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Optimal Lehmer Mean Bounds for the Geometric and Arithmetic Combinations of Arithmetic and Seiffert Means

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Abstract. For any $\beta \in (0,1)$, we answer the questions: what are the greatest values p and r, and the least values q and s, such that the inequalities $L_p(a,b) < \beta T(a,b) + (1-\beta)A(a,b) < L_q(a,b)$ and $L_r(a,b) < T^{\beta}(a,b)A^{1-\beta}(a,b) < L_s(a,b)$ hold for all $a,b>0$ with $a \neq b$. Here, $A(a,b)$, $T(a,b)$ and $L_r(a,b)$ denote the arithmetic, Seiffert, and r-th Lehmer means of two positive numbers a and b, respectively.

Key Words and Phrases: Lehmer mean, arithmetic mean, Seiffert mean
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1. Introduction

For $r \in \mathbb{R}$, the $r-$th Lehmer mean $L_r(a,b)$ [5] and Seiffert mean $T(a,b)$ [8] of two positive numbers a and b are defined by

$$L_r(a,b) = \frac{a^{r+1}+b^{r+1}}{a^r+b^r}, \tag{1}$$

and

$$T(a,b) = \begin{cases} \frac{a-b}{2\arctan\left(\frac{a-b}{a+b}\right)}, & a \neq b, \\ a, & a = b, \end{cases} \tag{2}$$

respectively.

It is well known that $L_r(a,b)$ is strictly increasing with respect to $r \in \mathbb{R}$ for fixed $a,b>0$ with $a \neq b$. Many means are the special case of Lehmer mean, for example

$A(a,b) = (a+b)/2 = L_0(a,b)$ is the arithmetic mean,
$G(a,b) = \sqrt{ab} = L_{-1/2}(a,b)$ is the geometric mean,
$H(a,b) = 2ab/(a+b) = L_{-1}(a,b)$ is the harmonic mean.

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Let $I(a,b)=1/e(b^b/a^a)^{1/(b-a)}$, $L(a,b)=(b-a)/(\log b-\log a)$, and $M_p(a,b)=((a^p+b^p)/2)^{1/p}$ $(p\neq 0)$ and $M_0(a,b)=\sqrt{ab}$ be the identric, logarithmic, and $p-$th power means of two positive numbers a and b with $a\neq b$, respectively. Then

$$\min\{a,b\}<H(a,b)=L_{-1}(a,b)=M_{-1}(a,b)<G(a,b)=L_{-\frac{1}{2}}(a,b)=M_0(a,b)$$
$$<L(a,b)<I(a,b)<A(a,b)=L_0(a,b)=M_1(a,b)<\max\{a,b\},$$

for all $a,b>0$ with $a\neq b$.

Recently, the inequalities for the Lehmer, Seiffert and other bivariate means were investigated in papers [1], [2], [3], [4], [6], [7], [8], [9], [10].

In [8], Seiffert proved that

$$M_1(a,b)=A(a,b)<T(a,b)<M_2(a,b), \tag{3}$$

for all $a,b>0$ with $a\neq b$.

Chu et al. [3] found the greatest value $p=\log 3/\log(\pi/2)=2.4328\ldots$ and least value $q=5/2$ such that

$$H_p(a,b)<T(a,b)<H_q(a,b),$$

for all $a,b>0$ with $a\neq b$. Here, $H_p(a,b)=[(a^p+(ab)^{p/2}+b)/3]^{1/p}$ $(p\neq 0)$ and $H_0(a,b)=\sqrt{ab}$ is the p-th power-type Heron mean of two positive numbers a and b .

The following sharp upper and lower Lehmer mean bounds for L, I, $(LI)^{1/2}$, and $(L+I)/2$ were presented in [2]:

$$L_{-1/3}(a,b)<L(a,b)<L_0(a,b),$$
$$L_{-1/6}(a,b)<I(a,b)<L_0(a,b),$$
$$L_{-1/4}(a,b)<I^{1/2}(a,b)L^{1/2}(a,b)<L_0(a,b),$$

and

$$L_{-1/4}(a,b)<\frac{1}{2}(I(a,b)+L(a,b))<L_0(a,b),$$

for all $a,b>0$ with $a\neq b$.

Very recently, Wang et al. [10] found the following sharp bounds for Seiffert mean $T(a,b)$ in terms of Lemhmer mean

$$L_0(a,b)<T(a,b)<L_{1/3}(a,b), \tag{4}$$

for $a,b>0$ with $a\neq b$.

The purpose of this paper is to present the best possible upper and lower Lehmer mean bounds for the sum $\beta T(a,b)+(1-\beta)A(a,b)$ and product $T^{\beta}(a,b)$ $A^{1-\beta}(a,b)$ for any $\beta\in(0,1)$ and all $a,b>0$ with $a\neq b$.

2. Main Results

In order to establish our main results we need a lemma, which we present in this seciton.

Lemma 1. *If $\beta \in (0,1)$, then the double inequality*

$$\beta L_{1/3}(a,b) + (1-\beta)A(a,b) < L_{\beta/3}(a,b),$$

holds for all $a, b > 0$ with $a \neq b$.

Proof. Without loss of generality, we assume that $a > b$. Let $\alpha = \beta/3 \in (0, 1/3)$ and $t = \sqrt[3]{a/b} > 1$. Then from (1.1) and (1.2) one has

$$\begin{aligned} & L_{\beta/3}(a,b) - (1-\beta)A(a,b) - \beta L_{1/3}(a,b) \\ = \;& b\left[\frac{1+t^{3\alpha+3}}{1+t^{3\alpha}} - (1-3\alpha)\frac{1+t^3}{2} - 3\alpha\frac{1+t^4}{1+t}\right] \\ = \;& b\left[\frac{g(t)}{2(1+t^{3\alpha})(1+t)}\right], \end{aligned} \tag{5}$$

where

$$\begin{aligned} g(t) = \;& (1-3\alpha)t^{3\alpha+4} + (1+3\alpha)t^{3\alpha+3} - (1-3\alpha)t^{3\alpha+1} - (1+3\alpha)t^{3\alpha} \\ & -(1+3\alpha)t^4 - (1-3\alpha)t^3 + (1+3\alpha)t + 1 - 3\alpha. \end{aligned} \tag{6}$$

Let $g_1(t) = g''(t)/(3t)$, $g_2(t) = g_1{}'(t)/(1+3\alpha)$ and $g_3(t) = t^{5-3\alpha}g_2{}'(t)/[3\alpha(1-3\alpha)]$. Then simple computations lead to

$$g(1) = 0, \tag{7}$$

$$\begin{aligned} g'(t) = \;& (1-3\alpha)(4+3\alpha)t^{3\alpha+3} + 3(1+3\alpha)(1+\alpha)t^{3\alpha+2} - (1-3\alpha)(1+3\alpha)t^{3\alpha} \\ & -3\alpha(1+3\alpha)t^{3\alpha-1} - 4(1+3\alpha)t^3 - 3(1-3\alpha)t^2 + 1 + 3\alpha, \; g'(1) = 0, \end{aligned} \tag{8}$$

$$\begin{aligned} g_1(t) = \;& (1-3\alpha)(4+3\alpha)(1+\alpha)t^{3\alpha+1} + (1+3\alpha)(1+\alpha)(2+3\alpha)t^{3\alpha} \\ & -\alpha(1-3\alpha)(1+3\alpha)t^{3\alpha-2} + \alpha(1+3\alpha)(1-3\alpha)t^{3\alpha-3} \\ & -4(1+3\alpha)t - 2(1-3\alpha), \end{aligned}$$

$$g_1(1) = 0, \tag{9}$$

$$\begin{aligned} g_2(t) = \;& (1-3\alpha)(4+3\alpha)(1+\alpha)t^{3\alpha} + 3\alpha(1+\alpha)(2+3\alpha)t^{3\alpha-1} \\ & +\alpha(1-3\alpha)(2-3\alpha)t^{3\alpha-3} - 3\alpha(1-\alpha)(1-3\alpha)t^{3\alpha-4} - 4, \end{aligned}$$

$$g_2(1) = 0, \tag{10}$$

and

$$\begin{aligned} g_3(t) &= (4+3\alpha)(1+\alpha)t^4 - (1+\alpha)(2+3\alpha)t^3 - (1-\alpha)(2-3\alpha)t \\ &\quad +(1-\alpha)(4-3\alpha) \\ &> 2(1+\alpha)t^3 - (1-\alpha)(2-3\alpha)t + (1-\alpha)(4-3\alpha) \\ &> \alpha(7-3\alpha)t + (1-\alpha)(4-3\alpha) > 0, \end{aligned} \tag{11}$$

for $\alpha \in (0, 1/3)$.

Therefore, Lemma 1 follows from equations (2.1)-(2.6) and inequality (2.7). ◀

Theorem 1. *If $\beta \in (0,1)$, then the double inequality*

$$L_p(a,b) < \beta T(a,b) + (1-\beta)A(a,b) < L_q(a,b),$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq 0$ and $q \geq \beta/3$.

Proof. From inequalities (1.3) and (1.4) together with Lemma 1 we clearly see that

$$L_0(a,b) = A(a,b) < \beta T(a,b) + (1-\beta)A(a,b),$$

and

$$L_{\beta/3}(a,b) > \beta L_{1/3}(a,b) + (1-\beta)A(a,b) > \beta T(a,b) + (1-\beta)A(a,b),$$

for all $a, b > 0$ with $a \neq b$.

Next, we prove that $L_0(a,b)$ and $L_{\beta/3}(a,b)$ are the best possible lower and upper Lehmer mean bounds for the sum $(1-\beta)A(a,b) + \beta T(a,b)$.

For any $\varepsilon > 0$ and $x > 0$, from (1.1) and (1.2) we have

$$\begin{aligned} &\lim_{x\to+\infty} \frac{L_\varepsilon(1,x)}{(1-\beta)A(1,x) + \beta T(1,x)} \\ = &\lim_{x\to+\infty} \frac{(x^{-1}+x^\varepsilon)/(1+x^\varepsilon)}{(1-\beta)(x^{-1}+1)/2 + \beta(1-x^{-1})/[2arctan((x-1)/(x+1))]} \\ = &\frac{2\pi}{\pi + (4-\pi)\beta} > 1, \end{aligned} \tag{12}$$

and

$$\begin{aligned} &(1-\beta)A(1,1+x) + \beta T(1,1+x) - L_{\beta/3-\varepsilon}(1,1+x) \\ = &(1-\beta)(1+\frac{x}{2}) + \frac{\beta x}{2\arctan\left(\frac{x}{x+2}\right)} - \frac{1+(1+x)^{\beta/3-\varepsilon+1}}{1+(1+x)^{\beta/3-\varepsilon}} \end{aligned}$$

$$= \frac{J(x)}{2[1+(1+x)^{\beta/3-\varepsilon}]\arctan\left(\frac{x}{x+2}\right)}, \tag{13}$$

where $J(x) = 2(1-\beta)(1+x/2)[1+(1+x)^{\beta/3-\varepsilon}]\arctan[x/(x+2)] + \beta x[1+(1+x)^{\beta/3-\varepsilon}] - 2[1+(1+x)^{\beta/3-\varepsilon+1}]\arctan[x/(x+2)]$.

Letting $x \to 0$ and making use of Taylor expansion one has

$$\begin{aligned} J(x) &= x(1-\beta)\left(1+\frac{x}{2}\right)\left[2+\left(\frac{\beta}{3}-\varepsilon\right)x+\frac{(\beta-3\varepsilon)(\beta-3\varepsilon-3)}{18}x^2+o(x^2)\right] \\ &\quad \times\left[1-\frac{1}{2}x+\frac{1}{6}x^2+o(x^2)\right]+\beta x \\ &\quad \times\left[2+\left(\frac{\beta}{3}-\varepsilon\right)x+\frac{(\beta-3\varepsilon)(\beta-3\varepsilon-3)}{18}x^2+o(x^2)\right] \\ &\quad -x\left[2+\left(\frac{\beta}{3}-\varepsilon+1\right)x+\frac{(\beta-3\varepsilon)(\beta-3\varepsilon+3)}{18}x^2+o(x^2)\right] \\ &\quad \times\left[1-\frac{1}{2}x+\frac{1}{6}x^2+o(x^2)\right] \\ &= \frac{\varepsilon}{2}x^3+o(x^3). \end{aligned} \tag{14}$$

Inequality (2.8) and equations (2.9) and (2.10) imply that for any $\varepsilon > 0$ there exist $X_1 > 1$ and $\delta_1 > 0$, such that $L_\varepsilon(1,x) > (1-\beta)A(1,x) + \beta T(1,x)$ for $x \in (X_1, +\infty)$ and $L_{\beta/3-\varepsilon}(1,1+x) < (1-\beta)A(1,1+x) + \beta T(1,1+x)$ for $x \in (0,\delta_1)$. ◄

Theorem 2. *If $\beta \in (0,1)$, then the double inequality*

$$L_r(a,b) < T^\beta(a,b)A^{1-\beta}(a,b) < L_s(a,b),$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $r \leq 0$ and $s \geq \beta/3$.

Proof. From (1.3) and Theorem 1 we know that

$$L_0(a,b) < T^\beta(a,b)A^{1-\beta}(a,b) < L_{\beta/3}(a,b),$$

for all $a,b > 0$ with $a \neq b$.

Next, we prove that $L_0(a,b)$ and $L_{\beta/3}$ are the best possible lower and upper Lehmer mean bounds for the product $A^{1-\beta}(a,b)T^\beta(a,b)$.

For any $\varepsilon > 0$ and $x > 0$, from (1.1) and (1.2) we have

$$\lim_{x\to+\infty}\frac{L_\varepsilon(1,x)}{A^{1-\beta}(1,x)T^\beta(1,x)}$$

$$
\begin{aligned}
&= \lim_{x\to+\infty} \frac{(x^{-1}+x^{\varepsilon})/(1+x^{\varepsilon})}{[(x^{-1}+1)/2]^{1-\beta}\{(1-x^{-1})/\arctan[(x-1)/(x+1)]/2\}^{\beta}} \\
&= 2\left(\frac{\pi}{4}\right)^{\beta} > 1, \qquad (15)
\end{aligned}
$$

and

$$
\begin{aligned}
& A^{1-\beta}(1,1+x)T^{\beta}(1,1+x) - L_{\beta/3-\varepsilon}(1,1+x) \\
&= \left(1+\frac{x}{2}\right)^{1-\beta}\left[\frac{x}{2\arctan\left(\frac{x}{x+2}\right)}\right]^{\beta} - \frac{1+(1+x)^{\beta/3-\varepsilon+1}}{1+(1+x)^{\beta/3-\varepsilon}} \\
&= \frac{H(x)}{[1+(1+x)^{\beta/3-\varepsilon}][2\arctan\left(\frac{x}{x+2}\right)]^{\beta}}, \qquad (16)
\end{aligned}
$$

where $H(x) = x^{\beta}\,(1+x/2)^{1-\beta}\,[1+(1+x)^{\beta/3-\varepsilon}] - \{2\arctan[x/(x+2)]\}^{\beta}[1+(1+x)^{\beta/3-\varepsilon+1}]$.

Letting $x \to 0$ and making use of Taylor expansion one has

$$
\begin{aligned}
H(x) &= x^{\beta}\left[1+\frac{1-\beta}{2}x - \frac{\beta(1-\beta)}{8}x^2 + o(x^2)\right] \\
&\quad \times \left[2+\left(\frac{\beta}{3}-\varepsilon\right)x + \frac{(\beta-3\varepsilon)(\beta-3\varepsilon-3)}{18}x^2 + o(x^2)\right] \\
&\quad - x^{\beta}\left[2+\left(\frac{\beta}{3}-\varepsilon+1\right)x + \frac{(\beta-3\varepsilon)(\beta-3\varepsilon+3)}{18}x^2 + o(x^2)\right] \\
&\quad \times \left[1-\frac{\beta}{2}x + \frac{\beta(1+3\beta)}{24}x^2 + o(x^2)\right] \\
&= \frac{\varepsilon}{2}x^{\beta+2} + o(x^{\beta+2}). \qquad (17)
\end{aligned}
$$

Inequality (2.11) and equations (2.12) and (2.13) imply that for any $\varepsilon > 0$ there exist $X_2 > 1$ and $\delta_2 > 0$, such that $L_{\varepsilon}(1,x) > A^{1-\beta}(1,x)T^{\beta}(1,x)$ for $x \in (X_2, +\infty)$ and $L_{\beta/3-\varepsilon}(1,1+x) < A^{1-\beta}(1,1+x)T^{\beta}(1,1+x)$ for $x \in (0, \delta_2)$. ◄

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