p+1<p+1>} = -iW^-(k_{p+1})A^{-1}_p(k_{p+1})\frac{2k_{p+1}}{z^{[p+1]+}_{11}a^p_{11}(k_{p+1})}Z^+_{p+1}C_p(k_{p+1}) =$$
$$= \frac{1}{z^{[p+1]+}_{11}a^p_{11}(k_{p+1})^2}W\Big\{\widetilde{E}^\wedge_-(x,k_{p+1});\ E^p_+(x,k_{p+1}) -$$
$$\left. -\frac{e^{p+}_{11}(x,k_{p+1})}{1+z^{[p+1]+}_{11}\int\limits_x^\infty(e^{p+}_{11}(t,k_{p+1}))^2dt}Z^+_{p+1}\int\limits_x^\infty \widetilde{E}^p_+(y,k_{p+1})E^p_+(y,k_{p+1})dy\right\}Z^+_{p+1}C_p(k_{p+1}) =$$
$$= \frac{W\left\{e^\wedge_{11}(x,k_{p+1});\ \frac{e^{p+}_{11}(x,k_{p+1})}{z^{[p+1]+}_{11}\int\limits_x^\infty(e^{p+}_{11}(t,k_{p+1}))^2dt}\right\}}{z^{[p+1]+}_{11}a^p_{11}(k_{p+1})}Z^+_{p+1}C_p(k_{p+1}).$$

Supposing $x \to -\infty$ and using the asymptotics $e_{11}^{p+}(x,k_{p+1}) \sim e^{ik_{p+1}x} a_{11}^p(k_{p+1})$, that is with $x \to -\infty$ one has

$$Z^-_{p+1<p+1>} = \frac{W\left\{ e^{ik_{p+1}x}; e^{ik_{p+1}x} a_{11}^p(k_{p+1}) \frac{2ik_{p+1}e^{ik_{p+1}x}}{-z_{11}^{[p+1]+} a_{11}^p(k_{p+1})^2} \right\}}{z_{11}^{[p+1]+} a_{11}^p(k_{p+1})^2} Z^+_{p+1} C_p(k_{p+1}) = \\ = \frac{-4k_{p+1}^2}{\left(z_{11}^{[p+1]+}\right)^2 a_{11}^p(k_{p+1})^3} Z^+_{p+1} C_p(k_{p+1}).$$

It follows from the relation (3.13) with $\tau = t$ and (4.50) that

$$Z^+_{p+1<p+1>} = -A_{-1}^{<k_{p+1}>} (Z^-_{p+1<p+1>} + Q_{p+1})^{-1} C_{-1}^{<k_{p+1}>} = \\ = -\frac{2k_{p+1}}{z_{11}^{[p+1]+} a_{11}^p(k_{p+1})} A_p^{-1}(k_{p+1}) Z^+_{p+1} C_p(k_{p+1}) \frac{\left(z_{11}^{[p+1]+}\right)^2 a_{11}^p(k_{p+1})^3}{-4k_{p+1}^2} \cdot \\ \cdot \left(Z^+_{p+1} C_p(k_{p+1}) + Q_{p+1} \frac{z_{11}^{[p+1]+} a_{11}^p(k_{p+1})^3}{-4k_{p+1}^2} \right)^{-1} \frac{2k_{p+1}}{z_{11}^{[p+1]+} a_{11}^p(k_{p+1})} Z^+_{p+1} = \\ = \frac{1}{a_{11}^p(k_{p+1})^2} a_{11}^p(k_{p+1}) \left(z_{11}^{[p+1]+}\right)^2 a_{11}^p(k_{p+1})^3 \frac{1}{z_{11}^{[p+1]+} a_{11}^p(k_{p+1})} \frac{1}{z_{11}^{[p+1]+} a_{11}^p(k_{p+1})} Z^+_{p+1} = \\ = Z^+_{p+1},$$

and in the Case II) Lemma 13 is proved too.

Case III). $Z^+_{p+1}(t) = \begin{pmatrix} z_{11}^{[p+1]+} & z_{12}^{[p+1]+}(t) \\ 0 & z_{22}^{[p+1]+} \end{pmatrix}$, where $z_{11}^{[p+1]+} > 0$, $z_{22}^{[p+1]+} > 0$, then, similarly to the Case I), we obtain

$$A(k) = \gamma(k) A_p(k), \qquad B(k) = \gamma(-k) B_p(k), \qquad k \in \mathbb{R}, \tag{4.51}$$

with

$$\gamma(k) = \frac{k - k_{p+1}}{k + k_{p+1}} I + i \left\{ \frac{a_{11}^p(k_{p+1})}{(k + k_{p+1}) z_{22}^{[p+1]+} a_{22}^p(k_{p+1})} + \right. \\ \left. + \frac{a_{22}^p(k_{p+1})}{(k + k_{p+1}) z_{11}^{[p+1]+} a_{11}^p(k_{p+1})} - \frac{2k_{p+1} a_{11}^p(k_{p+1})}{(k + k_{p+1})^2 z_{11}^{[p+1]+} a_{11}^p(k_{p+1})} \right\} (Z^+_{p+1})'(0). \tag{4.52}$$

The eigenvalues k_j^2, $(j = \overline{1,p})$, of the equations (4.42) and (4.25) coincide, and (4.42) has one more eigenvalue k_{p+1}^2 by (4.51) and (4.52).

(3.7) and (2.12) imply that the reflection coefficients for the equations (4.42) and (4.25) coincide $R^+_{p+1}(k) = R^+(k)$, and

$$C(k) = C_p(k) \gamma^{-1}(-k). \tag{4.53}$$

The coincidence of the p normalizing polynomials of (4.42) and (4.25) can be deduced similarly to Case I).

In this context

$$Z^{-}_{j<p+1>}(t) = -i\left\{\dot{W}^{-}(k_j)A^{p<k_j>}_{-2}\gamma^{-1}(k_j) + W^{-}(k_j)A^{p<k_j>}_{-2}\frac{d}{dk}(\gamma^{-1}(k))_{k_j} + \right.$$
$$\left. + W^{-}(k_j)A^{p<k_j>}_{-1}\gamma^{-1}(k_j)\right\} - tW^{-}(k_j)A^{p<k_j>}_{-2}\gamma^{-1}(k_j). \quad (4.54)$$

Using $k_j < k_{p+1}$, $(j = \overline{1,p})$, as $x \to -\infty$, we get

$$Z^{-}_{j<p+1>}(t) = \gamma(-k_j)Z^{-}_j(t)\gamma^{-1}(k_j) - \frac{4ik_{p+1}(k_j+k_{p+1})}{(k_j-k_{p+1})^3}(Z^{-}_j)'(0).$$

Thus, (3.13) with $\tau = t$, (4.51) and (4.52) imply

$$Z^{+}_{j<p+1>}(t) = -\left[A^{p<k_j>}_{-1}\gamma^{-1}(k_j) + A^{p<k_j>}_{-2}\frac{d}{dk}(\gamma^{-1}(k))_{k_j}\right]\cdot$$
$$\cdot\gamma(k_j)\left\{(Z^{-}_j(t) + \gamma(k_j)Q_j\gamma^{-1}(-k_j))^{-1} + \right.$$
$$\left. + \frac{4ik_{p+1}(k_j+k_{p+1})(Z^{-}_j)'(0)}{(k_j-k_{p+1})^3\left(z^{[j]-}_{11} + \left(\frac{k_j-k_{p+1}}{k_j+k_{p+1}}\right)^2 q^{[j]}_{11}\right)\left(z^{[j]-}_{22} + \left(\frac{k_j-k_{p+1}}{k_j+k_{p+1}}\right)^2 q^{[j]}_{22}\right)}\left(\frac{k_j-k_{p+1}}{k_j+k_{p+1}}\right)^2\right\}\cdot$$
$$\cdot\gamma^{-1}(-k_j)\left[\gamma(-k_j)C^{p<k_j>}_{-1} - \dot{\gamma}(-k_j)C^{p<k_j>}_{-2}\right] =$$
$$= Z^{+}_j(t) + A^{p<k_j>}_{-1}(Z^{-}_j(t) + \gamma(k_j)Q_j\gamma^{-1}(-k_j))^{-1}\gamma^{-1}(-k_j)\dot{\gamma}(-k_j)C^{p<k_j>}_{-2} -$$
$$- A^{p<k_j>}_{-2}\frac{d}{dk}(\gamma^{-1}(k))_{k_j}\gamma(k_j)\left(Z^{-}_j(t) + \gamma(k_j)Q_j\gamma^{-1}(-k_j)\right)^{-1}C^{p<k_j>}_{-1} -$$
$$- \frac{4ik_{p+1}A^{p<k_j>}_{-1}(Z^{-}_j)'(0)C^{p<k_j>}_{-1}}{(k_j^2-k_{p+1}^2)\left(z^{[j]-}_{11} + \left(\frac{k_j-k_{p+1}}{k_j+k_{p+1}}\right)^2 q^{[j]}_{11}\right)\left(z^{[j]-}_{22} + \left(\frac{k_j-k_{p+1}}{k_j+k_{p+1}}\right)^2 q^{[j]}_{22}\right)} = Z^{+}_j(t),$$

in view of (3.3) and (3.5) and $Q_j = 0$, $j = \overline{1,p}$, with the definitions of Q_j of Lemma 3 being taken into account, as $z^{[j]-}_{11} > 0$ and $z^{[j]-}_{22} > 0$. On the other hand, if either $z^{[j]-}_{11} = 0$ or $z^{[j]-}_{22} = 0$, then by assumption 5) of Theorem 1 $(Z^{-}_j)'(0) = 0$, $A^{p<k_j>}_{-2} = (A_p(k)^{-1}(k-k_j)^2)_{k_j} = 0$; $C^{p<k_j>}_{-2} = (C_p(k)^{-1}(k-k_j)^2)_{k_j} = 0$, hence $Z^{+}_{j<p+1>}(t) = Z^{+}_j(t)$, $j = \overline{1,p}$.

Prove that $Z^{+}_{p+1}(t)$ is a normalizing polynomial of the problem (4.42), (1.2) with $V(x) = V_{p+1}(x)$.

Using (4.51) and (4.52) similarly to (4.54), one has

$$Z^-_{p+1<p+1>}(t) =$$

$$= -2ik_{p+1}W^-(k_{p+1})A^-_p(k_{p+1}) - \frac{2k_{p+1}}{z_{22}^{[p+1]+}a_{22}^p(k_{p+1})}\dot{W}^-(k_{p+1})(Z^+_{p+1})'(0) -$$

$$-\left\{\frac{1}{z_{22}^{[p+1]+}a_{22}^p(k_{p+1})} + \frac{a_{22}(k_{p+1})}{z_{11}^{[p+1]+}a_{11}^p(k_{p+1})^2} - \right.$$

$$\left. -\frac{2k_{p+1}\dot{a}_{11}^p(k_{p+1})}{a_{11}^p(k_{p+1})z_{22}^{[p+1]+}a_{22}^p(k_{p+1})}\right\} W^-(k_{p+1})(Z^+_{p+1})'(0)+$$

$$+ \frac{2ik_{p+1}t}{z_{22}^{[p+1]+}a_{22}^p(k_{p+1})}W^-(k_{p+1})(Z^+_{p+1})'(0).$$

It is easy to show using the asymptotics as $x \to -\infty$,
$E^p_+(x, k_{p+1}) \sim e^{ik_{p+1}x}A_p(k_{p+1})$; $\widetilde{E}^p_+(x, k_{p+1}) \sim e^{ik_{p+1}x}C_p(k_{p+1})$,
that with $x \to -\infty$

$$Z^-_{p+1<p+1>}(t) = (2ik_{p+1})^2[A_p(k_{p+1})Z^+_{p+1}(t)C_p(k_{p+1})]^{-1} +$$

$$+ 2ik_{p+1}\left[\frac{1}{z_{11}^{[p+1]+}z_{22}^{[p+1]+}a_{11}^p(k_{p+1})a_{22}^p(k_{p+1})}\left(2 - 2k_{p+1}\left(\frac{\dot{a}_{22}^p(k_{p+1})}{a_{22}^p(k_{p+1})} + \frac{\dot{a}_{11}^p(k_{p+1})}{a_{11}^p(k_{p+1})}\right)\right) + \right.$$

$$\left. + \frac{a_{11}^p(k_{p+1})}{\left(z_{22}^{[p+1]+}\right)^2 a_{22}^p(k_{p+1})^3} + \frac{a_{22}^p(k_{p+1})}{\left(z_{11}^{[p+1]+}\right)^2 a_{11}^p(k_{p+1})^3}\right](Z^+_{p+1})'(0). \quad (4.55)$$

(3.13) with $\tau = t$ implies

$$Z^+_{p+1<p+1>}(t) = -A_{-1}^{<k_{p+1}>}(Z^-_{p+1<p+1>}(t))^{-1}C_{-1}^{<k_{p+1}>}. \quad (4.56)$$

Use (4.51) – (4.53) and the definition to get

$$A_{-1}^{<k_{p+1}>} = \frac{d}{dk}(A^{-1}(k)(k-k_{p+1})^2)_{k_{p+1}} = 2k_{p+1}A_p^{-1}(k_{p+1}) - i\left[\frac{1}{z_{22}^{[p+1]+}a_{22}^p(k_{p+1})} + \right.$$

$$\left. + \frac{a_{22}(k_{p+1})}{z_{11}^{[p+1]+}a_{11}^p(k_{p+1})^2} - \frac{2k_{p+1}\dot{a}_{11}^p(k_{p+1})}{z_{22}^{[p+1]+}a_{22}^p(k_{p+1})a_{11}^0(k_{p+1})}\right](Z^+_{p+1})'(0), \quad (4.57)$$

and in a similar way

$$C_{-1}^{<k_{p+1}>} = 2k_{p+1}C_p^{-1}(k_{p+1}) - i\left[\frac{1}{z_{11}^{[p+1]+}a_{11}^p(k_{p+1})} + \right.$$

$$\left. + \frac{a_{11}(k_{p+1})}{z_{22}^{[p+1]+}a_{22}^p(k_{p+1})^2} - \frac{2k_{p+1}\dot{a}_{11}^p(k_{p+1})}{z_{11}^{[p+1]+}a_{11}^p(k_{p+1})a_{22}^p(k_{p+1})}\right](Z^+_{p+1})'(0). \quad (4.58)$$

Use (4.55), (4.57), and (4.58) to deduce from (4.56), after some obvious computations, that $Z^+_{p+1<p+1>}(t) = Z^+_{p+1}(t)$, and Lemma 13 is proved in Case III) too and therefore it is proved completely.

It remains to notice that the scattering problems constructed above do not have virtual levels since $R^+(0) = R^-(0) = -I$ for them.

The sufficiency of the conditions of Theorem 1 is proved in version 4a). ◄

4.4. Eliminating the discrete spectrum and completing the proof of Theorem 1

Statement 1. *It is established that conditions 1) – 6) of Theorem 1 in the version 4) are necessary (see Subsection 4.1), and in the version 4a) are sufficient (see subsections 4.2 and 4.3).*

Lemma 14. *Statement 1 implies that condition 4a) is necessary.*

We sketch here a *proof* of Lemma 14. This will allow us to claim that Theorem 1 is proved completely in both versions, that is, either with condition 4) only or with condition 4a) only. This is because if one proves that conditions 1) – 6) in the version 4) imply conditions 1) – 6) in the version 4a), this also proves that conditions 1) – 6) in the version 4) are sufficient, while the fact that they are necessary has already been established. It should be noted here that the conditions of Theorem 1 in the version 4a) has been shown to be sufficient earlier in subsections 4.2, 4.3.

So let (2.16) be an SD for the problem (1.1), (1.2) in question, hence it satisfies condition 4) of Theorem 1. Prove by discarding a single eigenvalue (either simple or multiple) that the values

$$\left\{R^+(k),\ k \in \mathbb{R};\ k_j^2 < 0, \quad Z_j^+(t),\ j = 1, \dots, p-1 < \infty\right\}, \tag{4.59}$$

also satisfy condition 4) of Theorem 1, along with conditions 1) – 3) and 5), 6), which are obviously valid. This procedure is to be repeated p times to prove that condition 4a) is necessary for SD (2.16).

Start with considering the diagonal elements of the values (4.59). To make our notation less cumbersome, we omit the indices ll of the diagonal elements. Prove that the functions of the form

$$f_R^{[p-1]-}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r_p^-(k)\left(\frac{k-k_p}{k+k_p}\right)^2 e^{-ikx}dk =$$

$$= f_R^{[p]-}(x) - 4\mu_p^2\int_{-\infty}^{x} f_R^{[p]-}(t)(x-t)e^{-\mu_p(x-t)}dt + 4\mu_p\int_{-\infty}^{x} f_R^{[p]-}(t)e^{-\mu_p(x-t)}dt,$$

with

$$\mu_j \equiv -ik_j > 0, \quad r_j^-(k) \equiv -\frac{r^+(-k)a_j(-k)}{a_j(k)}, \quad f_R^{[j]-}(x) \equiv \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} r_j^-(k)e^{-ikx}dk,$$

$j = 0, \ldots, p$, derived from the diagonal elements from (2.16), satisfy condition 4) of Theorem 1. Obviously, $f_R^{[p-1]-}(x)$ is absolutely continuous.

Prove that $\frac{d}{dx}f_R^{[p-1]-}(x)$ satisfies (4.5). An easy computation, which includes integration in parts and changing an integration order in some multiple integrals, yields

$$\int\limits_{-\infty}^{a}(1+x^2)\left|\frac{d}{dx}f_R^{[p-1]-}(x)\right|dx \leq$$

$$\leq 9\int\limits_{-\infty}^{a}(1+x^2)\left|\frac{d}{dx}f_R^{[p]-}(x)\right|dx + \frac{24}{\mu_p}\int\limits_{-\infty}^{a} t\left|\frac{d}{dt}f_R^{[p]-}(t)\right|dt + \frac{32}{\mu_p^2}\int\limits_{-\infty}^{a}\left|\frac{d}{dt}f_R^{[p]-}(t)\right|dt -$$

$$-4\left(\mu_p + \mu_p a^2 + 2a + \frac{2}{\mu_p}\right)e^{-\mu_p a}\int\limits_{-\infty}^{a} t\left|\frac{d}{dt}f_R^{[p]-}(t)\right|e^{\mu_p t}dt -$$

$$-\frac{4}{\mu_p^2}\left(\mu_p^3 a(1+a^2) + 4(a\mu_p+1)^2 + 2\mu_p^2 + 4\right)e^{-\mu_p a}\int\limits_{-\infty}^{a}\left|\frac{d}{dt}f_R^{[p]-}(t)\right|e^{\mu_p t}dt.$$

On the other hand, since condition 4) of Theorem 1 holds for $f_R^{[p]-}(x)$, one should have a condition of the form (4.5) for $\frac{d}{dx}f_R^{[p-1]-}(x)$, hence $f_R^{[p-1]-}(x)$ satisfies condition 4) of Theorem 1.

Now consider a non-diagonal element of the values (4.59). Prove that such an element satisfies condition 4) of Theorem 1.

One could encounter here three possibilities, depending on the specific form of the matrix polynomial $Z_p^+(t)$ to be discarded.

Case I) Let $s_p^1 = \text{sign } z_{11}^{[p]+} = 1$, $s_p^2 = \text{sign } z_{22}^{[p]+} = 0$, that is $Z_p^+(t) = \begin{pmatrix} z_{11}^{[p]+} & z_{12}^{[p]+}(t) \\ 0 & 0 \end{pmatrix}$.

Then

$$c_{11}^p(k) = c_{11}^{p-1}(k)\frac{k-k_p}{k+k_p}, \qquad c_{22}^p(k) = c_{22}^{p-1}(k), \qquad c_{12}^p(k) = c_{12}^{p-1}(k) + \frac{P_{\kappa-1}(k)}{Q_\kappa(k)}c_{11}^{p-1}(k),$$

where $\kappa = \sum\limits_{j=1}^{p}\kappa_j = \sum\limits_{j=1}^{p}\left(\text{sign } z_{11}^{[j]+} + \text{sign } z_{22}^{[j]+}\right)$,

$$P_{\kappa-1}(k) = \psi^+(0)\prod_{j=1}^{p-1}k_j^{\kappa_j} + a_1^p - a_1k_p +$$

$$+ k(a_2^p - a_1 - a_2k_p) + \ldots + k^{\kappa-2}(a_{\kappa-1}^p - a_{\kappa-2} - a_{\kappa-1}k_p) + k^{\kappa-1}(a_\kappa^p - a_{\kappa-1}),$$

$$\deg P_{\kappa-1}(k) \le \kappa - 1; \qquad Q_\kappa(k) = (k+k_p)\prod_{j=1}^{p-1}(k+k_j)^{s_j^2}(k-k_j)^{s_j^1}, \qquad \deg Q_\kappa(k) = \kappa.$$

Here $\kappa = \kappa(p)$.

Thus we have

$$r_{12}^{[p-1]-}(k) = \frac{k-k_p}{k+k_p} r_{12}^{[p]-}(k) - r_{11}^{[p]-}(k)\frac{P_{\kappa-1}(-k)}{Q_\kappa(-k)}\left(\frac{k-k_p}{k+k_p}\right)^2 + r_{22}^{[p]-}(k)\frac{P_{\kappa-1}(k)}{Q_\kappa(k)}.$$

Therefore one can use the specific form of the Fourier transform of a ratio of polynomials (see Lemma 8) to deduce that the function

$$f_{12}^{[p-1]-}(x) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} r_{12}^{[p-1]-}(k)e^{-ikx}dk =$$

$$= f_{12}^{[p]-}(x) + 2\mu_p e^{-\mu_p x}\int_{-\infty}^{x} f_{12}^{[p]-}(t)e^{\mu_p t}dt - \widetilde{\alpha}_p e^{-\mu_p x}\int_{-\infty}^{x} f_{11}^{[p]-}(t)(x-t)e^{\mu_p t}dt -$$

$$- \sum_{j=1}^{p} e^{-\mu_j x}\int_{-\infty}^{x}\left\{s_j^1\alpha_j f_{11}^{[p]-}(t) - s_j^2\gamma_j f_{22}^{[p]-}(t)\right\}e^{\mu_j t}dt -$$

$$- \sum_{j=1}^{p-1} e^{\mu_j x}\int_{x}^{+\infty}\left\{s_j^2\beta_j f_{11}^{[p]-}(t) - s_j^1\delta_j f_{22}^{[p]-}(t)\right\}e^{-\mu_j t}dt,$$

is absolutely continuous. Here α_j, β_j, γ_j, δ_j are constants, with $\widetilde{\alpha_p} \neq 0$, $\gamma_p \neq 0$. Assume $\beta_p = \delta_p = 0$ to deduce that for all $a < \infty$,

$$\int_{-\infty}^{a}(1+x^2)\left|\frac{d}{dx}f_{12}^{[p-1]-}(x)\right|dx \le$$

$$\le 3\int_{-\infty}^{a}(1+x^2)\left|\frac{d}{dx}f_{12}^{[p]-}(x)\right|dx + \frac{4}{\mu_p}\int_{-\infty}^{a} t\left|\frac{d}{dt}f_{12}^{[p]-}(t)\right|dt + \frac{4}{\mu_p^2}\int_{-\infty}^{a}\left|\frac{d}{dt}f_{12}^{[p]-}(t)\right|dt +$$

$$+ \int_{-\infty}^{a}(1+t^2)\left\{\left(\sum_{j=1}^{p}\frac{s_j^1|\alpha_j| + s_j^2|\beta_j|}{\mu_j} + \frac{|\widetilde{\alpha}_p|}{\mu_p^2}\right)\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right| + \right.$$

$$\left. + \sum_{j=1}^{p}\frac{s_j^2|\gamma_j| + s_j^1|\delta_j|}{\mu_j}\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right\}dt +$$

$$+ 2\int\limits_{-\infty}^{a} t\left\{\left(\sum_{j=1}^{p}\frac{s_j^1|\alpha_j| - s_j^2|\beta_j|}{\mu_j^2} + \frac{2|\widetilde{\alpha}_p|}{\mu_p^3}\right)\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right| + \right.$$

$$\left. + \sum_{j=1}^{p}\frac{s_j^2|\gamma_j| - s_j^1|\delta_j|}{\mu_j^2}\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right\} dt +$$

$$+ 2\int\limits_{-\infty}^{a}\left\{\left(\sum_{j=1}^{p}\frac{s_j^1|\alpha_j| + s_j^2|\beta_j|}{\mu_j^3} + \frac{3|\widetilde{\alpha}_p|}{\mu_p^4}\right)\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right| + \right.$$

$$\left. + \sum_{j=1}^{p}\frac{s_j^2|\gamma_j| + s_j^1|\delta_j|}{\mu_j^3}\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right\} dt + M_p(a) < \infty,$$

where

$$M_p(a) \equiv -2e^{-\mu_p a}[1 + a^2 + \frac{2a}{\mu_p} + \frac{2}{\mu_p^2}]\int\limits_{-\infty}^{a}|\frac{d}{dt}f_{12}^{[p]-}(t)|e^{\mu_p t}dt -$$

$$- \sum_{j=1}^{p}\frac{e^{-\mu_j a}}{\mu_j}\left[1 + a^2 + \frac{2a}{\mu_p} + \frac{2}{\mu_p^2}\right]\int\limits_{-\infty}^{a}\left(s_j^1|\alpha_j|\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right| + s_j^2|\gamma_j|\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right)e^{\mu_j t}dt +$$

$$+ \sum_{j=1}^{p-1}\frac{e^{\mu_j a}}{\mu_j}\left[1 + a^2 - \frac{2a}{\mu_p} + \frac{2}{\mu_p^2}\right]\int\limits_{a}^{+\infty}\left(s_j^2|\beta_j|\cdot\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right| + s_j^1|\delta_j|\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right)e^{-\mu_j t}dt -$$

$$- |\widetilde{\alpha}_p|\frac{e^{-\mu_p a}}{\mu_p}\left[(1 + a^2)a - \frac{3a^2 + 1}{\mu_p} + \frac{6a}{\mu_p^2} + \frac{6}{\mu_p^3}\right]\int\limits_{-\infty}^{a}\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right|e^{\mu_p t}dt +$$

$$+ |\widetilde{\alpha}_p|\frac{e^{-\mu_p a}}{\mu_p}\left[1 + a^2 - \frac{2a}{\mu_p} + \frac{2}{\mu_p^2}\right]\int\limits_{-\infty}^{a} t\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right|e^{\mu_p t}dt < \infty.$$

Case II) Now let $s_p^1 = 0$, $s_p^2 = 1$, that is $Z_p^+(t) = \begin{pmatrix} 0 & z_{12}^{[p]+}(t) \\ 0 & z_{22}^{[p]+} \end{pmatrix}$. Then

$$c_{11}^p(k) = c_{11}^{p-1}(k), \quad c_{22}^p(k) = c_{22}^{p-1}(k)\frac{k - k_p}{k + k_p}, \quad c_{12}^p(k) = c_{12}^{p-1}(k)\frac{k - k_p}{k + k_p} + \frac{P_{\kappa-1}(k)}{Q_\kappa(k)}c_{11}^{p-1}(k),$$

where

$$P_{\kappa-1}(k) = \psi^+(0)\prod_{j=1}^{p-1}k_j^{\kappa_j} + a_1^p + a_1 k_p + k(a_2^p - a_1 + a_2 k_p) + \ldots + k^{\kappa-1}(a_\kappa^p - a_{\kappa-1}) +$$

$$+\frac{2k_p\psi^-(-k_p)}{k}\left[\prod_{j=1}^{p-1}(k+k_j)^{\kappa_j}-\prod_{j=1}^{p-1}k_j^{\kappa_j}\right],$$

$$\deg P_{\kappa-1}(k)\le\kappa-1;\qquad Q_\kappa(k)=\prod_{j=1}^{p}(k+k_j)^{s_j^2}(k-k_j)^{s_j^1},\qquad \deg Q_\kappa(k)=\kappa.$$

This implies

$$r_{12}^{[p-1]-}(k)=\frac{k-k_p}{k+k_p}r_{12}^{[p]-}(k)-r_{11}^{[p]-}(k)\frac{P_{\kappa-1}(-k)}{Q_\kappa(-k)}\frac{k-k_p}{k+k_p}+r_{22}^{[p]-}(k)\frac{P_{\kappa-1}(k)}{Q_\kappa(k)}\frac{k-k_p}{k+k_p}.$$

Use again Lemma 8 cited above to deduce that the function

$$f_{12}^{[p-1]-}(x)\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}r_{12}^{[p-1]-}(k)e^{-ikx}dk=f_{12}^{[p]-}(x)+2\mu_pe^{-\mu_px}\int_{-\infty}^{x}f_{12}^{[p]-}(t)e^{\mu_pt}dt-$$
$$-\sum_{j=1}^{p}e^{-\mu_jx}\int_{-\infty}^{x}\left\{s_j^1\alpha_jf_{11}^{[p]-}(t)-s_j^2\gamma_jf_{22}^{[p]-}(t)\right\}e^{\mu_jt}dt-$$
$$-\sum_{j=1}^{p-1}e^{\mu_jx}\int_{x}^{+\infty}\left\{s_j^2\beta_jf_{11}^{[p]-}(t)-s_j^1\delta_jf_{22}^{[p]-}(t)\right\}e^{-\mu_jt}dt+\widetilde{\gamma}_pe^{-\mu_px}\int_{-\infty}^{x}f_{22}^{[p]-}(t)(x-t)e^{\mu_pt}dt,$$

is absolutely continuous. Here α_j, β_j, γ_j, δ_j are constants with $\alpha_p\ne0$, $\widetilde{\gamma}_p\ne0$. Again set $\beta_p=\delta_p=0$ to get for all $a<\infty$

$$\int_{-\infty}^{a}(1+x^2)\left|\frac{d}{dx}f_{12}^{[p-1]-}(x)\right|dx\le$$
$$\le3\int_{-\infty}^{a}(1+x^2)\left|\frac{d}{dx}f_{12}^{[p]-}(x)\right|dx+\frac{4}{\mu_p}\int_{-\infty}^{a}t\left|\frac{d}{dt}f_{12}^{[p]-}(t)\right|dt+\frac{4}{\mu_p^2}\int_{-\infty}^{a}\left|\frac{d}{dt}f_{12}^{[p]-}(t)\right|dt+$$
$$+\int_{-\infty}^{a}(1+t^2)\left\{\sum_{j=1}^{p}\frac{s_j^1|\alpha_j|+s_j^2|\beta_j|}{\mu_j}\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right|+\right.$$
$$\left.+\left(\sum_{j=1}^{p}\frac{s_j^2|\gamma_j|+s_j^1|\delta_j|}{\mu_j}+\frac{|\widetilde{\gamma}_p|}{\mu_p^2}\right)\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right\}dt+$$
$$+2\int_{-\infty}^{a}t\left\{\sum_{j=1}^{p}\frac{s_j^1|\alpha_j|-s_j^2|\beta_j|}{\mu_j^2}\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right|+\right.$$

$$+\left(\sum_{j=1}^{p}\frac{s_j^2|\gamma_j|-s_j^1|\delta_j|}{\mu_j^2}-\frac{2|\widetilde{\gamma}_p|}{\mu_p^3}\right)\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\Bigg\}dt+$$

$$+2\int\limits_{-\infty}^{a}\left\{\sum_{j=1}^{p}\frac{s_j^1|\alpha_j|+s_j^2|\beta_j|}{\mu_j^3}\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right|+\right.$$

$$\left.+\left(\sum_{j=1}^{p}\frac{s_j^2|\gamma_j|+s_j^1|\delta_j|}{\mu_j^3}+\frac{3|\widetilde{\gamma}_p|}{\mu_p^4}\right)\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right\}dt+M_p(a)<\infty,$$

where

$$M_p(a)\equiv -2e^{-\mu_p a}\left[1+a^2+\frac{2a}{\mu_p}+\frac{2}{\mu_p^2}\right]\int\limits_{-\infty}^{a}\left|\frac{d}{dt}f_{12}^{[p]-}(t)\right|e^{\mu_p t}dt-$$

$$-\sum_{j=1}^{p}\frac{e^{-\mu_j a}}{\mu_j}\left[1+a^2+\frac{2a}{\mu_p}+\frac{2}{\mu_p^2}\right]\int\limits_{-\infty}^{a}\left(s_j^1|\alpha_j|\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right|+s_j^2|\gamma_j|\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right)e^{\mu_j t}dt+$$

$$+\sum_{j=1}^{p-1}\frac{e^{\mu_j a}}{\mu_j}\left[1+a^2-\frac{2a}{\mu_p}+\frac{2}{\mu_p^2}\right]\int\limits_{a}^{+\infty}\left(s_j^2|\beta_j|\cdot\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right|+s_j^1|\delta_j|\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right)e^{-\mu_j t}dt-$$

$$-|\widetilde{\gamma}_p|\frac{e^{-\mu_p a}}{\mu_p}\left[(1+a^2)a-\frac{3a^2+1}{\mu_p}+\frac{6a}{\mu_p^2}+\frac{6}{\mu_p^3}\right]\int\limits_{-\infty}^{a}\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|e^{\mu_p t}dt+$$

$$+|\widetilde{\gamma}_p|\frac{e^{-\mu_p a}}{\mu_p}\left[1+a^2-\frac{2a}{\mu_p}+\frac{2}{\mu_p^2}\right]\int\limits_{-\infty}^{a}t\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|e^{\mu_p t}dt<\infty.$$

Case III) Finally, let now $s_p^1=1$, $s_p^2=1$, that is $Z_p^+(t)=\begin{pmatrix} z_{11}^{[p]+} & z_{12}^{[p]+}(t)\\ 0 & z_{22}^{[p]+}\end{pmatrix}$. Then

$$c_{11}^p(k)=c_{11}^{p-1}(k)\frac{k-k_p}{k+k_p},\qquad c_{22}^p(k)=c_{22}^{p-1}(k)\frac{k-k_p}{k+k_p},$$

$$c_{12}^p(k)=c_{12}^{p-1}(k)\frac{k-k_p}{k+k_p}+\frac{P_{\kappa-1}(k)}{Q_\kappa(k)}c_{11}^{p-1}(k),$$

where $\kappa=\sum\limits_{j=1}^{p}\kappa_j=\sum\limits_{j=1}^{p}\left(\text{sign } z_{11}^{[j]+}+\text{sign } z_{22}^{[j]+}\right)$,

$$P_{\kappa-1}(k)=a_1^p+a_1k_p^2+k\left(\psi^+(0)\prod_{j=1}^{p-1}k_j^{\kappa_j}+a_2^p+a_2k_p^2\right)+\ldots+k^{\kappa-1}(a_\kappa^p-a_{\kappa-2})+$$

$$+ 2k_p\psi^-(-k_p)\prod_{j=1}^{p-1}(k+k_j)^{\kappa_j} + \frac{2k_p^2\psi^-(-k_p)}{k}\left[\prod_{j=1}^{p-1}(k+k_j)^{\kappa_j} - \prod_{j=1}^{p-1}k_j^{\kappa_j}\right],$$

$$\deg P_{\kappa-1}(k) \le \kappa-1; \qquad Q_\kappa(k) = (k+k_p)^2\prod_{j=1}^{p-1}(k+k_j)^{s_j^2}(k-k_j)^{s_j^1}, \qquad \deg Q_\kappa(k) = \kappa.$$

Thus we obtain

$$r_{12}^{[p-1]-}(k) = r_{12}^{[p]-}(k) - \frac{4k_p}{k+k_p}r_{12}^{[p]-}(k) + \frac{4k_p^2}{(k+k_p)^2}r_{12}^{[p]-}(k) -$$

$$- r_{11}^{[p]-}(k)\frac{R_\kappa(k)}{(k+k_p)^3\prod\limits_{j=1}^{p-1}(k-k_j)^{s_j^2}(k+k_j)^{s_j^1}} + r_{22}^{[p]-}(k)\frac{T_\kappa(k)}{(k+k_p)^3\prod\limits_{j=1}^{p-1}(k+k_j)^{s_j^2}(k-k_j)^{s_j^1}},$$

where

$$R_\kappa(k) \equiv (-1)^\kappa P_{\kappa-1}(-k)(k-k_p), \qquad \deg R_\kappa(k) \le \kappa;$$
$$T_\kappa(k) \equiv P_{\kappa-1}(k)(k-k_p), \qquad \deg T_\kappa(k) \le \kappa.$$

Another application of Lemma 8 allows us to conclude that the function

$$f_{12}^{[p-1]-}(x) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} r_{12}^{[p-1]-}(k)e^{-ikx}dk =$$

$$= f_{12}^{[p]-}(x) + 4\mu_p e^{-\mu_p x}\int_{-\infty}^{x} f_{12}^{[p]-}(t)e^{\mu_p t}dt - 4\mu_p^2 e^{-\mu_p x}\int_{-\infty}^{x} f_{12}^{[p]-}(t)(x-t)e^{\mu_p t}dt -$$

$$- \sum_{j=1}^{p} e^{-\mu_j x}\int_{-\infty}^{x}\left\{s_j^1\alpha_j f_{11}^{[p]-}(t) - s_j^2\gamma_j f_{22}^{[p]-}(t)\right\}e^{\mu_j t}dt -$$

$$- \sum_{j=1}^{p-1} e^{\mu_j x}\int_{x}^{+\infty}\left\{s_j^2\beta_j f_{11}^{[p]-}(t) - s_j^1\delta_j f_{22}^{[p]-}(t)\right\}e^{-\mu_j t}dt -$$

$$- e^{-\mu_p x}\int_{-\infty}^{x}\left(\alpha_p^{[1]} f_{11}^{[p]-}(t) - \gamma_p^{[1]} f_{22}^{[p]-}(t)\right)(x-t)e^{\mu_p t}dt -$$

$$- e^{-\mu_p x}\int_{-\infty}^{x}\left(\alpha_p^{[2]} f_{11}^{[p]-}(t) - \gamma_p^{[2]} f_{22}^{[p]-}(t)\right)(x-t)^2 e^{\mu_p t}dt,$$

is absolutely continuous. Here α_j, β_j, γ_j, δ_j are constants with $\alpha_p^{[l]} \ne 0 \ne \gamma_p^{[l]}$, $l = 1, 2$. Set $\beta_p = \delta_p = 0$ to deduce that for all $a < \infty$ one has

$$\int_{-\infty}^{a}(1+x^2)\left|\frac{d}{dx}f_{12}^{[p-1]-}(x)\right|dx \le$$

$$\le 9\int_{-\infty}^{a}(1+x^2)\left|\frac{d}{dx}f_{12}^{[p]-}(x)\right|dx+\frac{24}{\mu_p}\int_{-\infty}^{a}t\left|\frac{d}{dt}f_{12}^{[p]-}(t)\right|dt+\frac{32}{\mu_p^2}\int_{-\infty}^{a}\left|\frac{d}{dt}f_{12}^{[p]-}(t)\right|dt+$$

$$+\int_{-\infty}^{a}(1+t^2)\left\{\left(\sum_{j=1}^{p}\frac{s_j^1|\alpha_j|+s_j^2|\beta_j|}{\mu_j}+\frac{|\alpha_p^{[1]}|}{\mu_p^2}+\frac{2|\alpha_p^{[2]}|}{\mu_p^3}\right)\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right|+\right.$$

$$\left.+\left(\sum_{j=1}^{p}\frac{s_j^2|\gamma_j|+s_j^1|\delta_j|}{\mu_j}+\frac{|\gamma_p^{[1]}|}{\mu_p^2}+\frac{2|\gamma_p^{[2]}|}{\mu_p^3}\right)\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right\}dt+$$

$$+2\int_{-\infty}^{a}t\left\{\left(\sum_{j=1}^{p}\frac{s_j^1|\alpha_j|-s_j^2|\beta_j|}{\mu_j^2}+\frac{2|\alpha_p^{[1]}|}{\mu_p^3}+\frac{6|\alpha_p^{[2]}|}{\mu_p^4}\right)\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right|+\right.$$

$$\left.+\left(\sum_{j=1}^{p}\frac{s_j^2|\gamma_j|-s_j^1|\delta_j|}{\mu_j^2}+\frac{2|\gamma_p^{[1]}|}{\mu_p^3}+\frac{6|\gamma_p^{[2]}|}{\mu_p^4}\right)\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right\}dt+$$

$$+2\int_{-\infty}^{a}\left\{\left(\sum_{j=1}^{p}\frac{s_j^1|\alpha_j|+s_j^2|\beta_j|}{\mu_j^3}+\frac{3|\alpha_p^{[1]}|}{\mu_p^4}+\frac{12|\alpha_p^{[2]}|}{\mu_p^5}\right)\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right|+\right.$$

$$\left.+\left(\sum_{j=1}^{p}\frac{s_j^2|\gamma_j|+s_j^1|\delta_j|}{\mu_j^3}+\frac{3|\gamma_p^{[1]}|}{\mu_p^4}+\frac{12|\gamma_p^{[2]}|}{\mu_p^5}\right)\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right\}dt+M_p(a)<\infty,$$

where

$$M_p(a)\equiv -4e^{-\mu_p a}\left[\mu_p a(1+a^2)-2a^2+\frac{8a}{\mu_p}+\frac{8}{\mu_p^2}\right]\int_{-\infty}^{a}\left|\frac{d}{dt}f_{12}^{[p]-}(t)\right|e^{\mu_p t}dt+$$

$$+4\mu_p e^{-\mu_p a}\left[1+a^2-\frac{2a}{\mu_p}+\frac{2}{\mu_p^2}\right]\int_{-\infty}^{a}t\left|\frac{d}{dt}f_{12}^{[p]-}(t)\right|e^{\mu_p t}dt-$$

$$-\sum_{j=1}^{p}\frac{e^{-\mu_j a}}{\mu_j}\left[1+a^2+\frac{2a}{\mu_p}+\frac{2}{\mu_p^2}\right]\int_{-\infty}^{a}\left(s_j^1|\alpha_j|\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right|+s_j^2|\gamma_j|\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right)e^{\mu_j t}dt+$$

$$+\sum_{j=1}^{p-1}\frac{e^{\mu_j a}}{\mu_j}\left[1+a^2-\frac{2a}{\mu_p}+\frac{2}{\mu_p^2}\right]\int_{a}^{+\infty}\left(s_j^2|\beta_j|\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right|+s_j^1|\delta_j|\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right)e^{-\mu_j t}dt-$$

$$-\frac{e^{-\mu_p a}}{\mu_p a}\left[(1+a^2)a-\frac{3a^2+1}{\mu_p}+\frac{6a}{\mu_p^2}+\frac{6}{\mu_p^3}\right]\cdot$$

$$\cdot\int\limits_{-\infty}^{a}\left(\left|\alpha_p^{[1]}\right|\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right|+\left|\gamma_p^{[1]}\right|\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right)e^{\mu_p t}dt+$$

$$+\frac{e^{-\mu_p a}}{\mu_p a}\left[1+a^2-\frac{2a}{\mu_p}+\frac{2}{\mu_p^2}\right]\int\limits_{-\infty}^{a}t\left(\left|\alpha_p^{[1]}\right|\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right|+\left|\gamma_p^{[1]}\right|\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right)e^{\mu_p t}dt-$$

$$-\frac{e^{-\mu_p a}}{\mu_p a}\left[(1+a^2)a^2+\frac{2(a^2+a(1+a^2))}{\mu_p}-\frac{2}{\mu_p^2}+\frac{24a}{\mu_p^3}+\frac{24}{\mu_p^4}\right]\cdot$$

$$\cdot\int\limits_{-\infty}^{a}\left(\left|\alpha_p^{[2]}\right|\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right|+\left|\gamma_p^{[2]}\right|\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right)e^{\mu_p t}dt+$$

$$+\frac{e^{-\mu_p a}}{\mu_p a}\left[2a(1+a^2)+\frac{2(3a^2+1)}{\mu_p}+\frac{4a}{\mu_p^2}+\frac{12}{\mu_p^3}\right]\cdot$$

$$\cdot\int\limits_{-\infty}^{a}t\left(\left|\alpha_p^{[2]}\right|\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right|+\left|\gamma_p^{[2]}\right|\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right)e^{\mu_p t}dt-$$

$$-\frac{e^{-\mu_p a}}{\mu_p a}\left[1+a^2+\frac{2a}{\mu_p}+\frac{2}{\mu_p^2}\right]\int\limits_{-\infty}^{a}t^2\left(\left|\alpha_p^{[2]}\right|\left|\frac{d}{dt}f_{11}^{[p]-}(t)\right|+\left|\gamma_p^{[2]}\right|\left|\frac{d}{dt}f_{22}^{[p]-}(t)\right|\right)e^{\mu_p t}dt<\infty.$$

This case completes proving the fact that condition 4a) is necessary for SD (2.16); this also completes the proof of Lemma 14 as well as the proof of Theorem 1 in both versions, namely either with condition 4) or 4a).

5. The cases with virtual level

5.1. The case of multiplicity two VL

Theorem 2. (See [27, Theorem 1]). *The collection of values* (2.16) *is the right SD for the problem* (1.1), (1.2) *with an upper triangular* 2×2 *matrix potential, which is real on the diagonal, has the second moment on the axis and determines a multiplicity two VL if and only if the following conditions 1) – 6) are satisfied:*

1)

$$R^+(k)=O(k^{-1}),\qquad dR^+(k)/dk=o(k^{-1})\qquad as\quad k\to\pm\infty.\tag{5.1}$$

The functions

$$\rho_{jl}(k)\equiv r_{jl}(k)k^{l-j},\qquad 1\le j\le l\le 2,\tag{5.2}$$

are continuously differentiable at all $k\in\mathbb{R}$*;*

$$\overline{r_{ll}^+(k)}=r_{ll}^+(-k),\quad |r_{ll}^+(k)|<1-\frac{C_l k^2}{1+k^2},\qquad where\quad C_l>0,\quad l=1,2.\tag{5.3}$$

2) *Set* $\gamma^+ = \lim_{k\to 0} k r_{12}^+(k) \equiv \rho_{12}(0)$,

$$R^{\gamma+}(k) = R^+(k) - \gamma^+ k^{-1} J, \qquad F_R^{\gamma+}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R^{\gamma+}(k) e^{ikx} dk. \tag{5.4}$$

Then the function

$$F_R^+(x) = F_R^{\gamma+}(x) - i\gamma^+\eta(-x)J, \tag{5.5}$$

is absolutely continuous, and for every $a > -\infty$ *one has*

$$(1+x^2)\left|\frac{d}{dx}F_R^+(x)\right| \in L^1(a,+\infty). \tag{5.6}$$

Here $\eta(x)$ *is the Heaviside function* (2.17). (Note that $F_R^+(x) \neq \frac{1}{2\pi}\int_{-\infty}^{\infty} R^+(k)e^{ikx}dk$ for $\gamma^+ \neq 0$.)

3) *The functions* $c_{ll}^0(z) \equiv a_{ll}^0(z)$, $l = 1, 2$, *given by*

$$c_{ll}^0(z) \equiv a_{ll}^0(z) := e^{-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln\left(1-|r_{ll}^+(k)|^2\right)}{k-z}dk}, \qquad \mathrm{Im}\ z > 0, \tag{5.7}$$

are continuously differentiable in the closed upper half-plane (being defined for z *with* $\mathrm{Im}\ z = 0$ *by continuity).*

4) *Set, in view of* (2.10),

$$R_0^{\gamma-}(k) \equiv -C_0(k)^{-1}R^+(-k)C_0(-k) - \gamma^- k^{-1}J, \tag{5.8}$$

where the diagonal elements $c_{ll}^0(k)$ *of the matrix* $C_0(k)$ *are defined by condition 3 of this theorem,* $c_{21}^0 \equiv 0$, $c_{12}^0(k) \equiv c_{12}^0(k+i0)$ *for* $k \in \mathbb{R}\backslash\{0\}$,

$$zc_{12}^0(z) \equiv \left\{\psi_0^+(z) - \psi_0^+(0) - \frac{\gamma^+}{\sqrt{1-r_{22}^+(0)^2}}\sqrt{\frac{1-r_{11}^+(0)}{1+r_{11}^+(0)}}\right\} a_{11}^0(z), \quad \mathrm{Im}\ z > 0, \tag{5.9}$$

$$\psi_0^\pm(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{h^0(k)}{k-z}dk, \qquad \pm\mathrm{Im}\ z > 0, \tag{5.10}$$

$$h^0(k) = k a_{11}^0(-k) a_{22}^0(k)\{r_{11}^+(-k)r_{12}^+(k) + r_{12}^+(-k)r_{22}^+(k)\}, \tag{5.11}$$

$$\gamma^- = \gamma^+ \prod_{l=1}^{2}\{1 - r_{ll}^+(0)\}^{1/2}\{1 + r_{ll}^+(0)\}^{-1/2}. \tag{5.12}$$

Then the function

$$F_{R_0}^{-}(x) = F_{R_0}^{\gamma-}(x) - i\gamma^{-}\eta(x)J, \qquad \textit{where} \quad F_{R_0}^{\gamma-}(x) \equiv \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} R_0^{\gamma-}(k)e^{-ikx}dk, \quad (5.13)$$

is absolutely continuous, and for every $a < +\infty$ *one has*

$$(1+x^2)\left|\frac{d}{dx}F_{R_0}^{-}(x)\right| \in L^1(-\infty, a). \qquad (5.14)$$

5) $\deg Z_j^{+}(t) \le \sum\limits_{l=1}^{2} \operatorname{sign} z_{ll}^{[j]+} - 1$, $j = \overline{1,p}$, *the elements* $z_{ll}^{[j]+}$ *are non-negative and constant.*

6) $\operatorname{rg} Z_j^{+}(t) = \operatorname{rg} \operatorname{diag} Z_j^{+}(t) = \operatorname{rg} \operatorname{diag} Z_j^{+}(0)$, $j = \overline{1,p}$.

Example 1. Set in (1.1) $V(x) = v(x)J$. In this simplest case it suffices to require that $V(x)$ has only the first moment: $(1+|x|)|V(x)| \in L^1(-\infty,\infty.)$ This problem determines a multiplicity two VL, while a discrete spectrum is absent. It is easy to see that in this case one has

$$K^{\pm}(x,t) = \widetilde{K}^{\pm}(x,t) = \pm\frac{1}{2}J\eta(\pm t \mp x)\int\limits_{\frac{x+t}{2}}^{\pm\infty} v(s)ds,$$

$$E^{\pm}(x,k) = \widetilde{E}^{\pm}(x,k) = e^{\pm ikx}I \pm \int\limits_{x}^{\pm\infty} K^{\pm}(x,t)e^{\pm ikt}dt,$$

$$A(k) = C(k) = I - \frac{\gamma^{\pm}}{k}J, \qquad \pm R^{\pm}(\pm k) = D(k) = -B(-k) = \frac{1}{2ik}J\int\limits_{-\infty}^{\infty} v(t)e^{-2ikt}dt.$$

Thus $kR^{+}(k)$ here is a Fourier transform of $v(x)$, which has the first moment on the axis, and a solution of the inverse problem is now given by

$$V(x) = \frac{2i}{\pi}\int\limits_{-\infty}^{\infty} kR^{+}(k)e^{2ikx}dk.$$

Let us derive $V(x)$ from $R^{+}(k)$ under the general scheme. Under our notation

$$\gamma^{\pm} = \frac{1}{2i}\int\limits_{-\infty}^{\infty} v(t)dt, \qquad R^{\gamma+}(k) = \frac{1}{2ik}J\int\limits_{-\infty}^{\infty} v(t)(e^{-2ikt}-1)dt,$$

$$F^{\gamma+}(x) = -\frac{1}{2}J \int\limits_{x/2}^{(\operatorname{sign} x)\infty} v(t)dt, \qquad F^{+}(x) = -\frac{1}{2}J \int\limits_{x/2}^{\infty} v(t)dt, \qquad x \in \mathbb{R}.$$

The Marchenko equation in our case acquires the form $K^+(x,y) + F^+(x+y) = 0$, since the product of matrices $K^+(x,t)F^+(t+y)$ here is identically zero. Thus $V(x) = -2dK^+(x,x)/dx = v(x)J$ according to the general theory.

Proof of Theorem 2.

The 'only if' part for item 1 of Theorem 2. The asymptotic estimates (5.1) as $k \to \pm\infty$ are established in the same way as this has been done under absence of virtual level in Theorem 1. The properties of $r^+_{ll}(k)$ (5.3) are direct consequences of the properties of $r^+(k)$ in the scalar problem [18]. The strict inequality in (5.3) is due to $|r^+(0)| < 1$ under presence of a VL (see [18], and also [14]).

Lemma 15. *The functions $\rho_{jl}(k)$ (5.2), $-\infty < k < \infty$, are continuously differentiable on the axis.*

Proof of Lemma 15 outside of a neighborhood of $k = 0$ coincides to that of Lemma 7 under absence of a VL.

Consider $r^+_{ll}(k)$ in the neighborhood of $k = 0$, which is equivalent to considering $r^+(k) = -\frac{b(-k)}{a(k)}$ for a scalar problem. We are about to apply the representations for $a(k)$, $b(k)$ under presence of a VL [14]:

$$a(k) = \frac{1}{2}\left\{1 + \int\limits_{0}^{\infty} \varphi^+(t)e^{ikt}dt\right\} e_-(0,k) - \frac{1}{2}\left\{-1 + \int\limits_{-\infty}^{0} \varphi^-(t)e^{-ikt}dt\right\} e_+(0,k), \quad (5.15)$$

$$b(k) = \frac{1}{2}\left\{1 - \int\limits_{-\infty}^{0} \varphi^-(t)e^{ikt}dt\right\} e_+(0,k) + \frac{1}{2}\left\{1 + \int\limits_{0}^{\infty} \varphi^+(t)e^{-ikt}dt\right\} e_-(0,k), \quad (5.16)$$

where $\varphi^\pm(z) \in L^1(0,\pm\infty)$ are bounded under existence of the first moment for $v(x)$, and satisfy the Marchenko equations

$$\varphi^\pm(z) \mp \int\limits_{0}^{\pm\infty} \varphi^\pm(t)F^\pm_R(t+z)dt = \pm F^\pm(z), \quad (5.17)$$

(which in the special case $e^\pm(0,0) = 0$ turn out to be homogeneous), $|F^\pm_R(x)| \le C\sigma^\pm(\frac{x}{2})$, $\sigma^\pm(x) \equiv \pm \int\limits_{x}^{\pm\infty} |v(t)|dt$ [18, p. 195].

Let us demonstrate that *if the potential has the n-th moment (in our case $n = 2$), then $\varphi^\pm(z)$ have $(n-1)$-th moment (the first moment in our case).*

We multiply (5.17) by z^{n-1} and then integrate to obtain

$$\pm\int_0^{\pm\infty}|\varphi^{\pm}(z)z^{n-1}|dz \le \int_0^{\pm\infty}|\varphi^{\pm}(t)|dt\int_t^{\pm\infty}|F_R^{\pm}(z)z^{n-1}|dz + \int_0^{\pm\infty}|F_R^{\pm}(z)z^{n-1}|dz < \infty.$$

The existence of the moment for $\varphi^{\pm}(z)$ is now established. Therefore one may differentiate the expressions for $a(k)$ (5.15) and $b(k)$ (5.16) in k, with $k = 0$ included, under the integral. Even more, $da(k)/dk$, $db(k)/dk$ turn out to be continuous in k, with $k = 0$ included. This implies continuous differentiability for $r^+(k)$ and $r^+_{ll}(k)$ on the axis $-\infty < k < \infty$ since $|a(k)|^2 = 1 + |b(k)|^2 \ne 0$, $k \in \mathbb{R}$. Now we demonstrate continuous differentiability for $\rho_{12}(k)$ (5.2). It follows from (2.9) that

$$r^+_{12}(k) = \{d_{12}(k) - c_{12}(k)r^+_{11}(k)\}a_{22}(k)^{-1} = -\{b_{12}(-k) + a_{12}(k)r^+_{22}(k)\}a_{11}(k)^{-1}. \quad (5.18)$$

Therefore, continuous differentiability for $kr^+_{12}(k)$ follows from the differentiability properties of $a_{11}(k) \ne 0$ and $r^+_{22}(k)$ proved above, together with continuous differentiability in k for $ka_{12}(k)$ and $kb_{12}(-k)$, which is itself due to the representation

$$\begin{aligned} A(k) &= I - \frac{1}{2ik}\left\{\int_{-\infty}^{\infty}V(x)dx + \int_{-\infty}^{0}A_1(t)e^{-ikt}dt\right\}, \\ B(k) &= \frac{1}{2ik}\int_{-\infty}^{\infty}B_1(t)e^{-ikt}dt, \end{aligned} \quad (5.19)$$

where $(1 + |t|)|A_1(t)| \in L^1(-\infty, 0)$, $(1 + |t|)|B_1(t)| \in L^1(-\infty, \infty)$. This representation is given by Lemma 6, which is an analog of Lemma 3.5.1 from [18] applied to the case of matrix potentials having the second moment on the axis. This proves Lemma 15. ◀

Hence the ‘only if’ part of Theorem 2 concerning all the conditions of item 1 of this Theorem is proved. ◀

The ‘only if’ part for item 2 of Theorem 2. Note that $r^{\gamma+}_{12}(k) = r^+_{12}(k) - \gamma^+k^{-1} = [\rho_{12}(k) - \rho_{12}(0)]k^{-1}$ is continuous due to the properties of $\rho_{12}(k)$, hence $R^{\gamma+}(k) \in L^2(-\infty, \infty)$ by item 1. Therefore there exists $F^{\gamma+}_R(x) \in L^2(-\infty, \infty)$.

Lemma 16. *Set*

$$F^+(x) = F^+_R(x) + \sum_{j=1}^{p} Z^+_j(x)e^{ik_jx}, \qquad \mathrm{Im}\, k > 0, \quad (5.20)$$

where $F^+_R(x)$ *is defined by* (5.5) *and* $Z_j(x)$ *by* (2.13). *Then* $F^+(x)$ (5.20) *satisfies the Marchenko equation*

$$K^+(x, y) + F^+(x + y) + \int_x^{\infty} K^+(x, t)F^+(t + y)\,dt = 0. \quad (5.21)$$

Hence $F^+(x)$ and $F^+_R(x)$ are absolutely continuous and $dF^+(x)/dx$, $dF^+_R(x)/dx$ have the second moment on (a,∞) for all $a>0$.

Proof of Lemma 16. The equation (5.21) for the diagonal elements $k^+_{ll}(x,y)$ and $f^+_{ll}(x)$ is known [18]. Similarly to [18, § 3.5], we use the matrix relation

$$E_-(x,k)\{C(k)^{-1}-I\}=E^+(x,-k)+E^+(x,k)R^+(k)-E_-(x,k), \tag{5.22}$$

to extract the scalar relations corresponding to the indices 1 2

$$-e^-_{11}(x,k)\frac{c_{12}(k)}{c_{11}(k)c_{22}(k)}+e^-_{12}(x,k)\left(\frac{1}{c_{22}(k)}-1\right)=$$
$$=e^+_{12}(x,-k)-e^-_{12}(x,k)+e^+_{11}(x,k)[r^{\gamma+}_{12}(k)+\gamma^+k^{-1}]+e^+_{12}(x,k)r^+_{22}(k). \tag{5.23}$$

Let us multiply this relation by $\frac{1}{2\pi}e^{iky}$ and then integrate in k from $-\infty$ to $+\infty$. We are going to apply the Fourier inversion formulas and representations for the Jost functions in terms of transformation operators. Using this techniques, we arrange a contour integration of the l.h.s. along the semicircles of radii N and ε in the upper half-plane, and the segments $(-N,-\varepsilon)$, (ε,N) ($\varepsilon\to 0$, $N\to\infty$). This procedure, which takes into account (2.13) – (2.15), allows one to deduce for $y>x$ that

$$-e^-_{11}(x,0)\frac{ic_{12}[-1]}{2c_{11}(0)c_{22}(0)}=k^+_{12}(x,y)+\int_x^\infty k^+_{12}(x,t)f^+_{22}(t+y)dt+f^{\gamma+}_{12}(x+y)+$$
$$+\int_x^\infty k^+_{11}(x,t)f^{\gamma+}_{12}(t+y)dt+\frac{i\gamma^+}{2}\left\{\text{sign }(x+y)+\int_x^\infty k^+_{11}(x,t)\text{sign }(t+y)dt\right\}, \tag{5.24}$$

where $f^{\gamma+}_{12}(t)$ is the matrix element of $F^{\gamma+}(t)\equiv F^{\gamma+}_R(t)+\sum_{j=1}^p Z^+_j(t)e^{ik_jt}$. Now multiply (5.23) by k. With $k\to 0$, we get for $-\infty<x<\infty$

$$-e^-_{11}(x,0)c_{11}(0)^{-1}c_{22}(0)^{-1}c_{12}[-1]=\gamma^+e^+_{11}(x,0), \tag{5.25}$$

which corresponds to the VL for the potential $v_{11}(x)$.

Now we multiply (5.25) by $\frac{i}{2}$ and subtract this from (5.24) to obtain (5.21) with $F^+(x)$ (5.20). This completes the proof of Lemma 16. ◄

Hence the 'only if' part of Theorem 2 concerning its condition 2) is proved.

The 'only if' part for item 3 of Theorem 2. The representations (5.7) for $a_{ll}(z)$ in terms of the reflection coefficients $r^+_{ll}(k)$ as in the direct problem (the Wronski determinants divided by $2ik$) are well known [18]. Those are continuously differentiable, as one can observe from the proof of Lemma 15, which was to be proved.

Remark 6. *As an additional explanation, we point out that, under some discrete spectrum being present, the expressions for* $a_{ll}(z) \equiv c_{ll}(z)$ *differ from those for* $a^0_{ll}(z) \equiv c^0_{ll}(z)$ (5.7) *by a product of linear-fractional terms* $c_{ll}(z) \equiv a_{ll}(z) = c^0_{ll}(z) \prod_{j=1}^{p} [(z-k_j)/(z+k_j)]^{\text{sign } z_{ll}^{[j]+}}$, $\text{Im } z > 0$. *This does not affect simultaneous smoothness of* $c_{ll}(z)$ *and* $c^0_{ll}(z)$ *for* $\text{Im } z \geq 0$. *Furthermore, it is obvious that* $c_{ll}(0) = (-1)^{p_l} c^0_{ll}(0)$, $a_{ll}(0) = (-1)^{p_l} a^0_{ll}(0)$, *where* $p_l = \sum_{j=1}^{p} \text{sign } z_{ll}^{[j]+}$.

The 'only if' part for item 4 of Theorem 2. Assume first that the problem (1.1), (1.2), hence also the SD (2.16) has no discrete spectrum. Use the properties of $a^0_{ll}(k)$ and $R^+(k)$ (item 1 of Theorem 2) to deduce that $h^0(k) = O(k^{-1})$, $dh^0(k)/dk = o(k^{-1})$ as $k \to \pm\infty$, and these are continuous on the entire k-axis.

Thus (5.9), (5.10), (5.11) imply that $zc^0_{12}(z)$ (in a pair with $-za^0_{12}(-z)$) gives a bounded solution of the Riemann-Hilbert problem in the half-plane (with the factorized coefficient $\frac{a^0_{11}(k)}{a^0_{22}(-k)}$)

$$\frac{kc^0_{12}(k)}{a^0_{11}(k)} - \psi_0^+(k) = -\frac{ka^0_{12}(-k)}{a^0_{22}(-k)} - \psi_0^-(k) \equiv \text{const}. \tag{5.26}$$

(It is implicit here that

$$\psi_0^+(k) - \psi_0^-(k) = h^0(k), \tag{5.27}$$

due to the Plemelj-Sokhotski formula.)

The constant in (5.26) can be computed via passage to a limit as $k \to 0$ using

$$\text{const} = a^0_{11}(0)^{-1} c^0_{12}[-1] - \psi_0^+(0) = a^0_{12}[-1] a^0_{22}(0)^{-1} - \psi_0^-(0), \tag{5.28}$$

and the subsequent application of the direct problem. Namely, (5.18) implies

$$c^0_{12}[-1] = -\gamma^+ a^0_{22}(0)\{1 + r^+_{11}(0)\}^{-1}, \qquad a^0_{12}[-1] = -\gamma^+ a^0_{11}(0)\{1 + r^+_{22}(0)\}^{-1}. \tag{5.29}$$

(Here we use the notation $g[-1]$ from (2.17).) Also, by (5.7) one has

$$a^0_{ll}(0) \equiv c^0_{ll}(0) \equiv (1 - |r^+_{ll}(0)|^2)^{-\frac{1}{2}}, \tag{5.30}$$

due to the Plemelj-Sokhotski formulas. Since the integrand in (5.7) is odd in k at $z = 0$, we deduce that const has the same value in (5.26) and in (5.9). Therefore, $zc^0_{12}(z)$ (5.9) is the only bounded solution of the problem (5.26) (with const being fixed). On the other hand, the direct scattering problem implies that the matrix elements $c^0_{12}(k)$ and $a^0_{12}(-k)$ satisfy the same equation (5.26). Hence $c^0_{12}(z)$ in (5.9), together with $c^0_{ll}(z)$, $l = 1, 2$ form the matrix $C_0(k) = C(k)$ (2.8) derived from the Wronski determinant divided by $2ik$ for the Jost solutions.

Thus the left reflection coefficient of the direct problem $R_0^-(k)$ (2.9) can be expressed via $R^+(k)$ by (2.10), where $C(k) = C_0(k)$ is given by (5.7) and (5.9). Then the function $F^-_{R_0}(x)$ (5.13) possesses the properties indicated in item 4, with (5.13) included, similarly to the function $F^+_R(x)$ (5.5), (5.6).

Let us verify (5.12). It follows from the direct problem, similarly to (5.29), that

$$\gamma^- \equiv r_{12}^{[0]-}[-1] = -c_{12}^0[-1]\left\{1 + r_{22}^{[0]-}(0)\right\} a_{11}^0(0)^{-1} = \\ = -a_{12}^0[-1]\left\{1 + r_{11}^{[0]-}(0)\right\} a_{22}^0(0)^{-1}. \quad (5.31)$$

This, together with (5.29), leads to (5.12), in view of $r_{ll}^{[0]-}(0) = -r_{ll}^+(0)$.

The 'only if' part for item 4 of Theorem 2 is already proved under absence of discrete spectrum. If some discrete spectrum is present, then the 'only if' part for item 4 of Theorem 2 could be established just as in Lemma 14. This lemma has been proved by the authors under absence of a VL, via an application of the consecutive elimination of eigenvalues method. (Note that Theorem 1 is proved in the two modifications labeled by conditions 4 or 4a, both without a VL. Condition 4 of Theorem 2 is an analog of condition 4a from Theorem 1. The 'only if' part for condition 4a in Theorem 1 was established after proving the 'only if' part for condition 4 of the same Theorem by the method of consecutive elimination of eigenvalues. This is because the condition 4 of Theorem 1 was formulated with discrete spectrum being used explicitly.)

The 'only if' part for items 5 and 6 of Theorem 2 under some discrete spectrum being present can be proved in the same way as it has been done in Theorem 1 under absence of VL.

Let us prove *the 'only if' part for conditions 1 – 4 of Theorem 2* under absence of discrete spectrum. (Since in this special case the values labeled by 0 coincide with those without index 0, the index 0 will be omitted throughout this proof to simplify notation.) As a consequence of conditions 1 – 2, in view of Lemma 16, we have the equation (5.21), which has for every x a single solution $K^+(x, y)$, similarly to [18]. This solution is a kernel of the transformation operator (2.3) for solutions $E^+(x, k)$ of equations of the form (1.1), (1.2), with the potential

$$V(x) = V^+(x) = -2dK^+(x, x)/dx, \quad (5.32)$$

having the second moment for all $a < x < +\infty$ and real diagonal elements $v_{ll}^+(x)$. In a similar way, conditions 1, 3, 4 imply the equation

$$K^-(x, y) + F_R^-(x + y) + \int_{-\infty}^{x} K^-(x, t)F_R^-(t + y)dt = 0, \quad (5.33)$$

with $F_R^-(x)$ (5.13). This equation is uniquely solvable with respect to $K^-(x, y)$ at every x, where $K^-(x, y)$ appears to be a kernel of the transformation operator (2.3) for solutions of an equation of the form (1.1), (1.2), with the potential

$$V(x) = V^-(x) = 2dK^-(x, x)/dx. \quad (5.34)$$

It remains to prove an important fact that

$$V^+(x) = V^-(x), \qquad \text{for} \quad -\infty < x < \infty. \quad (5.35)$$

For that, similarly to the case of scalar problem [18], [16], we introduce a matrix valued function $H^-(x,k)$

$$H^-(x,k) = \{E^+(x,-k) + E^+(x,k)R^+(k)\}C(k). \tag{5.36}$$

It is clear from (5.36) that $H^-(x,k)$ is a solution of (1.1) with $V(x) = V^+(x)$.

Let us prove that

$$H^-(x,k) = E^-(x,k), \tag{5.37}$$

which thus satisfies (1.1) also with $V(x) = V^-(x)$, hence satisfies (5.35) as well. Observe that in the scalar case the ISP with a real potential on the axis was solved [18], and it was also demonstrated that

$$h^-_{ll}(x,k) = e^-_{ll}(x,k), \tag{5.38}$$

(see [18, proof of Theorem 3.5.1], [16, (6.5.17)]). In view of this, it suffices to prove that

$$h^-_{12}(x,k) = e^-_{12}(x,k). \tag{5.39}$$

For doing this, we prove the following properties of $H^-(x,k)$, which are deducible from conditions $1-4$ of Theorem 2.

I) $H^-(x,k)$ admits an analytic continuation to the half-plane $\operatorname{Im} z > 0$, and there with $z \to \infty$ one has

$$|H^-(x,z) - e^{-ixz}I| = O\left(|z|^{-1}e^{x\operatorname{Im} z}\right). \tag{5.40}$$

II) $zH^-(x,z)$ is continuous in the closed upper half-plane $\operatorname{Im} z \geq 0$ and there $zH^-(x,z) \to 0$ as $z \to 0$ uniformly in x.

III)

$$\{H^-(x,k) - e^{-ikx}I\} \in L^2(-\infty,\infty;dk). \tag{5.41}$$

The principal distinction here from the scalar case [11], [16], [18] is a possible singularity of order k^{-1} as $k \to 0$ for the elements $r^+_{12}(k)$, $a_{12}(k)$, $c_{12}(k)$ (cf. [7]).

Set

$$G^+(x,y) \equiv F^{\gamma+}(x+y) + \int_x^\infty K^+(x,t)F^{\gamma+}(t+y)dt. \tag{5.42}$$

Then

$$\int_{-\infty}^{\infty} G^+(x,y)e^{-iky}dy = E^+(x,k)R^{\gamma+}(k). \tag{5.43}$$

Now (5.21) and (5.4) imply (5.5) for all $x \in \mathbb{R}$, taking into account that $K^+(x,y) = 0$ for $y < x$

$$H^-(x,k) = \left\{ e^{-ikx}I + \int_{-\infty}^{x} G^+(x,y)e^{-iky}dy \right\} C(k) +$$

$$+\gamma^+k^{-1}Jc_{22}(k)\left\{e_{11}^+(x,k)+\eta(-x)\left[-2i\sin kx+\int\limits_x^{-x}k_{11}^+(x,t)(e^{-ikx}-e^{ikt})dt\right]\right\}. \tag{5.44}$$

This formula implies property I) for $H^-(x,z)$ by virtue of items 1 and 3, together with (5.9) – (5.11) in the formulation of Theorem 2, whence

$$|C(z)-I|=O(|z|^{-1}) \qquad \text{for } |z|\to\infty, \quad \text{Im } z\geq 0,$$

using the Plemelj-Sokhotski and Plemelj-Privalov theorems [20, § 18], [12, § 5], and the representation (5.19) for the diagonal elements within the direct problem. We deduce from (5.44) via integration by parts which incorporates the jump of $G^+(x,y)$ in y at $y=-x$, that for $x>0$

$$H^-(x,z)-e^{-ixz}I=e^{-izx}\{C(z)-I\}+\left\{\frac{e^{-izy}}{-iz}G^+(x,y)\Big|_{y=-\infty}^{x}-\right.$$
$$\left.-\left[\int\limits_{-\infty}^{-x-0}+\int\limits_{-x+0}^{x}\right]G_y^{+'}(x,y)\frac{e^{-izy}}{-iz}dy-G(x,y)\Big|_{y=-x-0}^{y=-x+0}e^{izx}\right\}C(k)+$$
$$+\gamma^+z^{-1}Jc_{22}(z)e_{11}^+(x,z),$$

whence the required estimate.

On the contrary, if $x<0$, then the jump of $G^+(x,y)$ is outside the integration interval. However, this case requires an additional estimate for the term $\gamma^+z^{-1}Jc_{22}(z)\{\ldots\}$ in the r.h.s. of (5.44), taking into account that $\eta(-x)=1$ for $x<0$. Let us estimate the expression in the above braces $\{\ldots\}$

$$\{\ldots\}=e_{11}^+(x,z)-2i\sin zx+\int\limits_x^{-x}k_{11}^+(x,t)(e^{-izx}-e^{izt})dt=$$
$$=\int\limits_{-x}^{\infty}k_{11}^+(x,t)e^{izt}dt+e^{-izx}+e^{-izx}\int\limits_x^{-x}k_{11}^+(x,t)dt,$$

whence the required exponential estimate for $x<0$: $|\{\ldots\}|\leq e^{-izx}$const, where const depends on $x<0$, but not on z. This assures the validity of property I for the matrix valued function $H^-(x,z)$. Note that for the scalar equation, hence for the diagonal elements $h_{ll}^-(x,z)$, property I was established in [18] in the proof of Theorem 3.5.1.

Let us prove property II. We use the formulas (5.9) – (5.11) for $c_{12}(k)$, together with (5.30), to deduce from (5.44) via multiplying by k as $k\to 0$, in view of the relation

$$f_{ll}^{\gamma+}(x)=f_{ll}^+(x), \qquad l=1,2, \tag{5.45}$$

that

$$h^-_{12}(x,[-1]) \equiv \lim_{k\to 0}\{kh^-_{12}(x,k)\} = \\ = \left\{1+\int_{-\infty}^{x} g^+_{11}(x,y)dy\right\} c_{12}[-1] + \gamma^+ c_{22}(0)e^+_{11}(x,0). \quad (5.46)$$

On the other hand, (5.38) and (5.44) imply

$$e^-_{11}(x,0) = h^-_{11}(x,0) = \left\{1+\int_{-\infty}^{x} g^+_{11}(x,y)dy\right\} c_{11}(0). \quad (5.47)$$

Hence (5.46) yields

$$h^-_{12}(x,[-1]) = \gamma^+[1-r^+_{22}(0)^2]^{-1/2}\{e^+_{11}(x,0) - [1+r^+_{11}(0)]^{-1}e^-_{11}(x,0)\} = 0,$$

by virtue of (5.25), (5.29), (5.30). That is, for $\operatorname{Im} z \geq 0$ one has

$$\lim_{z\to 0}\{zh^-_{12}(x,z)\} = 0. \quad (5.48)$$

This already implies property II.

Let us prove property III. Due to (5.38), it suffices to establish that $h^-_{12}(x,k) \in L^2(-\infty,\infty;dk)$, where, in view of (5.36),

$$h^-_{12}(x,k) = \{e^+_{11}(x,-k) + e^+_{11}(x,k)r^+_{11}(k)\}c_{12}(k) + \\ + \{e^+_{12}(x,-k) + e^+_{11}(x,k)r^+_{12}(k) + e^+_{12}(x,k)r^+_{22}(k)\}c_{22}(k). \quad (5.49)$$

We use asymptotics of the terms as $k \to \pm\infty$, and, in particular, $e^+_{12}(x,-k) = \int_x^\infty k^+_{12}(x,t)e^{-ikt}dt \in L^2(-\infty,\infty;dk)$, to observe that it remains to demonstrate that $h^-_{12}(x,k) \in L^2$ in the neighborhood of $k=0$. Note that by (5.48), the singularities in (5.49) of order k^{-1} as $k\to 0$ annihilate each other (they appear due to $c_{12}(k)$ and $r^+_{12}(k)$). This implies that

$$h^-_{12}(x,k) \equiv \{x,k\}c_{12}(k) - \{x,0\}\frac{c_{12}[-1]}{k} + \\ + \frac{\gamma^+}{k}[e^+_{11}(x,k)c_{22}(k) - e^+_{11}(x,0)c_{22}(0)] + O(1) \qquad \text{for} \quad k\to 0, \quad (5.50)$$

where $\{x,k\}$ here stands for the first braces in (5.49) and, in particular, $\{x,0\} = e^+_{11}(x,0)[1+r^+_{11}(0)]$. We have

$$h^-_{12}(x,k) = [\{x,k\} - \{x,0\}]c_{12}(k) + \{x,0\}\left[c_{12}(k) - \frac{c_{12}[-1]}{k}\right] +$$

$$+\frac{\gamma^+}{k}[e_{11}^+(x,k)c_{22}(k)-e_{11}^+(x,0)c_{22}(0)]+O(1). \quad (5.51)$$

It is easy to see that the first and the third terms are bounded in k in a neighborhood of zero, which is, in particular, due to, continuous differentiability in k of $r_{ll}^+(k)$ and $c_{ll}(k)\equiv a_{ll}(k)$. It remains to observe that the second term in (5.51) allows by virtue of (5.9) – (5.11) an estimate

$$|c_{12}(k)-c_{12}[-1]k^{-1}|=|k^{-1}\{kc_{12}(k)-c_{12}[-1]\}|=O(|k|^{-\varepsilon}), \quad (5.52)$$

with $\varepsilon>0$ being arbitrarily small, as $h^0(k)$ in (5.10) is continuously differentiable by (5.11) and conditions of item 1 of Theorem 2. Therefore, $c_{12}(k)$ (5.9) belongs to the Hölder class with exponential $\mu=1-\varepsilon$ by the Plemelj-Privalov theorem mentioned above [12], [20]. This already implies the estimate (5.52), which completes the proof of property III for $H^-(x,k)$ (5.41).

Now we are in a position to prove (5.37). Due to properties I – III we have for some $P^-(x,t)\in L^2(-\infty,x)$

$$H^-(x,k)=e^{-ikx}I+\int_{-\infty}^{x}P^-(x,t)e^{-ikt}dt. \quad (5.53)$$

On the other hand, it follows from (5.36) and (2.10), (2.12) that

$$E^+(x,k)A(k)^{-1}-e^{ikx}I=H^-(x,k)R^-(k)+H^-(x,-k)-e^{ikx}I. \quad (5.54)$$

Also, (5.54) implies in view of (5.53) that

$$E^+(x,k)A(k)^{-1}-e^{ikx}I=$$
$$=\int_{-\infty}^{x}P^-(x,t)e^{ikt}dt+e^{-ikx}R^-(k)+\int_{-\infty}^{x}P^-(x,t)e^{-ikt}dtR^-(k). \quad (5.55)$$

Let us multiply both sides of (5.55) by $\frac{1}{2\pi}e^{-iky}$ and integrate in dk for $y<x$ from $-\infty$ to $+\infty$. In the l.h.s. of (5.55), a contour integration along $(-N,-\varepsilon)$, (ε,N), and upper semicircles of radii ε and N, give zeros for the diagonal elements as $\varepsilon\to 0$, $N\to\infty$, similarly to the scalar case [18]. Thus for $y<x$ we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\{E^+(x,k)A(k)^{-1}-e^{ikx}I\}e^{-iky}dk=$$
$$=\frac{1}{2\pi}J\int_{-\infty}^{\infty}\left\{-\frac{a_{12}(k)}{c_{11}(k)c_{22}(k)}e_{11}^+(x,k)+\frac{1}{c_{22}(k)}e_{12}^+(x,k)\right\}e^{-iky}dk=$$
$$=-\frac{i}{2}J\frac{a_{12}[-1]}{c_{11}(0)c_{22}(0)}e_{11}^+(x,0)=-\frac{i}{2}J\frac{a_{12}[-1]}{c_{22}(0)}(1+r_{11}^-(0))e_{11}^-(x,0)=$$

$$= \frac{i}{2}\gamma^- e_{11}^-(x,0)J = \frac{i}{2}\gamma^- J\left(1 + \int_{-\infty}^{x} p_{11}^-(x,t)dt\right). \quad (5.56)$$

It is implicit here that

$$e_{11}^+(x,0)c_{11}(0) = e_{11}^-(x,0)(1 + r_{11}^-(0)), \quad (5.57)$$

which is derived from (5.25) in view of (5.29) – (5.31) and (5.12). On the other hand, an integration in the r.h.s. of (5.55) for $y < x$ and in the notation of (5.12), (5.13) gives

$$P^-(x,y) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{R^{\gamma-}(k) + J\frac{\gamma^-}{k}\right\}e^{-ik(x+y)}dk +$$
$$+ \int_{-\infty}^{x} P^-(x,t)\frac{dt}{2\pi}\int_{-\infty}^{\infty}\left\{R^{\gamma-}(k) + J\frac{\gamma^-}{k}\right\}e^{-ik(t+y)}dk =$$
$$= P^-(x,y) + F^{\gamma-}(x+y) - J\frac{i}{2}\gamma^-\text{sign }(x+y) +$$
$$+ \int_{-\infty}^{x} P^-(x,t)\left\{F^{\gamma-}(t+y) - J\frac{i}{2}\gamma^-\text{sign }(t+y)\right\}dt. \quad (5.58)$$

Now we equate (5.56) and (5.58) to obtain

$$\frac{i}{2}\gamma^- J\left(1 + \int_{-\infty}^{x} p_{11}^-(x,t)dt\right) = P^-(x,y) + F^-(x+y) +$$
$$+ \frac{i}{2}\gamma^- J + \int_{-\infty}^{x} P^-(x,t)\left\{F^-(t+y) + J\frac{i}{2}\gamma^-\right\}dt,$$

i.e., $P^-(x,y)$ satisfies the Marchenko equation

$$P^-(x,y) + F^-(x+y) + \int_{-\infty}^{x} P^-(x,t)F^-(t+y)dt = 0.$$

This means that $P^-(x,t)$ is a transformation operator such that $P^-(x,t) = K^-(x,y)$, hence (5.37) holds, and Theorem 2 is proved completely under absence of discrete spectrum.

Now consider the case when the data (2.16) contains finitely many $k_j^2 < 0$, $j = 1,\dots,p < \infty$, and the corresponding matrix polynomials $Z_j^+(t)$. Then Theorem 2 can be proved, in view of conditions 5 and 6, by the subsequent adding the eigenvalues method, which is well-known for the scalar selfadjoint case (see, e.g., [16] or [3]). In our case the eigenvalues k_j^2 can be simple or multiplicity two, according to ranks of normalizing polynomials $Z_j(t)$. For $p = 0$ the set $\{k_j^2, \quad Z_j(t)\}$ is empty, so that in this case Theorem 2 is proved, with the corresponding potential $V(x) = V_0(x)$ being uniquely determined by (5.32) using the solution $K^+(x,y) = K_0^+(x,y)$ of the equation (5.21).

Lemma 17. *Suppose that the assumptions of Theorem 2 are satisfied for a data of the form (2.16) with $j = 1,\dots,p,p+1$, then they are satisfied for a part of this data with $j = 1,\dots,p$, where $k_{j+1}^2 < k_j^2 < 0$. Suppose also that for given p the ISP is uniquely solvable. Denote the corresponding potential by $V_p(x)$. Then the ISP with $p+1$ instead of p in* (2.16) *is also uniquely solvable and*

$$V_{p+1}(x) = V_p(x) - 2dB_p(x,x)/dx. \tag{5.59}$$

Here $B_p(x,y)$ is determined from the degenerate integral equation

$$B_p(x,y) + F_p(x,y) + \int_x^\infty B_p(x,t)F_p(t,y)dt = 0, \qquad x < y, \tag{5.60}$$

where

$$F_p(x,y) = E_+^{<p>}(x,k_{p+1})Z_{p+1}^+(0)\widetilde{E}_+^{<p>}(y,k_{p+1}) - \\ - i\frac{d}{dk}\left\{E_+^{<p>}(x,k)Z_{p+1}^{+'}(0)\widetilde{E}_+^{<p>}(y,k)\right\}_{k=k_{p+1}}, \tag{5.61}$$

and $E_+^{<p>}(x,k)$, $\widetilde{E}_+^{<p>}(y,k)$ stand for the Jost solutions of the equation (1.1) *with $V(x) = V_p(x)$.*

An explicit solution of the equation (5.60) and its investigation has been done by the authors in subsection 4.3, cf. (4.27), (4.28), (4.29), under absence of a VL. However, the argument and the result under some VL being present (both multiple and simple) remain intact, so we do not reproduce them here.

Let us note only that, under the procedure of adding eigenvalues, starting from $p = 0$ up to the given p, the right reflection coefficient $R^+(k)$ does not vary. In particular, the values $r_{11}^+(0)$ and $r_{22}^+(0)$ are the same for all p, hence the potentials $V_p(x)$ produced within this induction argument, starting from $p = 0$ up to given p have (or have not) a VL of the same multiplicity. This completes the proof of Theorem 2. ◄

5.2. The case of multiplicity one VL

Theorem 3. (See [27, Theorem 2]). *The values* (2.16) *are the right SD for the problem* (1.1), (1.2)*, with a matrix potential of the form considered in Theorem 2, but with precisely multiplicity one VL if and only if the following conditions 1 – 6 are satisfied:*

1) *All the claims in item 1 of Theorem 2 are valid with the refinement as follows. Looking at the inequalities* (5.3) *for* $|r^+_{ll}(k)|$*, one should choose* l *(either* $l = 1$ *or* $l = 2$*) such that for this* l *there exists a VL. The corresponding inequality should be left strict, and in another one (with the different* l*) one should replace '<' by '≤'.*

Besides that, in the case of a VL being present for the potential $v_{11}(x)$*, one should have* $dr^+_{12}(k)/dk \in C(\mathbb{R})$*, and in the case of a VL for the potential* $v_{22}(x)$ *it should be* $r^+_{12}(k) \in C(\mathbb{R}) \bigcap C^1(\mathbb{R}\backslash\{0\})$*.*

2) *All the claims in item 2 of Theorem 2 are valid with* $\gamma^+ = 0$*, hence also with* $R^+(k) = R^{\gamma+}(k)$ *and with* $F^+_R(x) = F^{\gamma+}_R(x)$*.*

3) *The claim in item 3 of Theorem 2 on continuous differentiability of* $c^0_{ll}(z) \equiv a^0_{ll}(z)$ (5.7) *for* $\operatorname{Im} z \geq 0$ *is valid for exactly that* $l = 1$ *or* 2*, for which a VL is present, and for another* l *these are functions* $zc^0_{ll}(z) \equiv za^0_{ll}(z)$ *which are continuously differentiable for* $\operatorname{Im} z \geq 0$*.*

4) *According to* (2.10)*, set*

$$R^-_0(k) \equiv -C_0(k)^{-1}R^+(-k)C_0(-k). \tag{5.62}$$

Here the diagonal elements $c^0_{ll}(k)$ *of the matrix* $C_0(k)$ *are determined by condition 3 of this Theorem,* $c^0_{21} \equiv 0$*,* $c^0_{12}(k) = c^0_{12}(k+i0)$*,* $k \in \mathbb{R}\backslash\{0\}$*. Also, if a VL corresponds to the potential* $v_{11}(x)$ *then*

$$zc^0_{12}(z) = [\psi^+_0(z) - \psi^+_0(0) + h^0(0)]a^0_{11}(z), \quad \operatorname{Im} z > 0. \tag{5.63}$$

On the other hand, if a VL is present for the potential $v_{22}(x)$ *then*

$$zc^0_{12}(z) = [\psi^+_0(z) - \psi^+_0(0)]a^0_{11}(z), \qquad \operatorname{Im} z > 0. \tag{5.64}$$

In both cases $\psi^\pm_0(z)$ *and* $h^0(k)$ *are given by* (5.10) *and* (5.11)*.*

Then the function $F^-_{R_0}(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} R^-_0(k)e^{-ikx}dk$ *is absolutely continuous, and for every* $a < \infty$ *one has* (5.14)*.*

5)*,* **6)** *The items 5 and 6 of Theorem 2 are to be reproduced literally.*

Proof of Theorem 3.

The 'only if' part. In item 1 making the inequality (5.3) non-strict for one of the values of l (either 1 or 2) is due to the fact that under the presence of a VL one has $|r^+_{ll}(0)| < 1$, while with a VL being absent it should be $r^+_{ll}(0) = -1$.

It follows from (5.18) that, under a VL being present only for $v_{11}(x)$, one has

$$r^+_{12}(k) = \{kd_{12}(k) - kc_{12}(k)r^+_{11}(k)\}\{kc_{22}(k)\}^{-1},$$

where all the products in braces are continuously differentiable, and $\{kc_{22}(k)\}|_{k=0} \neq 0$. Hence $r^+_{12}(k) \in C^1(\mathbb{R})$.

On the other hand, if a VL is present for $v_{22}(x)$ (only!), then $r^+_{12}(k) = \{d_{12}(k)+c_{12}(k)-c_{12}(k)[r^+_{11}(k)+1]\}c_{22}(k)^{-1}$. Here $c_{22}(k)^{-1} \in C^1(\mathbb{R})$ by Lemma 15, $\{c_{12}(k)[r^+_{11}(k)+1]\} \in C(\mathbb{R})$ since $r^+_{11}(0)+1 = 0$, $r^+_{11}(k) \in C^1(\mathbb{R})$ and $kc_{12}(k) \in C^1(\mathbb{R})$.

Lemma 18. *The matrix valued functions* $\{D(k)+C(k)\}$ *and* $\{A(k)+B(k)\}$ *(see* (2.8)*) are continuous on the axis.*

Proof of Lemma 18. We have

$$
\begin{aligned}
&c_{12}(k)+d_{12}(k) = \\
&= \frac{1}{2ik}[w\{e^+_{11}(x,-k)-e^+_{11}(x,k), e^-_{12}(x,k)\} + w\{\widetilde{e}^+_{12}(x,-k)-\widetilde{e}^+_{12}(x,k), e^-_{22}(x,k)\}] = \\
&\qquad = -w\left\{\frac{\sin kx}{k} + \int_x^\infty k^+_{11}(x,t)\frac{\sin kt}{k}dt, e^-_{12}(x,k)\right\} - \\
&\qquad\quad - w\left\{\int_x^\infty \widetilde{k}^+_{12}(x,t)\frac{\sin kt}{k}dt, e^-_{22}(x,k)\right\} \xrightarrow[k\to 0]{} \\
&\xrightarrow[k\to 0]{} -w\left\{x + \int_x^\infty k^+_{11}(x,t)t\,dt, e^-_{12}(x,0)\right\} - w\left\{\int_x^\infty \widetilde{k}^+_{12}(x,t)t\,dt, e^-_{22}(x,0)\right\} = \text{const.}
\end{aligned}
$$

Similarly, one can establish the existence of $\lim_{k\to 0}\{c_{ll}(k)+d_{ll}(k)\}$, hence continuity of the sum $C(k)+D(k)$ for $k=0$, and therefore for $k \in \mathbb{R}$. In the same way, one can establish continuity for $A(k)+B(k)$. Lemma 18 is proved. ◄

Hence the 'only if' part for item 1 of Theorem 3 is proved. (Note that in the scalar case boundedness for the sum $a(k)+b(k) = O(1)$ as $k \to 0$ has been proved in [16, lemma 6.1.6].)

The 'only if' part for item 2 of Theorem 3 can be proved similarly to the 'only if' part for item 2 of Theorem 2. This requires an application of Lemma 16 with appropriate simplifications being introduced, because $\gamma^+ = 0$ (by virtue of item 1 of Theorem 3).

The 'only if' part for item 3 follows, just as in Theorem 2, from the known [18] representation (5.7) for $a^0_{ll}(k)$ in terms of $r^+_{ll}(k)$, and the proof of Lemma 15 (with a VL being present for a given l) or the proof of item 3 of Theorem 1 under absence of a VL for $v_{ll}(x)$ for a given l.

The 'only if' part for item 4 of Theorem 3 can be proved similarly to that for item 4 of Theorem 2 (with some simplifications). It should be taken into account that $|a^0_{11}(0)| < \infty$ with a VL being present for $v_{11}(x)$ and $|a^0_{11}(k)| \asymp |k|^{-1}$ $(k \to 0)$ with a VL being present for $v_{22}(x)$.

The 'only if' part for items 5 and 6 of Theorem 3 can be proved in the same way as in Theorem 1 or in Theorem 2.

The 'if' part for conditions 1 – 4 of Theorem 3 under absence of discrete spectrum can be proved similarly to that for conditions 1 – 4 of Theorem 2 (with some simplifications).

If finitely many negative discrete levels are present, it can be proved just as Theorem 2, with conditions 5 and 6 being taken into account, using the method of subsequent adding simple or multiplicity two eigenvalues (see Lemma 17 and subsection 4.3). The proof of Theorem 3 is complete. ◀

5.3. The Parseval equality in the case with VL

If a VL is absent, hence also in the case when the reflection coefficient $R^+(k)$ has no pole, i.e., with $\gamma^+ = 0$, the Parseval equality, or the expansion of the Dirac δ-function for the system (1.1), (1.2), has the form

$$\delta(x-t)I = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_+(x,k)A^{-1}(k)\widetilde{E}_-(t,k)dk + \\ + \sum_{j=1}^{p}\sum_{l=0}^{1}\frac{d^l}{i^l dk^l}\left\{E_+(x,k)\left(\widetilde{Z}_j^+\right)^{(l)}(0)\widetilde{E}_+(t,k)\right\}_{k_j}, \quad (5.65)$$

see Lemma 4. It can be also rewritten in the form

$$\int_{-\infty}^{\infty}\Phi(x)\Psi(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_-(\Phi,k)C^{-1}(k)\widetilde{E}_+(\Psi,k)dk + \\ + \sum_{j=1}^{p}\sum_{l=0}^{1}\frac{d^l}{i^l dk^l}\left\{E_-(\Phi,k)\left(Z_j^+\right)^{(l)}(0)\widetilde{E}_+(\Psi,k)\right\}_{k=k_j}. \quad (5.66)$$

Here $\Phi(x)$ and $\Psi(x)$ are square 2×2-matrix valued functions with compact support, which are continuous in x:

$$E_\pm(\Phi,k) = \int_{-\infty}^{\infty}\Phi(x)E_\pm(x,k)dx, \qquad \widetilde{E}_\pm(\Psi,k) = \int_{-\infty}^{\infty}\widetilde{E}_\pm(x,k)\Psi(x)dx. \quad (5.67)$$

(A modern approach to the Dirac δ-function can be found in [2].)

Now it should be noted that even in the case of a multiple VL and $\gamma^+ = 0$, the relation

$$F_R^+(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R^+(k)e^{ikx}dx, \quad (5.68)$$

is still valid by virtue of (5.4), (5.5). This, together with Lemma 16, allows one also in the matrix case with multiple VL and $\gamma^+ = 0$ considered in this paper, to apply the argument used to prove the Parseval equality (5.65), (5.66) in Lemma 4 and our earlier works. Namely, the argumentation which was used in the scalar case in [18, Problem 4 to Chapter 3, § 5], and which was also used in the matrix case under absence of VL in [29].

This argumentation is based on application of the Marchenko equation (5.21) with $F^+(x)$ of the form (5.20), where in $F_R^+(x)$ is given by (5.68). Thus the formulas (5.65), (5.66) are also valid under a multiple VL with $\gamma^+ = 0$.

On the other hand, the simplest Example 1 clearly indicates that for $\gamma^+ \neq 0$ the above relations (5.65), (5.66) become invalid. In particular, in addition to $\delta(x-t)$, an excessive term appears in the r.h.s. of (5.65). This term vanishes only with $\gamma^+ = 0$, i.e., under $\int\limits_{-\infty}^{\infty} v_{12}(x)dx = 0$ in the example mentioned above. In order to get again formulas for the expansion of the Dirac δ-function, starting from (5.65), (5.66), we replace the potential $V(x)$ in (1.1) by the potential

$$V_\alpha(x) = V(x) + \alpha J \delta(x). \tag{5.69}$$

After that, we choose $\alpha = \alpha_0$ in such a way that the perturbed equation

$$-Y'' + V_\alpha(x)Y = k^2 Y, \tag{5.70}$$

determines $\gamma^+_{\alpha_0} = 0$. To be rephrased, the equation (1.1) with $V_\alpha(x)$ instead of $V(x)$, determines the reflection coefficient $R^+_{\alpha_0}(k)$ with no pole at $k=0$, in spite of a multiplicity two VL being present.

Remark 7. *The multiplicity of a VL for all α remains the same. Also, the discrete spectrum and its algebraic multiplicity does not depend on α.*

For the perturbed equation with the potential $V_\alpha(x)$ and $\gamma^+_{\alpha_0} = 0$, the relations (5.65), (5.66) still hold if one replaces the solutions $E_\pm(x,k)$, $\widetilde{E}_\pm(x,k)$ of the non-perturbed equation (1.1), the matrices $A(k)$, $C(k)$, and other values involved therein, by the corresponding solutions of the perturbed equation (5.70), $E^\pm_{\alpha_0}(x,k)$, $\widetilde{E}^\pm_{\alpha_0}(x,k)$; the matrices $A(k)$, $C(k)$ are to be replaced by $A_{\alpha_0}(k)$, $C_{\alpha_0}(k)$, etc. Since $E^\pm_\alpha(x,k)$, $\widetilde{E}^\pm_\alpha(x,k)$ etc. are easily expressible in terms of $E_\pm(x,k)$, $\widetilde{E}_\pm(x,k)$ etc., we substitute these expressions to (5.65), (5.66) instead of $E^\pm_\alpha(x,k)$, $\widetilde{E}^\pm_\alpha(x,k)$ to deduce the modified Parseval equality for the equation (1.1) in the case when the reflection coefficient $R^+(k)$ has a pole, that is, for $\gamma^+ \neq 0$.

Lemma 19. *The equation* (5.70) *with a multiple VL determines $\gamma^+_{\alpha_0} = 0$ when*

$$\alpha_0 = \frac{c_{12}[-1]}{e^+_{11}(0,0)e^-_{22}(0,0)}\,\frac{1+r^+_{11}(0)}{1-r^+_{11}(0)}. \tag{5.71}$$

In this case, the Jost solutions of (5.70) *have the form*

$$E^-_{\alpha_0}(x,k) = \\ = \begin{cases} E_-(x,k), & x<0, \\ E_-(x,k) + \\ +\mu_0 k^{-1}\left\{E_+(x,k)A^{-1}(k)\widetilde{E}_-(0,k) - E_-(x,k)C^{-1}(k)\widetilde{E}_+(0,k)\right\}J, & x>0, \end{cases} \tag{5.72}$$

where

$$\mu_0 = \frac{-ic_{12}[-1]}{2e^+_{11}(0,0)} \cdot \frac{1+r^+_{11}(0)}{1-r^+_{11}(0)}, \tag{5.73}$$

and the tilde-Jost solutions have the form

$$\widetilde{E}^+_{\alpha_0}(x,k) = \begin{cases} \widetilde{E}_+(x,k), & x>0, \\ \widetilde{E}_+(x,k) + \widetilde{\mu}_0 k^{-1} J \Big\{ E_-(0,k)C^{-1}(k)\widetilde{E}_+(x,k) - \\ \qquad - E_+(0,k)A^{-1}(k)\widetilde{E}_-(x,k)\Big\}, & x<0, \end{cases} \tag{5.74}$$

where

$$\widetilde{\mu}_0 = -\mu_0 e^+_{11}(0,0)e^-_{22}(0,0)^{-1} = \frac{ic_{12}[-1]}{2e^-_{22}(0,0)} \times \frac{1+r^+_{11}(0)}{1-r^+_{11}(0)}, \tag{5.75}$$

and α_0 *is the same as in* (5.71). *Besides that*

$$C_\alpha(k) = C(k) - \frac{\alpha}{2ik} J e^+_{11}(0,k)e^-_{22}(0,k), \tag{5.76}$$

$$C_\alpha(k)^{-1} = C(k)^{-1} + \frac{\alpha}{2ik}\, \frac{e^+_{11}(0,k)e^-_{22}(0,k)}{c_{11}(k)c_{22}(k)} J. \tag{5.77}$$

Proof. Let us substitute $E^-_\alpha(x,k)$ to (5.70), and then integrate from $x=-0$ to $x=+0$. This procedure, which takes into account the fact that $E^-_\alpha(x,k)$ is continuous in x, yields

$$-\, E^{-\prime}_\alpha(x,k)\Big|_{x=-0}^{x=+0} + \alpha J E^-_\alpha(0,k) = 0. \tag{5.78}$$

Therefore, the following system should be satisfied

$$\begin{cases} E^-_\alpha(+0,k) = E^-(0,k), \\ E^{-\prime}_\alpha(+0,k) = E^{-\prime}(0,k) + \alpha J E^-_\alpha(0,k). \end{cases} \tag{5.79}$$

One can verify, via a comparison to the ordinary construction for the kernel of resolvent (the Green function) of the problem (1.1), that the system (5.79) has the solution $E^-_\alpha(x,k)$ given by (5.72) with arbitrary α and μ without indices 0, subject to the only condition

$$\alpha = 2i\mu e^-_{22}(0,0). \tag{5.80}$$

The values $\mu = \mu_0$ (5.73) and, by virtue of (5.80), also $\alpha = \alpha_0$, are deducible from $\gamma^+_{\alpha_0} = 0$. The latter condition is equivalent to $r^+_{12}[-1] = 0$, or, by (5.18), to

$$d^\alpha_{12}[-1] - c^\alpha_{12}[-1] r^+_{11}(0) = 0. \tag{5.81}$$

The tilde-solution $\widetilde{E}^+_\alpha(x,k)$ of the equation (2.1) with $V = V_\alpha(x)$ is found in a similar way from (5.74). Within the process, one should discard the indices '0' at α and $\widetilde{\mu}$, but to keep the relation between $\widetilde{\mu}$ and α in view of (5.75) (with the indices '0' being also discarded) and (5.80).

Now (5.81), together with (2.8) and (5.72), implies that

$$0 = d_{12}[-1] - c_{12}[-1]r^{+}_{11}(0) + 2i\mu_0 e^{+}_{11}(0,0)\left[1 - r^{+}_{11}(0)\right].$$

This allows us to find μ_0 (5.73) via observing that $c_{12}[-1] + d_{12}[-1] = 0$ by Lemma 18. Then we use (5.73) to find α_0 (5.71) by virtue of (5.80). The formulas (5.76), (5.77) are established via a direct computation.

The value α_0 for the tilde-solution (5.74) is the same as (5.71), because the right SD of the problem (5.70) and the tilde-problem with the same potential coincide by Lemma 2. The value $\widetilde{\mu}_0$ is derived from μ_0 (5.73) using (5.75). Lemma 19 is proved. ◀

Now let us write down the Parseval equality (5.66) for the equation (5.70), (5.69) with $\alpha = \alpha_0$, hence with $\gamma^{+}_{\alpha_0} = 0$. We express this Parseval equality in terms of transforms of the matrix valued functions $\Phi(x)$ and $\Psi(x)$ in solutions of the form (5.72) – (5.75). The latter transforms are in turn expressible through transforms in non-perturbed solutions of (1.1) and (2.1), where $\alpha = 0$. To simplify matters, we assume that there is no discrete spectrum. In fact, such spectrum can be very well added (eliminated) via subsequent adding (eliminating) eigenvalues (see subsections 4.3, 4.4).

Theorem 4. (See [27, Theorem 3]) *The Parseval equality for transforms in solutions of the problems* (1.1) – (2.1) *for square* 2×2 *continuous matrix valued functions* $\Phi(x)$ *and* $\Psi(x)$ *with compact supports, in the case when the reflection coefficient* $R^{+}(k)$ *has a pole at* $k = 0$ *(hence a multiplicity two VL being present) and no discrete spectrum, can be written in the form:*

$$\int\limits_{-\infty}^{\infty}\Phi(x)\Psi(x)dx = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}E_{-}(\Phi,k)\left\{C^{-1}(k) + \frac{\alpha_0}{2ik}J\frac{e^{+}_{11}(0,k)e^{-}_{22}(0,k)}{c_{11}(k)c_{22}(k)}\right\}\widetilde{E}_{+}(\Psi,k)dk +$$

$$+\frac{\mu_0}{2\pi}\int\limits_{-\infty}^{\infty}E_{-}(\Phi,k)\frac{dk}{k}c^{-1}_{11}(k)c^{-1}_{22}(k)\int\limits_{-\infty}^{0}\left\{e^{-}_{22}(0,k)e^{+}_{22}(t,k) - e^{+}_{22}(0,k)e^{-}_{22}(t,k)\right\}J\Psi(t)dt +$$

$$+\frac{\widetilde{\mu}_0}{2\pi}\int\limits_{-\infty}^{\infty}\frac{dk}{k}c^{-1}_{11}(k)c^{-1}_{22}(k)\cdot$$

$$\cdot\int\limits_{0}^{\infty}\Phi(x)J\left\{e^{+}_{11}(x,k)e^{-}_{11}(0,k) - e^{-}_{11}(x,k)e^{+}_{11}(0,k)\right\}\widetilde{E}_{+}(\Psi,k)dx. \quad (5.82)$$

Here $E^{\pm}(\Phi,k)$, $\widetilde{E}^{\pm}(\Psi,k)$ *are determined by* (5.67), *and the values* α_0, μ_0, $\widetilde{\mu}_0$ *are given by* (5.71), (5.73), (5.75). *In this setting* α_0, μ_0, $\widetilde{\mu}_0$ *vanish for* $\gamma^{+} = 0$ *(i.e., in the case of no pole for* $R^{+}(k)$ *at* $k = 0$*), when the relation* (5.82) *acquires the form* (5.66) *established in Lemma 4 under absence of a VL.*

Proof of Theorem 4 is based on Lemma 19 and an application of the Marchenko equation (5.21), (5.20), (5.5) (cf. [18, Problem 4 to Chapter 3, § 5]). The method of proving is described above, so we need not reproduce it again.

Example 2. Let us apply the general form of the Parseval equality (5.82) in the case $\gamma^+ \neq 0$ to the simplest example 1, where $V(x) = v(x)J$. In this case

$$E_-(x,k) = \widetilde{E}_-(x,k) = e^{-ikx}I - \frac{1}{2ik}J\int\limits_{-\infty}^{x} v(s)\left[e^{-ikx} - e^{ik(x-2s)}\right]ds,$$

$$E_+(x,k) = \widetilde{E}_+(x,k) = e^{ikx}I + \frac{1}{2ik}J\int\limits_{x}^{\infty} v(s)\left[e^{ik(2s-x)} - e^{ikx}\right]ds,$$

$$\alpha_0 = -\int\limits_{-\infty}^{\infty} v(s)ds = -2i\gamma^{\pm}, \qquad C_{\alpha_0} = I, \qquad C(k)^{-1} = I + \gamma^+k^{-1}J.$$

Under the above setting the expansion of the Dirac δ-function acquires the form

$$\delta(x-t)I = \\ = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} E_-(x,k)\widetilde{E}_+(t,k)dk + \frac{\gamma^+}{2\pi}\int\limits_{-\infty}^{\infty} E_-(x,k)\frac{J}{k}\eta(-t)\left[\widetilde{E}_+(t,k) - \widetilde{E}_-(t,k)\right]dk- \\ - \frac{\gamma^+}{2\pi}\int\limits_{-\infty}^{\infty} k^{-1}\eta(x)\left[E_+(x,k) - E_-(x,k)\right]J\widetilde{E}^+(t,k)dk,$$

which is deducible by a direct computation.

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References

[1] Z.S. Agranovich and V.A. Marchenko. *The Inverse Problem of Scattering Theory.* Kharkov State University, Kharkov, 1960 (Russian) (Engl. transl. N.Y.-London, Gordon & Breach, 1963.).

[2] S. Albeverio, F. Gesztesy, R. Høegh Krohn, and H. Holden. *Solvable Models in Quantum Mechanics.* Springer-Verlag, N.-Y., Berlin, Heidelberg, London, Paris, Tokyo, 1988 (Rus. transl.: Mir, Moscow, 1991).

[3] V.De Alfaro and T. Regge. *Potential Scattering.* New York, John Wiley and Sons.1965, (Russian transl.: Mir, Moscow, 1966).

[4] L. Martinez Alonso and E. Olmedilla. Trace identities in the inverse scattering transform method associated with matrix schrödinger operator. *J. Math. Phys.*, 23(11):2116–2121, 1982.

[5] Ju.M. Berezanskii. *Expansions in eigenfunctions of selfadjoint operators.* Naukova Dumka, Kiev , 1965, Russian. (Engl. Transl.: AMS, **17**, Providence, RI, 1968.).

[6] F.A. Berezin and M.A. Shubin. *The Schrödinger Equation.* Moscow University, 1983, Russian. (Engl. transl.: Kluwer, 1991).

[7] V.A. Blashchak. An analogue of inverse problem in scattering theory for a non-selfadjoint operator, I, II. *Differential Equations*, 4(8): 1519 – 1533(4(10): 1915 – 1924), 1968, Russian.

[8] E.I. Bondarenko and F.S. Rofe-Beketov. Phase equivalent matrix potentials. *Electromagnetic waves and electronic systems .*, 5(3):6–24, 2000, Russian. (Engl. transl.: *Telecommun. and Radio Eng.* 56(8 and 9): 4 - 29, 2001).).

[9] E.I. Bondarenko and F.S. Rofe-Beketov. Inverse scattering problem on the semiaxis for a system with a triangular matrix potential. *Math. Physics, Analysis, and Geometry*, 10(3):412 – 424, 2003, Russian.

[10] K. Chadan and P.C. Sabatier. *Inverse Problems in Quantum Scattering Theory. With Foreword by R.G. Newton.* Springer-Verlag, New York, Heidelberg, Berlin, 1977 (Russian transl.: Mir, Moscow, 1980).

[11] L.D. Faddeev. Inverse scattering problem in quantum theory II. *Modern Probl. Math.VINITI, Moscow*, 3:93–180, 1974, (Russian) (Engl. Transl.: *J. Soviet Math.* 5: 334 – 396, 1976.).

[12] F.D. Gakhov. *Boundary Problems.* Fizmatgiz, Moscow, 1958, Russian.

[13] F.D. Gakhov and Yu.I. Cherskiy. *Convolution Type Equations.* Nauka, Moscow, 1978, Russian.

[14] I.M. Guseynov. On the continuity of the coefficient of reflection of the Schrödinger one-dimensional equation. *Differencialnye uravnenya*, 21(11):1993 –1995, 1985, Russian.

[15] B.Ya. Levin. Fourier and Laplace type transformations via solutions of second order differential equations. *Dokl. Acad. Sci. USSR*, 106(2):187–190, 1956, Russian.

[16] B.M. Levitan. *Inverse Sturm-Liouville Problems.* Nauka, Moscow, 1984, Russian, (Engl. transl.: VSP, Zeist, 1987.).

[17] V.E. Lyantse. An analogue for inverse problem of scattering theory for nonselfadjoint operator. *Mat. Sb.*, 72(4):537–557, 1967, Russian.

[18] V.A. Marchenko. *Sturm-Liouville Operators and Applications.* Naukova Dumka, Kiev, 1977, Russian. (Engl. transl.: Basel, Birkhäuser, 1986.).

[19] V.A. Marchenko. The inverse scattering problem and its applications to NLPDE. *Chapter 6.2.1 in: Scattering. V. 1, 2. Scattering and Inverse Scattering in Pure and Applied Science / ed. R. Pike and P. Sabatier. San Diego: Academic Press.*, 1695-1706, 2002.

[20] N.I. Muskhelishvili. *Singular Integral Equations.* Fizmatgiz, Moscow, 1962, Russian.

[21] M.A. Naimark. *Linear differential operators.* 2nd ed. Nauka, Moscow,1969, with Addendum by V. E. Lyantse: Non-selfadjoint second order differential operator on the semiaxis, Russian. (Engl. transl.: Ungar., New York, 1968).

[22] R.G. Newton. *Inverse Schrödinger Scattering in Three Dimensions.* Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo-Hong Kong, 1989.

[23] L.P. Nizhnik. *Inverse Scattering Problem for Hyperbolic Equations.* Naukova Dumka, Kiev, 1991, Russian.

[24] A.Ya. Povzner. On eigenfunction expansions in terms of scattering solutions. *Dokl. Akad. Nauk SSSR*, 104(3):360–363, 1955, Russian.

[25] A.G. Ramm. *Inverse problems.* Springer, New York, 2005.

[26] M. Reed and B. Simon. *Methods of modern mathematical physics. III. Scattering theory.* Acad Press, N.-Y., San Francisco, London, 1979(Russ. transl.: Mir, Moscow 1982.).

[27] F.S. Rofe-Beketov and E.I. Zubkova. Inverse scattering problem on the axis for the triangular 2×2 matrix potential with a virtual level. *Methods Funct. Anal. Topology*, 15(4):301–321, 2009.

[28] I.P. Syroyid. *The Composite Method of Inverse Scattering Problem and Studying Non-selfadjoint Lax Pairs for Korteveg-de Vries systems.* Lviv, Ya. S. Pidstrygach Institute of Applied Problems in Mechanics and Mathematics, Ukrainian National Acad. Sci., Ukrainian, 2005.

[29] E.I. Zubkova and F.S. Rofe-Beketov. Inverse scattering problem on the axis for the Schrödinger operator with triangular 2×2 matrix potential. I. main theorem. *J. Math. Phys., Anal., Geom.*, 3(1):47–60, 2007.

[30] E.I. Zubkova and F.S. Rofe-Beketov. Inverse scattering problem on the axis for the Schrödinger operator with triangular 2×2 matrix potential. II. addition of the discrete spectrum. *J. Math. Phys., Anal., Geom.*, 3(2):176–195, 2007.

[31] E.I. Zubkova and F.S. Rofe-Beketov. Necessary and sufficient conditions in inverse scattering problem on the axis for the triangular 2×2 matrix potential. *J. Math. Phys., Anal., Geom.*, 5(3):296–309, 2009.

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