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Approximation of some classes of holomorphic functions and properties of generating kernels

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Abstract. Let $\psi = \sum_{k=0}^{\infty} \widehat{\psi}_k z^k$ be a holomorphic function in the unit disk and $H_p^{\psi} := \{f = g * \psi : \|g\|_{H_p} \le 1\}$ be a functional class with generating kernel ψ (under Hadamard convolution $*$). We construct some method for approximation of holomorphic functions from H_p^{ψ}. We explore the question of when the introduced method will be best linear method of approximation on H_p^{ψ}. In this case we also find an asymptotic formula for the upper bounds of deviations of partial sums of Taylor's series on the classes H_p^{ψ}. Some interesting properties of generating kernels ψ are indicated.

Key Words and Phrases: Best linear method of approximation, Approximation by partial sums, Hardy space, Bergman space, Generating kernels, Functions with positive real part
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1. Introduction

Let $\mathcal{H}$ be a set of functions holomorphic in the disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

If functions $g(z) = \sum_{k=0}^{\infty} \widehat{g}_k z^k$ and $\psi(z) = \sum_{k=0}^{\infty} \widehat{\psi}_k z^k$ belong to $\mathcal{H}$, then the sum of the series

$$(g * \psi)(z) := \sum_{k=0}^{\infty} \widehat{g}_k \widehat{\psi}_k z^k, \quad \widehat{g}_k := \frac{g^{(k)}(0)}{k!},$$

defines a function from $\mathcal{H}$ and is called the Hadamard convolution (product) of the functions f and g.

For a given function $\psi \in \mathcal{H}$ and $p \in [1, \infty]$ we define the class H_p^{ψ} as follows

$$H_p^{\psi} := \{f = g * \psi : \|g\|_{H_p} \le 1\},$$

where H_p is a Hardy space endowed with the norm

$$\|g\|_{H_p} := \begin{cases} \sup\limits_{0 \le \varrho < 1} \left(\displaystyle\int_0^{2\pi} \left| g(\varrho e^{it}) \right|^p \frac{dt}{2\pi} \right)^{1/p}, & 1 \le p < \infty, \\ \sup\limits_{z \in \mathbb{D}} |g(z)|, & p = \infty. \end{cases}$$

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We say that a function $\psi \in \mathcal{H}$ is the generating kernel of the class H_p^ψ if $|\widehat{\psi}_k| > 0$ for all $k \in \mathbb{Z}_+$.

It is clear that if ψ is the generating kernel and $f \in H_p^\psi$, then the function

$$f^\psi(z) := \sum_{k=0}^{\infty} \frac{\widehat{f}_k}{\widehat{\psi}_k} z^k \in \mathcal{H},$$

$\left\|f^\psi\right\|_{H_p} \leq 1$ and $f = f^\psi * \psi$.

The aim of this paper is to investigate the polynomial approximations of functions from classes H_p^ψ. More precisely, we find out what properties of generating kernels ψ influence on the rate of approximation on class H_p^ψ. For this reason we construct the polynomial linear method of approximation $U_{n,\mu}$ and find exact values of the upper bounds for the deviations of $U_{n,\mu}(f)$ from functions $f \in H_p^\psi$ in the norm $\|\cdot\|_{H_p}$.

The paper is written according to the following scheme. In Sec. 2, we consider the sequence of linear operators $\{U_{n,\mu}\}$ that is acting from H_p^ψ into the set $\mathcal{P}_n$ of algebraic polynomials of degree n at most. These operators essentially are the sequence of multipliers of partial sums of Taylor series generated by a fixed sequence of complex numbers depending on the kernel ψ.

In Sec. 3, we introduce the set $\mathcal{R}_n$ of generating kernels ψ. It is shown that the best polynomial approximation on H_∞^ψ coincides with the best linear approximation and to take a minimum iff $\psi \in \mathcal{R}_n$.

In Sec. 4, we indicate how the techniques developed in previous sections may be used to study the rate of convergence of the Taylor series for functions from the classes H_p^ψ.

In Sec. 5, we give certain properties of generating kernels from $\mathcal{R}_n$. The result is closely related to the properties of holomorphic functions with positive real part. They also have of independent interest.

2. On some linear method of approximation of holomorphic functions

Suppose ψ is a generating kernel and denote

$$m_0 := \inf_{z\in\mathbb{D}} \operatorname{Re} \frac{\psi(z)}{\widehat{\psi}_0},\ m_n := \inf_{z\in\mathbb{D}} \operatorname{Re} \frac{\psi(z) - \sum_{k=0}^{n-1} \widehat{\psi}_k z^k}{\widehat{\psi}_n z^n},\ n \in \mathbb{N}.$$

Further assume that $\inf_{n\in\mathbb{Z}_+} m_n > -\infty$.

Let $\mu := \{\mu_n\}_{n=0}^\infty$ be a minorant sequence for $\mathbf{m} := \{m_n\}_{n=0}^\infty$ i.e. $\mu_n \leq m_n$ for all $n \in \mathbb{Z}_+$.

Consider the sequence of linear operators $\{U_{n,\mu}\}_{n=0}^\infty$ defined on H_p^ψ and acting according to the rule

$$U_0(f) = (2\mu_0 - 1)\widehat{f}_0,$$

$$U_{n,\mu}(f)(z) = \sum_{k=0}^{n-1}\left(1 - \frac{\overline{\widehat{\psi}_{2n-k}}}{\widehat{\psi}_k}\; e^{2i\arg\psi_n}\right)\widehat{f}_k z^k + (2\mu_n - 1)\widehat{f}_n z^n,\ n \in \mathbb{N}.$$

Theorem 1. *Let* $\mathbf{m} = \{m_k\}_{k=0}^{\infty}$ *be as above,* $1 \le q \le p \le \infty$*. Then for any minorant sequence* μ *of* $\mathbf{m}$

$$\max_{f\in H_p^{\psi}} \|f - U_{k,\mu}(f)\|_{H_q} = 2(1-\mu_k)\left|\widehat{\psi}_k\right|, \quad \forall\ k \in \mathbb{Z}_+. \tag{1}$$

For each $k \in \mathbb{Z}_+$ *the maximum is attained for a function* $f_k(z) = \omega\widehat{\psi}_k z^k$, $|\omega| = 1$.

To prove of (1) we use the following lemma due to Goluzin [6, pp. 515. 516].

Lemma 1. *Let* $g \in H_1$ *and* $\sum_{\nu=0}^{\infty} c_\nu z^\nu \in \mathcal{H}$, $c_0 = 1$*. Then for all* $k \in \mathbb{N}$ *and* $z \in \mathbb{D}$

$$\int_0^{2\pi} g(e^{it})e^{-ikt}\left(1 + 2\,\mathrm{Re}\sum_{\nu=1}^{\infty} c_\nu z^\nu e^{-i\nu t}\right)\frac{dt}{2\pi} = \sum_{\nu=0}^{k-1}\widehat{g}_\nu \overline{c}_{k-\nu}\overline{z}^{k-\nu} + \sum_{\nu=k}^{\infty}\widehat{g}_\nu c_{\nu-k} z^{\nu-k}.$$

Proof. Suppose $f = g * \psi$, so is $g = f^{\psi}$. If we set in lemma $c_\nu = \widehat{\psi}_{\nu+k}/\widehat{\psi}_k$ we get

$$\sum_{\nu=0}^{k-1}\widehat{g}_\nu \overline{\left(\frac{\widehat{\psi}_{2k-\nu}}{\widehat{\psi}_k}\right)}\overline{z}^{k-\nu} + \sum_{\nu=k}^{\infty}\widehat{g}_\nu \frac{\widehat{\psi}_\nu}{\widehat{\psi}_k} z^{\nu-k} = \int_0^{2\pi} g(e^{it})e^{-ikt}\left(1 + 2\,\mathrm{Re}\sum_{\nu=1}^{\infty}\frac{\widehat{\psi}_{k+\nu}}{\widehat{\psi}_k} z^\nu e^{-i\nu t}\right)\frac{dt}{2\pi},$$

which is equivalent to

$$f(z) - \sum_{\nu=0}^{k-1}\left(1 - \frac{\overline{\widehat{\psi}_{2k-\nu}}}{\widehat{\psi}_\nu}\; e^{2i\arg\widehat{\psi}_k}|z|^{2(k-\nu)}\right)\widehat{f}_\nu z^\nu =$$
$$= \widehat{\psi}_k z^k \int_0^{2\pi} g(e^{it})e^{-ikt}\left(2\,\mathrm{Re}\sum_{\nu=0}^{\infty}\frac{\widehat{\psi}_{k+\nu}}{\widehat{\psi}_k} z^\nu e^{-i\nu t} - 1\right)\frac{dt}{2\pi}.$$

Combining this with the formula

$$(2\mu_k - 1)\widehat{f}_k z^k = \widehat{\psi}_k z^k \int_0^{2\pi} g(e^{it})e^{-ikt}(2\mu_k - 1)\frac{dt}{2\pi},$$

we get

$$f(z) - \sum_{\nu=0}^{k-1}\left(1 - \frac{\overline{\widehat{\psi}_{2k-\nu}}}{\widehat{\psi}_\nu}\; e^{2i\arg\widehat{\psi}_k}|z|^{2(k-\nu)}\right)\widehat{f}_\nu z^\nu - (2\mu_k - 1)\widehat{f}_k z^k =$$
$$= 2\widehat{\psi}_k z^k \int_0^{2\pi} g(e^{it})e^{-ikt}\left(\mathrm{Re}\sum_{\nu=0}^{\infty}\frac{\widehat{\psi}_{k+\nu}}{\widehat{\psi}_k} z^\nu e^{-i\nu t} - \mu_k\right)\frac{dt}{2\pi}.$$

Applying Minkowski's integral inequality, we obtain

$$\left\| f(\varrho\,\cdot) - \sum_{\nu=0}^{k-1} \left(1 - \frac{\overline{\widehat{\psi}_{2k-\nu}}}{\widehat{\psi}_\nu}\; e^{2i\arg\widehat{\psi}_k} \varrho^{2(k-\nu)}\right) \widehat{f}_\nu (\varrho\,\cdot)^\nu - (2\mu_k - 1)\widehat{f}_k(\varrho\,\cdot)^k \right\|_{L_p} =:$$

$$=: I(\varrho) \le 2(1-\mu_k)\varrho^k \left|\widehat{\psi}_k\right|, \quad \forall\ \varrho \in [0,1),$$

where $\|\cdot\|_{L_p}$ is usual L_p-norm on the interval $[0, 2\pi]$.

Since for any function $f \in H_p^\psi$, $\left|\widehat{f}_\nu\right| \le \left|\widehat{\psi}_\nu\right|$, we have

$$\|f(\varrho\,\cdot) - U_{k,\mu}(f)(\varrho\,\cdot)\|_{L_p} \le$$

$$\le I(\varrho) + \left\| \sum_{\nu=0}^{k-1} (1-\varrho^{2(k-\nu)}) \frac{\overline{\widehat{\psi}_{2k-\nu}}}{\widehat{\psi}_\nu} e^{2i\arg\widehat{\psi}_k} \widehat{f}_\nu (\varrho\,\cdot)^\nu \right\|_{L_p} \le$$

$$\le 2(1-\mu_k)\varrho^k \left|\widehat{\psi}_k\right| + (1-\varrho^{2k}) \sum_{\nu=k+1}^{2k} \left|\widehat{\psi}_\nu\right|.$$

Passing to the limit as $\varrho \to 1-$ in this inequality and applying the Riesz theorem (see, for example, [6, Ch. XI]) $\|f(\cdot) - f(\varrho\,\cdot)\|_{L_p} \to 0$, $\varrho \to 1-$, we obtain the relation

$$\|f - U_{k,\mu}(f)\|_{H_p} \le 2(1-\mu_k)\left|\widehat{\psi}_k\right|.$$

This yields an upper bound for

$$\sup_{f\in H_p^\psi} \|f - U_{k,\mu}(f)\|_{H_q}.$$

For a lower bound we take the function $f_k(z) = \omega\widehat{\psi}_k z^k$, $|\omega| = 1$. It is readily verified that $U_{k,\mu}(f_k)(z) = \omega(2\mu_k - 1)\widehat{\psi}_k z^k$. Thus

$$\|f_k - U_{k,\mu}(f_k)\|_{H_p} = 2(1-\mu_k)\left|\widehat{\psi}_k\right|.$$

◀

Remark 1. *For a function* $f_*(z) = \widehat{\psi}_{n+1} z^{n+1}$ *we have* $f_*(z) - U_{n,\mathbf{m}}(f_*)(z) = \widehat{\psi}_{n+1} z^{n+1}$. *Thus*

$$\left|\widehat{\psi}_{n+1}\right| = \|f_* - U_{n,\mathbf{m}}(f_*)\|_{H_p} \le \max_{f\in H_p^\psi} \|f - U_{n,\mathbf{m}}(f)\|_{H_p} = 2(1-m_n)\left|\widehat{\psi}_n\right|.$$

It follows that

$$\mu_n \le m_n \le 1 - \frac{\left|\widehat{\psi}_{n+1}\right|}{2\left|\widehat{\psi}_n\right|} < 1, \quad \forall\ n \in \mathbb{Z}_+.$$

Remark 2. *For any generating kernel*

$$m_n = \inf_{w\in\mathbb{D}} \left(1 + \sum_{k=n+1}^{\infty} \left|\frac{\widehat{\psi}_k}{\widehat{\psi}_n}\right| |w|^{k-n} \cos(\theta_k + (k-n)t)\right) \geq$$

$$\geq 1 - \sum_{k=n+1}^{\infty} \left|\frac{\widehat{\psi}_k}{\widehat{\psi}_n}\right|, \quad \forall\ n \in \mathbb{Z}_+,$$

where $\theta_k = \arg\widehat{\psi}_k - \arg\widehat{\psi}_n$ *and* $t = \arg w$.

Remark 3. *Suppose* ψ *is a generating kernel. Then* $m_n \to 1$ *as* $n \to \infty$ *if and only if* $\left|\widehat{\psi}_{n+1}\right| / \left|\widehat{\psi}_n\right| \to 0$, $n \to \infty$.

Corollary 1. *Let* $\mathbf{m} = \{m_n\}_{n=0}^{\infty}$ *be as above,* $1 \leq p \leq \infty$. *If*

$$M := \sup_{n\in\mathbb{Z}_+} \sum_{k=n+1}^{\infty} \left|\frac{\widehat{\psi}_k}{\widehat{\psi}_n}\right| < \infty,$$

then

$$m := \inf_{n\in\mathbb{Z}_+} m_n \geq 1 - M,$$

and

$$2(1-m)\left|\widehat{\psi}_n\right| \leq \max_{f\in H_p^\psi} \|f - U_{n,\mu}(f)\|_{H_p} = 2\sum_{k=n+1}^{\infty} \left|\widehat{\psi}_k\right| \leq 2M\left|\widehat{\psi}_n\right|, \quad \forall\ k \in \mathbb{Z}_+,$$

where

$$\mu_n = 1 - \sum_{k=n+1}^{\infty} \left|\frac{\widehat{\psi}_k}{\widehat{\psi}_n}\right|,\ n \in \mathbb{N}.$$

3. Best approximation

Let $\mathcal{P}_n$ be a set of algebraic polynomials of degree $n, n \in \mathbb{Z}_+$, at most.

The quantity

$$E_0(H_p^\psi; H_q) := \sup_{f\in H_p^\psi} \|f\|_{H_q},$$

$$E_n(H_p^\psi; H_q) := \sup_{f\in H_p^\psi} \inf_{P_{n-1}\in\mathcal{P}_{n-1}} \|f - P_{n-1}\|_{H_q},\ n \in \mathbb{N},$$

is called the best polynomial approximation of order n of the class H_p^ψ in the space H_q.

Consider the sequence of linear operators $\{T_n\}_{n=0}^{\infty}$ defined on $\mathcal{H}$ and acting according to the rule

$$T_0(f)(z) = 0,\ T_n(f)(z) = T_{n,\Lambda}(f)(z) = \sum_{k=0}^{n-1} \lambda_{k,n}\widehat{f}_k z^k,\ n \in \mathbb{N},$$

where $\lambda_{k,n}$ are elements of the infinite lower triangular matrix $\Lambda := \{\lambda_{k,n}\}$, $n \in \mathbb{N}$, $k = 0, 1, \ldots, n-1$, over the field of complex numbers.

The quantity

$$\mathcal{E}_n(H_p^\psi; H_q) := \inf_\Lambda \sup_{f \in H_p^\psi} \|f - T_{n,\Lambda}(f)\|_{H_q},$$

where the lower bound is taken over the set of all lower triangular numerical matrices Λ, is called the best linear approximation of the class H_p^ψ in the space H_q. If there exists a matrix Λ^* that generates a sequence of operators $\{T_n^*\}_{n=0}^\infty$ such that

$$\sup_{f \in H_p^\psi} \|f - T_n^*(f)\|_{H_q} = \mathcal{E}_n(H_p^\psi; H_q),$$

then one says that the matrix Λ^* generates the best linear method of approximation of the class H_p^ψ in the space H_q.

It is easy to see that for any generating kernel ψ

$$\left|\widehat{\psi}_n\right| \le E_n(H_p^\psi; H_p) \le \mathcal{E}_n(H_p^\psi; H_p), \quad n = 0, 1, 2, \ldots. \tag{2}$$

It follows from (1) that equality occurs here if $m_n \ge 1/2$. Indeed

$$\mathcal{E}_n(H_p^\psi; H_p) \le \max_{f \in H_p^\psi} \|f - U_{n,\mu}(f)\|_{H_p} = \left|\widehat{\psi}_n\right|,$$

for $\mu_n = 1/2$.

Moreover, for $p = \infty$ equality in (2) implies a relation $m_n \ge 1/2$.

Set

$$\mathcal{R}_n := \left\{\psi \in \mathcal{H} : \inf_{k \ge n} m_k \ge 1/2\right\}.$$

Theorem 2. *Suppose ψ is a generating kernel and $\mathbf{m} = \{m_n\}_{n=0}^\infty$ is as above. Then:*

i)

$$\psi \in \mathcal{R}_n \Longleftrightarrow E_k\left(H_\infty^\psi; H_\infty\right) = \mathcal{E}_k\left(H_\infty^\psi; H_\infty\right) = \left|\widehat{\psi}_k\right|, \quad \forall\ k \ge n;$$

ii) for $1 \le q \le p < \infty$

$$\psi \in \mathcal{R}_n \Longrightarrow E_k\left(H_p^\psi; H_q\right) = \mathcal{E}_k\left(H_p^\psi; H_q\right) = \left|\widehat{\psi}_k\right|, \quad \forall\ k \ge n.$$

This statement is known. First time it was formulated without proof in [4].

From the Theorem 2, we can easily deduce the results of Babenko [1], Taikov [17], [16], [18], Scheick [13], Belyi and Dveirin [3], where the quantities $E_n(H_p^\psi, H_p)$ and $\mathcal{E}_n(H_p^\psi, H_p)$ were evaluated on the classes H_p^ψ for specific values of the parameter ψ and in the general case.

It should be noted that the statement *i)* is essentially contained in a Goluzin's theorem [6, p. 515] obtained in 1950, but this fact was not noticed.

Remark 4. *It was shown in [12] that for $\psi \in \mathcal{R}_n$ the linear method $U_{n,\mu}$ with $\mu_n = 1/2$ is unique best linear method for the approximation of the classes H^ψ_∞.*

Proof. Clearly, it is sufficient to prove the implication " $\Leftarrow$ ".
Set $H^\psi_{\infty,0} = H^\psi_\infty$ and

$$H^\psi_{\infty,k} := \left\{ f \in H^\psi_\infty : \widehat{f}_\nu = 0, \nu = 0, 1, \dots, k-1 \right\}, \ k \geq n.$$

Fix $k \geq n$ and consider the function $f_k = g * \psi$, where

$$g(z) = z^k \frac{z-\alpha}{1-\overline{\alpha}z}, \ |\alpha| < 1.$$

It is easy to verify that $f_k \in H^\psi_{\infty,k}$ and

$$f_k(z) = -\widehat{\psi}_k \alpha z^k + (1-|\alpha|^2) \sum_{\nu=k+1}^{\infty} \widehat{\psi}_\nu \overline{\alpha}^{\nu-k-1} z^\nu.$$

It follows from Schwarz lemma that $|f_k(z)| \leq |z|^k \|f_k\|_{H_\infty}$. On the other hand, according to the duality relation (see, for example, [9, p. 25, 81]),

$$\|f_k\|_{H_\infty} \leq \max_{h \in H^\psi_{\infty,k}} \|h\|_{H_\infty} = E_k(H^\psi_\infty, H_\infty) = \left|\widehat{\psi}_k\right|.$$

Thus, $|f_k(z)| \leq |z|^k \left|\widehat{\psi}_k\right|$ for all $z \in \mathbb{D}$, or equivalently

$$\left| -\alpha + \frac{1-|\alpha|^2}{\widehat{\psi}_n z^n} \sum_{\nu=n+1}^{\infty} \widehat{\psi}_\nu \overline{\alpha}^{\nu-n-1} z^\nu \right| \leq 1.$$

Exponentiating both parts of this inequality, we obtain

$$|\alpha|^2 - 2\,\mathrm{Re}\left(\frac{1-|\alpha|^2}{\widehat{\psi}_n z^n} \sum_{\nu=n+1}^{\infty} \widehat{\psi}_\nu \overline{\alpha}^{\nu-n} z^\nu \right) + \left| \frac{1-|\alpha|^2}{\widehat{\psi}_n z^n} \sum_{\nu=n+1}^{\infty} \widehat{\psi}_\nu \overline{\alpha}^{\nu-n-1} z^\nu \right|^2 \leq 1,$$

and consequently

$$-2\,\mathrm{Re}\left(\sum_{\nu=1}^{\infty} \frac{\widehat{\psi}_{\nu+n}}{\widehat{\psi}_n} \overline{\alpha}^\nu z^\nu \right) + (1-|\alpha|^2) \left| \sum_{\nu=n+1}^{\infty} \frac{\widehat{\psi}_\nu}{\widehat{\psi}_n} \overline{\alpha}^{\nu-n-1} z^{\nu-n} \right|^2 \leq 1.$$

Passing to the limit as $|\alpha| \to 1-$, we obtain

$$2\,\mathrm{Re}\left(\sum_{\nu=0}^{\infty} \frac{\widehat{\psi}_{\nu+n}}{\widehat{\psi}_n} e^{i\nu\theta} z^\nu \right) \geq 1, \quad \forall\ z \in \mathbb{D},\ \theta \in [0, 2\pi].$$

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4. Approximation by partial sums

In this section we apply a previous results to examine a remainder of Taylor series on classes H_p^ψ. More precisely, we consider the quantity

$$R_k\left(H_p^\psi; H_q\right) := \sup_{f\in H_p^\psi} \|f - S_k(f)\|_{H_q},$$

where

$$S_0(f)(z) = 0,\ S_k(f)(z) := \sum_{\nu=0}^{k-1} \widehat{f}_\nu z^\nu,\ k\in\mathbb{N}.$$

The following is immediate from Theorem 2.

Theorem 3. *Suppose* $1 \le q \le 2 \le p \le \infty$. *If* $\psi \in \mathcal{R}_n$, $n\in\mathbb{Z}_+$, *then*

$$R_k\left(H_p^\psi; H_q\right) = \left|\widehat{\psi}_k\right| \quad \forall\ k \ge n.$$

Proof. By Hölder inequality and Theorem 2, we get

$$\|f - S_k(f)\|_{H_q} \le \|f - S_k(f)\|_{H_2} \le E_k(H_p^\psi; H_2) \le$$

$$\le E_k(H_p^\psi; H_2) \le \left|\widehat{\psi}_k\right|.$$

The inequalities are sharp. They become equalities for $f(z) = \omega\widehat{\psi}_k z^k$, $|\omega| = 1$.

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Let UH_p be a unit ball of Hardy space. Set

$$G_0(\psi)_{p,q} := 0,\ G_k(\psi)_{p,q} := \sup_{g\in UH_p} \left\| \sum_{\nu=0}^{k-1} \overline{\widehat{\psi}_{2k-\nu}}\widehat{g}_\nu z^\nu \right\|_{H_q},\ k\in\mathbb{N},$$

and

$$G_{0,p,q} := 0,\ G_{k,p,q} := \sup_{g\in UH_p} \|S_k(g)\|_{H_q}.$$

The quantity $G_{k,\infty,\infty}$ is called the Landau constant. It is well known [10] that

$$G_{k,p,p} = \frac{1}{\pi}\ln k + O(1), \quad k\to\infty,\ p = 1,\infty. \tag{3}$$

Suppose $\psi \in \mathcal{R}_n$, $n\in\mathbb{Z}_+$. For a given function $f \in H_p^\psi$, $f = g*\psi$, consider a polynomial $U_{k,\mu}(f), k \ge n$, where $\mu_n = 1/2$. It is easy to see that

$$f(z) - S_k(f)(z) = f(z) - U_{k,\mu}(f)(z) - e^{2i\arg\widehat{\psi}_k} \sum_{\nu=0}^{k-1} \frac{\overline{\widehat{\psi}_{2k-\nu}}}{\widehat{\psi}_\nu}\widehat{f}_\nu z^\nu.$$

Hence, according to Theorems 1 and 2, we have

$$R_k\left(H_p^{\psi};H_q\right)\le \max_{f\in H_p^{\psi}}\|f-U_{k,\mu}(f)\|_{H_q}+G_k(\psi)_{p,q}=\left|\widehat{\psi}_k\right|+G_k(\psi)_{p,q}.$$

On the other hand

$$G_k(\psi)_{p,q}\le R_k\left(H_p^{\psi};H_q\right)+\left|\widehat{\psi}_k\right|.$$

Summing these relations we get

Theorem 4. *Suppose* $1\le q\le p\le\infty$. *If* $\psi\in\mathcal{R}_n$, $n\in\mathbb{Z}_+$, *then*

$$\left|G_k(\psi)_{p,q}-\left|\widehat{\psi}_k\right|\right|\le R_k\left(H_p^{\psi};H_q\right)\le G_k(\psi)_{p,q}+\left|\widehat{\psi}_k\right|,\ \forall\ k\ge n.$$

Corollary 2. *Suppose* $p=1,\infty$. *If* $\psi\in\mathcal{R}_n$, $n\in\mathbb{Z}_+$, *and*

$$\lim_{k\to\infty}\left|\frac{\widehat{\psi}_{2k}}{\widehat{\psi}_k}\right|=1,$$

then

$$R_k\left(H_p^{\psi};H_p\right)=\frac{1}{\pi}\left|\widehat{\psi}_k\right|\ln n+O\Bigg(\left|\widehat{\psi}_k\right|\Bigg),\ k\to\infty.$$

Such an asymptotic relation was obtained in [15] in a case when sequences $\left\{\operatorname{Re}\widehat{\psi}_\nu\right\}_{\nu=0}^{\infty}$ and $\left\{\operatorname{Im}\widehat{\psi}_\nu\right\}_{\nu=0}^{\infty}$ are convex and decrease to zero. Under the same conditions and in addition when $\operatorname{Im}\widehat{\psi}_\nu=0$, simple proof was suggested by Babenko [1].

In the case when $\widehat{\psi}_\nu=\nu!/(\nu+r)!, r\in\mathbb{Z}_+$, this equality was obtained by Stechkin [14].

Proof. Setting in lemma $c_\nu=\widehat{\psi}_{k+\nu+1}/\widehat{\psi}_{k+1}$, $k:=k-1$ and $g:=S_k(g)$, we have

$$\sum_{\nu=0}^{k-1}\overline{\widehat{\psi}_{2k-\nu}}\widehat{g}_\nu e^{i\nu\theta}=$$

$$=\overline{\widehat{\psi}_{k+1}}e^{i(k-1)\theta}\int_0^{2\pi}S_k\left(g\right)(e^{i(t+\theta)})e^{-i(k-1)t}\left(2\operatorname{Re}\sum_{\nu=0}^{\infty}\frac{\widehat{\psi}_{k+\nu+1}}{\widehat{\psi}_{k+1}}e^{-i\nu t}-1\right)\frac{dt}{2\pi}.$$

Applying Minkowski's integral inequality and (3), we obtain

$$G_k(\psi)_{p,q}\le\left|\widehat{\psi}_{k+1}\right|G_{k,p,q}=\left|\widehat{\psi}_{k+1}\right|\left(\frac{1}{\pi}\ln k+O(1)\right).$$

Observe that for $|z|=1$

$$\left|\sum_{\nu=0}^{k-1}\overline{\widehat{\psi}_{2k-\nu}}\widehat{g}_\nu z^\nu\right|=\left|\sum_{\nu=0}^{k-1}\overline{\widehat{\psi}_{k+\nu+1}}\widehat{g}_{k-\nu-1}z^\nu\right|.$$

Thus, applying in consecutive order the Hölder's and Hardy's inequalities, we get

$$G_k(\psi)_{\infty,\infty} \ge G_k(\psi)_{1,1} \ge$$
$$\ge \frac{1}{\pi} \sup_{g\in UH_1} \sum_{\nu=0}^{k-1} \frac{\left|\widehat{\psi}_{k+\nu+1}\widehat{g}_{k-\nu-1}\right|}{\nu+1} \ge \frac{1}{\pi}\left|\widehat{\psi}_{2k}\right| \sup_{g\in UH_1} \sum_{\nu=0}^{k-1} \frac{|\widehat{g}_{k-\nu-1}|}{\nu+1}. \tag{4}$$

Let $k \ge 2$. Consider a function

$$g(z) = z^{k-1} \sum_{\nu=-k+1}^{k-1} \left(1-\frac{|\nu|}{k}\right) z^\nu = \sum_{\nu=0}^{k-1} \frac{\nu+1}{k} z^\nu + \sum_{\nu=k}^{2k-2} \left(2-\frac{\nu+1}{k}\right) z^\nu.$$

It easy to see that

$$\|g\|_{H_1} = \int_0^{2\pi} \left| \sum_{\nu=-k+1}^{k-1} \left(1-\frac{|\nu|}{k}\right) e^{i\nu t} \right| \frac{dt}{2\pi} = \|F_k\|_{L_1} = 1,$$

where $F_k(t) := 1 + 2\sum_{\nu=1}^{k-1}(1-k/n)\cos\nu t$ is a Fejer kernel. Thus, $g \in UH_1$.

We have

$$\sum_{\nu=0}^{k-1} \frac{|\widehat{g}_{k-\nu-1}|}{\nu+1} = \frac{1}{k}\sum_{\nu=0}^{k-1} \frac{k-\nu}{\nu+1} \ge \sum_{\nu=1}^{k-1} \frac{1}{\nu}.$$

Combining this with (4), we obtain

$$G_k(\psi)_{p,p} \ge \left|\widehat{\psi}_{2k}\right| \frac{1}{\pi} \sum_{\nu=1}^{k-1} \frac{1}{\nu} > \left|\widehat{\psi}_{2k}\right| \frac{1}{\pi} \ln k, \quad p = 1, \infty.$$

Hence

$$\lim_{k\to\infty} \frac{G_k(\psi)_{p,p}}{\frac{1}{\pi}\left|\widehat{\psi}_k\right| \ln k} = \lim_{k\to\infty} \frac{\left|\widehat{\psi}_{2k}\right|}{\left|\widehat{\psi}_k\right|} = 1$$

and result follows.

◀

Let A_p denote the Bergman space of holomorphic functions in $\mathbb{D}$ with finite norm

$$\|f\|_{A_p} = \left(\int_0^1 \int_0^{2\pi} \left|f(\varrho e^{it})\right|^p \frac{dt}{\pi}\, \varrho d\varrho\right)^{1/p}, \quad 1 \le p < \infty.$$

Set

$$r_n(\psi)(z) = \psi(z) - S_n(\psi)(z), \quad n \in \mathbb{Z}_+.$$

Theorem 5. *Suppose $n \in \mathbb{Z}_+$ and $\psi \in \mathcal{R}_n$. Then*

$$\sup_{\psi\in\mathcal{R}_n} \frac{\|r_k(\psi)\|_{A_1}}{\left|\widehat{\psi}_k\right|} = \frac{2}{\pi}\frac{\ln(k+2)}{k+1} + O\left(\frac{1}{k+1}\right), \quad \forall\ k \ge n.$$

Proof. By the Riesz–Herglotz theorem

$$2\frac{r_k(\psi)(z)}{\widehat{\psi}_k z^k} - 1 = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu_k(t), \quad \forall\, z \in \mathbb{D},$$

where μ_k is a positive measure on $[0, 2\pi)$ with total variation =1.
Since

$$\frac{1+z}{1-z} + 1 = \frac{2}{1-z},$$

we see that

$$r_k(\psi)(z) = \widehat{\psi}_k z^k \int_0^{2\pi} \frac{1}{e^{it}-z} d\mu_k(t) \quad \forall\, z \in \mathbb{D}.$$

Hence

$$\int_0^{2\pi} \left| r_k(\psi)(\varrho e^{i\theta}) \right| \frac{d\theta}{2\pi} = \varrho^k \left| \widehat{\psi}_k \right| \int_0^{2\pi} \left| \int_0^{2\pi} \frac{1}{1-\varrho e^{i(\theta - t)}} d\mu_k(t) \right| \frac{d\theta}{2\pi} \leq$$
$$\leq \varrho^k \left| \widehat{\psi}_k \right| \int_0^{2\pi} \frac{1}{|1-\varrho e^{it}|} \frac{dt}{2\pi} = \varrho^k \left| \widehat{\psi}_k \right| \left(1 + \sum_{\nu=1}^{\infty} \left(\frac{(2\nu-1)!!}{(2\nu)!!} \right)^2 \varrho^{2\nu} \right).$$

It is known [7], [8], that

$$\left(\frac{(2\nu-1)!!}{(2\nu)!!} \right)^2 = \frac{1}{\pi(\nu + \varepsilon(\nu))},$$

where $\varepsilon(\nu)$ satisfies

$$\frac{1}{4} < \varepsilon(\nu) < \frac{1}{2}, \ \nu = 1, 2, \ldots.$$

Thus

$$\int_0^{2\pi} \left| r_k(\psi)(\varrho e^{i\theta}) \right| \frac{d\theta}{2\pi} < \varrho^k \left| \widehat{\psi}_k \right| \left(1 + \frac{1}{\pi} \sum_{\nu=1}^{\infty} \frac{\varrho^\nu}{\nu} \right).$$

Integrating this inequality on $[0, 1)$, we obtain

$$\|r_k(\psi)\|_{A_1} = 2 \int_0^1 \int_0^{2\pi} \left| r_k(\psi)(\varrho e^{it}) \right| \frac{dt}{2\pi} \varrho d\varrho <$$
$$< 2 \left| \widehat{\psi}_k \right| \int_0^1 \left(\varrho^{k+1} + \frac{1}{\pi} \sum_{\nu=1}^{\infty} \frac{\varrho^{\nu+k+1}}{\nu} \right) d\varrho = 2 \left| \widehat{\psi}_k \right| \left(\frac{1}{k+2} + \frac{1}{\pi} \sum_{\nu=1}^{\infty} \frac{1}{\nu(\nu+k+2)} \right).$$

Applying to the last sum the formula (see [2, pp. 15,16])

$$\sum_{\nu=1}^{\infty} \frac{k+2}{\nu(\nu+k+2)} = \frac{\Gamma'(k+2)}{\Gamma(k+2)} + \gamma + \frac{1}{k+2} = \sum_{\nu=1}^{k+2} \frac{1}{\nu}, \tag{5}$$

where γ is a Euler's constant, we obtain the upper estimate

$$\frac{1}{\left| \widehat{\psi}_k \right|} \|r_k(\psi)\|_{A_1} < \frac{2}{\pi(k+2)} \left(\sum_{\nu=1}^{k+2} \frac{1}{\nu} + \pi \right) < \frac{2}{\pi(k+1)} (\ln(k+2) + \pi + 1).$$

For lower estimate we consider function

$$\psi^*(z) := S_n(\psi)(z) + \frac{\widehat{\psi}_n z^n}{1-z}.$$

Obvious $\widehat{\psi}^*_k = \widehat{\psi}_k$ for all $k \geq n$ and

$$2\,\mathrm{Re}\,\frac{r_k(\psi^*)(z)}{\widehat{\psi}^*_k z^k} - 1 = 2\,\mathrm{Re}\,\frac{1}{1-z} - 1 = \frac{1-|z|^2}{|1-z|^2} \geq 0, \quad \forall\ z \in \mathbb{D}.$$

Thus $\psi^* \in \mathcal{R}_n$.

Using the Hardy's inequality and formula (5), we get

$$\|r_k(\psi^*)\|_{A_1} = \left|\widehat{\psi}_k\right| \frac{1}{\pi}\int_0^1\int_0^{2\pi} \frac{\varrho^k}{|1-\varrho e^{it}|} dt\varrho d\varrho \geq \frac{2}{\pi}\left|\widehat{\psi}_k\right| \int_0^1 \sum_{\nu=0}^{\infty} \frac{\varrho^{\nu+k+1}}{\nu+1} d\varrho =$$

$$= \frac{2}{\pi}\left|\widehat{\psi}_k\right| \sum_{\nu=1}^{\infty} \frac{1}{\nu(\nu+n+1)} = \frac{2}{\pi}\left|\widehat{\psi}_k\right| \frac{1}{k+1}\sum_{\nu=1}^{k+1}\frac{1}{\nu} > \frac{2}{\pi}\left|\widehat{\psi}_k\right| \frac{\ln(k+2)}{k+1}.$$

◀

For a given function $\psi \in \mathcal{R}_n$ and $p \in [1,\infty]$ we define the class A_p^ψ as follows

$$A_p^\psi := \{f = g * \psi : \|g\|_{A_p} \leq 1\}.$$

Set

$$D(\psi)(z) := (z\psi(z))' = \sum_{k=0}^{\infty}(k+1)\widehat{\psi}_k z^k.$$

Corollary 3. *Suppose $D\psi \in \mathcal{R}_n$, $n \in \mathbb{Z}_+$. Then*

$$R_k(A_1^\psi; A_1) \leq \frac{2}{\pi}\left|\widehat{\psi}_k\right| \ln(k+2) + O\left(\left|\widehat{\psi}_k\right|\right), \quad \forall\ k \geq n. \tag{6}$$

Proof. We have

$$r_k(f)(z) = \int_0^1\int_0^{2\pi} f^\psi(\varrho e^{it}) r_k(D(\psi))(z\varrho e^{-it})\frac{dt}{\pi}\ \varrho d\varrho, \quad z \in \mathbb{D}.$$

Using the Minkowski's integral inequality, we obtain the estimate

$$\|r_k(f)\|_{A_1} \leq \left\|f^\psi\right\|_{A_1} \|r_k(D(\psi))\|_{A_1} \leq \|r_k(D(\psi))\|_{A_1}.$$

Since by theorem 5

$$\|r_k(D(\psi))\|_{A_1} \leq \frac{2}{\pi}\left|\widehat{\psi}_k\right| \ln(k+2) + O\left(\left|\widehat{\psi}_k\right|\right),$$

(6) follows.

◀

5. Properties of generating kernels from class $\mathcal{R}_n$

Proposition 1. *Suppose $n \in \mathbb{Z}_+$. The following statements are equivalent:*
i) $\psi \in \mathcal{R}_n$;
ii) $|r_n(\psi)(z)| \geq |r_{n+1}(\psi)(z)| \geq |r_{n+2}(\psi)(z)| \geq \ldots, \forall\, z \in \mathbb{D}$.

Proof. This follows immediately from obvious identity

$$2\,\mathrm{Re}\,\zeta - 1 = |\zeta|^2 - |1-\zeta|^2 \quad \forall\, \zeta \in \mathbb{C}. \tag{7}$$

Let $\mathcal{M}_n, n \in \mathbb{Z}_+$, denote the class of null-sequences of real numbers $\{a_\nu\}_{\nu=0}^\infty$, for which

$$\Delta^0 a_\nu := a_\nu \geq 0,\ \Delta^m a_\nu := \Delta^{m-1} a_\nu - \Delta^{m-1} a_{\nu+1} \geq 0, m = 1, 2, \ldots, n,\ \forall\, \nu \in \mathbb{Z}_+.$$

Obviously, $\mathcal{M}_0 \supset \mathcal{M}_1 \supset \mathcal{M}_2 \supset \ldots$.

Proposition 2. *Let ψ be a generating kernel and $n \in \mathbb{Z}_+$. Then:*
i)

$$\left\{\widehat{\psi}_{\nu+n}\right\}_{\nu=0}^\infty \in \mathcal{M}_2 \Longrightarrow \psi \in \mathcal{R}_n;$$

ii) if $\widehat{\psi}_\nu$ is positive for all $\nu \geq n$ and

$$\mathrm{Re}\,\frac{r_k(\psi)(z)}{\widehat{\psi}_k z^k} \geq \mathrm{Re}\,\frac{r_{k+1}(\psi)(z)}{\widehat{\psi}_{k+1} z^{k+1}} \quad \forall\, k \geq n, z \in \mathbb{D}, \tag{8}$$

then

$$\psi \in \mathcal{R}_n \Longrightarrow \left\{\widehat{\psi}_{\nu+n}\right\}_{\nu=0}^\infty \in \mathcal{M}_2.$$

Remark 5. *Under condition (8) the sequence $\left\{\widehat{\psi}_{\nu+n}\right\}_{\nu=0}^\infty$ is logarithmically concave, that is, satisfies*

$$\widehat{\psi}_\nu \widehat{\psi}_{\nu+2} \leq \widehat{\psi}_{\nu+1}^2, \quad \forall\, \nu \geq n.$$

Proof. i) Fix $k \geq n$ and consider the trigonometric series

$$\frac{\widehat{\psi}_k}{2} + \sum_{\nu=1}^\infty \widehat{\psi}_{\nu+k} \cos \nu x.$$

Denote their sum by $\Psi_k(x)$.

It is well known that under condition $\{\widehat{\psi}_{\nu+n}\}_{\nu=0}^\infty \in \mathcal{M}_2$, $\Psi(x)$ exist for almost all $x \in (0, 2\pi)$ and $\Psi(x) \geq 0$.

Therefore

$$2\,\mathrm{Re}\left(\sum_{\nu=0}^\infty \frac{\widehat{\psi}_{\nu+k}}{\widehat{\psi}_k} z^\nu\right) - 1 = \frac{1}{\widehat{\psi}_k} \int_0^{2\pi} \Psi_k(x) \frac{1-|z|^2}{|e^{ix} - z|^2} \frac{dx}{\pi} \geq 0, \quad \forall\, z \in \mathbb{D}.$$

ii) Fix $k \geq n$. We have

$$2\operatorname{Re}\frac{\sum_{\nu=k}^{\infty}(\widehat{\psi}_\nu - \widehat{\psi}_{\nu+1})z^\nu}{(\widehat{\psi}_k - \widehat{\psi}_{k+1})z^k} - 1 =$$

$$= \frac{\psi_k}{\psi_k - \psi_{k+1}}\left(2\operatorname{Re}\frac{r_k(\psi)(z)}{\widehat{\psi}_k z^k} - 1\right) - \frac{\psi_{k+1}}{\psi_k - \psi_{k+1}}\left(2\operatorname{Re}\frac{r_{k+1}(\psi)(z)}{\widehat{\psi}_{k+1} z^{k+1}} - 1\right) \geq$$

$$\geq \left(2\operatorname{Re}\frac{r_{k+1}(\psi)(z)}{\widehat{\psi}_{k+1} z^{k+1}} - 1\right)\frac{\widehat{\psi}_k - \widehat{\psi}_{k+1}}{\widehat{\psi}_k - \widehat{\psi}_{k+1}} \geq 0, \quad \forall\, z \in \mathbb{D}.$$

Thus

$$\frac{\widehat{\psi}_{k+1} - \widehat{\psi}_{k+2}}{\widehat{\psi}_k - \widehat{\psi}_{k+1}}\varrho = \left|\int_0^{2\pi} e^{it}\left(2\operatorname{Re}\sum_{\nu=k}^{\infty}\frac{\widehat{\psi}_\nu - \widehat{\psi}_{\nu+1}}{\widehat{\psi}_k - \widehat{\psi}_{k+1}}\varrho^{\nu-k}e^{i(\nu-k)t} - 1\right)\frac{dt}{2\pi}\right| \leq$$
$$\leq \int_0^{2\pi}\left(2\operatorname{Re}\sum_{\nu=k}^{\infty}\frac{\widehat{\psi}_\nu - \widehat{\psi}_{\nu+1}}{\widehat{\psi}_k - \widehat{\psi}_{k+1}}\varrho^{\nu-k}e^{i(\nu-k)t} - 1\right)\frac{dt}{2\pi} = 1, \quad \forall\, \varrho \in [0,1),$$

and the result follows.

◀

Proposition 3. *Suppose $\psi \in \mathcal{R}_n \cap H_1, n \in \mathbb{Z}_+$. Then*

$$\lim_{k\to\infty}\|r_k(\psi)\|_{H_1} = 0.$$

Proof. It follows from proposition 1 that

$$\|r_n(\psi)\|_{H_1} \geq \|r_{n+1}(\psi)\|_{H_1} \geq \|r_{n+2}(\psi)\|_{H_1} \geq \dots. \tag{9}$$

It is known (see, for example, [11, p. 96]) that

$$\varliminf_{k\to\infty}\|r_k(\psi)\|_{H_1} = 0.$$

Combining this with (9), we arrive at

$$\varlimsup_{k\to\infty}\|r_k(\psi)\|_{H_1} = \varliminf_{k\to\infty}\|r_k(\psi)\|_{H_1} = 0.$$

◀

We have several corollaries from these statements.

Corollary 4. *Suppose $\psi \in \mathcal{R}_n$, and $n \in \mathbb{Z}_+$. Then for any integer $k \geq n$, polynomial $\sum_{\nu=0}^{k-n}\widehat{\psi}_{\nu+n}z^\nu$ have no zeros in $\mathbb{D}$.*

Indeed, $|z|^{-\nu}|r_\nu(\psi)(z)| \geq |z|^{-\nu}|r_{\nu+1}(\psi)(z)|$ for all $\nu \geq n$ and $z \in \mathbb{D}$. Because of domain conservation principle, the equality does not occur for all $z \in \mathbb{D}$. Consequently

$$\left|\sum_{\nu=n}^{k} \widehat{\psi}_\nu z^{\nu-n}\right| = \frac{1}{|z|^n}\,|r_n(\psi)(z) - r_{k+1}(\psi)(z)| \geq$$
$$\geq \frac{1}{|z|^n}\left(|r_n(\psi)(z)| - |r_{k+1}(\psi)(z)|\right) > 0, \quad \forall\, z \in \mathbb{D}.$$

Corollary 5. *Suppose $\psi \in \mathcal{R}_n, n \in \mathbb{Z}_+$. Then*

$$2\,\mathrm{Re}\,\frac{r_n(\psi)(z)}{\sum\limits_{\nu=n}^{k} \widehat{\psi}_\nu z^\nu} \geq 1, \quad \forall\, k \geq n,\; z \in \mathbb{D}. \tag{10}$$

In particular

$$\left|\sum_{\nu=n}^{k} \widehat{\psi}_\nu z^\nu\right| \leq 2|r_n(\psi)(z)|, \quad \forall\, k \geq n,\; z \in \mathbb{D}. \tag{11}$$

It follows immediately from proposition 1 and (7) that if we put $\zeta = r_n(\psi)(z)/(\sum_{\nu=n}^{k} \widehat{\psi}_\nu z^\nu)$.

In the case when $\psi \in \mathcal{M}_3$ the relations (10) and (11) were proved by Fejer [5]. It was shown by Wirths [19] that inequalities (10) and (11) holds for $\psi \in \mathcal{M}_2$.

Corollary 6. *Suppose $\psi \in \mathcal{R}_n \cap H_1, n \in \mathbb{Z}_+$. Then $\left|\widehat{\psi}_k\right| \log k = o(1),\; k \to \infty$.*

It follows from proposition 3 and Hardy's inequality that

$$0 \leftarrow \|r_n(\psi)\|_1 \geq \frac{1}{\pi}\sum_{\nu=0}^{\infty}\frac{\left|\widehat{\psi}_{\nu+n}\right|}{\nu+1} \geq \frac{1}{\pi}\sum_{\nu=0}^{n}\frac{\left|\widehat{\psi}_{\nu+n}\right|}{\nu+1} \geq$$
$$\geq \frac{1}{\pi}\left|\widehat{\psi}_{2n}\right|\sum_{\nu=0}^{n}\frac{1}{\nu+1} > \frac{1}{\pi}\left|\widehat{\psi}_{2n}\right| \ln n.$$

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