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Approximating Common Fixed Points for a Finite Family of Uniformly Quasi-Lipschitzian Mappings in Banach Spaces

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Abstract. In this paper we study an implicit iteration scheme with errors for a finite family of uniformly quasi-Lipschitzian mappings and give the necessary and sufficient condition to converge to a common fixed point for the said mappings in Banach spaces and also prove some strong convergence theorems in uniformly convex Banach spaces. Our results extend and improve some recent results.

Key Words and Phrases: Uniformly quasi-Lipschitzian mapping, implicit iteration scheme with bounded errors, uniformly convex Banach space, strong convergence

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1. Introduction

Let K be a nonempty subset of a Banach space E and $T: K \to K$ be a given mapping. Let $F(T)$ denotes the set of fixed points of T, i.e., $F(T) = \{x \in K : Tx = x\}$.

Definition 1. *A mapping $T: K \to K$ is said to be*

(1) asymptotically nonexpansive if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \tag{1}$$

for all $x, y \in K$ and $n \geq 1$.

(2) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - p\| \leq k_n \|x - p\|, \tag{2}$$

for all $x \in K$, $p \in F(T)$ *and* $n \geq 1$.

(3) uniformly quasi-Lipschitzian if $F(T) \neq \emptyset$ *and there exists a constant* $L > 0$ *such that*

$$\|T^n x - p\| \quad \leq \quad L\,\|x - p\|\,, \tag{3}$$

for all $x \in K$, $p \in F(T)$ *and* $n \geq 1$.

Remark 1. *It is easy to see that the asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive mapping, and the latter is uniformly quasi-Lipschitzian mapping with* $L = \sup_{n\geq 1}\{k_n\} < +\infty$. *However, the converse doesn't hold in general.*

In 1973, Petryshyn and Williamson [7] established a necessary and sufficient condition for a Mann [6] iterative sequence to converge strongly to a fixed point of a quasi-nonexpansive mapping. Subsequently, Ghosh and Debnath [2] extended the results of [7] and obtained some necessary and sufficient condition for an Ishikawa-type iterative sequence to converge to a fixed point of a quasi-nonexpansive mapping. In 2001, Liu in [4, 5] extended the results of Ghosh and Debnath [2] to the more general asymptotically quasi-nonexpansive mappings. In 2006, Shahzad and Udomene [9] gave the necessary and sufficient condition for convergence of common fixed point of two-step modified Ishikawa iterative sequence for two asymptotically quasi-nonexpansive mappings in real Banach space.

In 2001, Xu and Ori [15] have introduced an implicit iteration process for a finite family of nonexpansive mappings in a Hilbert space H. Let C be a nonempty subset of H. Let $T_1, T_2, \ldots, T_N$ be self-mappings of C and suppose that $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$, the set of common fixed points of $T_i, i = 1, 2, \ldots, N$. An implicit iteration process for a finite family of nonexpansive mappings is defined as follows, with $\{t_n\}$ a real sequence in $(0, 1)$, $x_0 \in C$:

$$\begin{aligned}
x_1 &= t_1 x_0 + (1 - t_1) T_1 x_1, \\
x_2 &= t_2 x_1 + (1 - t_2) T_2 x_2, \\
&\vdots \\
x_N &= t_N x_{N-1} + (1 - t_N) T_N x_N, \\
x_{N+1} &= t_{N+1} x_N + (1 - t_{N+1}) T_1 x_{N+1}, \\
&\vdots
\end{aligned}$$

which can be written in the following compact form

$$x_n \quad = \quad t_n x_{n-1} + (1 - t_n) T_n x_n, \quad n \geq 1, \tag{4}$$

where $T_k = T_{k \bmod N}$. (Here the mod N function takes values in $\mathcal{N}$). And they proved the weak convergence of the process (4).

In 2003, Sun [10] extend the process (4) to a process for a finite family of asymptotically quasi-nonexpansive mappings, with $\{\alpha_n\}$ a real sequence in $(0,1)$ and an initial point $x_0 \in C$ which is defined as follows

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1-\alpha_1) T_1 x_1, \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1-\alpha_N) T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1}) T_1^2 x_{N+1}, \\ &\vdots \\ x_{2N} &= \alpha_{2N} x_{2N-1} + (1-\alpha_{2N}) T_N^2 x_{2N}, \\ x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1}) T_1^3 x_{2N+1}, \\ &\vdots \end{aligned}$$

which can be written in the following compact form

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_i^k x_n, \quad n \geq 1, \tag{5}$$

where $n = (k-1)N + i$, $i \in \mathcal{N}$.

Sun [10] proved the strong convergence of the process (5) to a common fixed point, requiring only one member T in the family $\{T_i : i \in \mathcal{N}\}$ to be semi-compact. The result of Sun [10] generalized and extended the corresponding main results of Wittmann [12] and Xu and Ori [15].

The purpose of this paper is to study an implicit iteration process with errors which converges strongly to a common fixed point of a finite family of uniformly quasi-Lipschitzian mappings in Banach spaces. Also we prove some strong convergence theorems for said mappings in uniformly convex Banach spaces.

Let E be a Banach space, K be a nonempty closed convex subset of E, and $T_i\colon K \to K$, $i \in \{1, 2, \ldots, N\} = \mathcal{N}$ be N uniformly quasi-Lipschitzian mappings. Define a sequence $\{x_n\}$ in K as follows

$$\begin{aligned} x_1 &= \alpha_1 x_0 + \beta_1 T_1 x_1 + \gamma_1 u_1, \\ x_2 &= \alpha_2 x_1 + \beta_2 T_2 x_2 + \gamma_2 u_2, \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + \beta_N T_N x_N + \gamma_N u_N, \end{aligned}$$

$$
\begin{aligned}
x_{N+1} &= \alpha_{N+1}x_N + \beta_{N+1}T_1^2 x_{N+1} + \gamma_{N+1}u_{N+1},\\
&\vdots\\
x_{2N} &= \alpha_{2N}x_{2N-1} + \beta_{2N}T_N^2 x_{2N} + \gamma_{2N}u_{2N},\\
x_{2N+1} &= \alpha_{2N+1}x_{2N} + \beta_{2N+1}T_1^3 x_{2N+1} + \gamma_{2N+1}u_{2N+1},\\
&\vdots
\end{aligned}
$$

which can be written in the following compact form

$$
x_n = \alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n, \quad n \geq 1, \tag{6}
$$

where $n = (k-1)N + i$, $i \in \mathcal{N}$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and $\{u_n\}$ is a bounded sequence in K.

Definition 2. *(see [10]) Let K be a closed subset of a normed space E and let $T\colon K \to K$ be a mapping. Then T is said to be semi-compact if for any bounded sequence $\{x_n\}$ in K with $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x^* \in K$ as $n_k \to \infty$.*

In the sequel we shall need the following lemmas to prove our main results in this paper.

Lemma 1. *(see [11]) Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$
a_{n+1} \leq (1 + \delta_n)a_n + b_n, \qquad n \geq 1.
$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2. *(see [14]) Let $p > 1$ and $r > 0$ be two fixed real numbers and E a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g\colon [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$
\|\lambda x + (1-\lambda)y\|^p \leq \lambda \|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x-y\|),
$$

for all $x, y \in B_r(0)$ where $0 \leq \lambda \leq 1$ and $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

2. Main Results

In this section we prove strong convergence theorems of an implicit iteration scheme with bounded errors for a finite family of uniformly quasi-Lipschitzian mappings in a real Banach space.

Theorem 1. *Let E be a real arbitrary Banach space, K be a nonempty closed convex subset of E. Let $T_i \colon K \to K$, $i \in \{1, 2, \dots, N\} = \mathcal{N}$ be N uniformly quasi-Lipschitzian mappings. Let $\{x_n\}$ be the sequence defined by (6) with the restriction $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$. If $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i \in \mathcal{N}\}$ if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$, where $d(x, \mathcal{F})$ denotes the distance between x and the set $\mathcal{F}$.*

Proof. The necessity is obvious and it is omitted. Now we prove the sufficiency. Let $p \in \mathcal{F}$ and $L = \max_{1 \le i \le N} L_i$. Using $x_n = \alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n$, where $n = (k-1)N + i$, it follows that

$$
\begin{aligned}
\|x_n - p\| &= \left\|\alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n - p\right\| \\
&= \left\|\alpha_n (x_{n-1} - p) + \beta_n (T_i^k x_n - p) + \gamma_n (u_n - p)\right\| \\
&\le \alpha_n \|x_{n-1} - p\| + \beta_n \left\|T_i^k x_n - p\right\| + \gamma_n \|u_n - p\| \\
&\le \alpha_n \|x_{n-1} - p\| + \beta_n L \|x_n - p\| + \gamma_n \|u_n - p\| \\
&\le \alpha_n \|x_{n-1} - p\| + \beta_n L \|x_n - p\| + \gamma_n \|u_n - p\| \\
&\le \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n) L \|x_n - p\| \\
&\quad + \gamma_n \|u_n - p\| \\
&\le \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n L) \|x_n - p\| \\
&\quad + \gamma_n \|u_n - p\|, \qquad (7)
\end{aligned}
$$

since $\lim_{n\to\infty} \gamma_n = 0$, there exists a natural number n_1 such that for $n > n_1$, $\gamma_n \le \frac{s}{2}$. Hence

$$\alpha_n = 1 - \beta_n - \gamma_n \ge 1 - (1 - s) - \frac{s}{2} = \frac{s}{2}, \qquad (8)$$

for $n > n_1$. Thus, we have from (7) and (8) that

$$\alpha_n L \|x_n - p\| \le \alpha_n \|x_{n-1} - p\| + \gamma_n \|u_n - p\|,$$

and

$$\|x_n - p\| \le \frac{1}{L} \|x_{n-1} - p\| + \frac{\gamma_n}{\alpha_n L} \|u_n - p\|$$

$$
\begin{aligned}
&\leq \frac{1}{L}\|x_{n-1}-p\|+\frac{2}{sL}\gamma_n\|u_n-p\| \\
&\leq \frac{1}{L}\|x_{n-1}-p\|+\frac{2}{sL}\gamma_n M, \qquad (9)
\end{aligned}
$$

where, $M=\sup_{n\geq 1}\{\|u_n-p\|\}$, since $\{u_n\}$ is a bounded sequence in K. This implies that

$$
d(x_n,\mathcal{F}) \leq \frac{1}{L}d(x_{n-1},\mathcal{F})+\frac{2}{sL}\gamma_n M. \qquad (10)
$$

Since $\sum_{n=1}^{\infty}\gamma_n<\infty$, it follows from Lemma 1, we know that $\lim_{n\to\infty} d(x_n,\mathcal{F})=0$.

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. From (9) we have

$$
\begin{aligned}
\|x_{n+m}-p\| &\leq \frac{1}{L}\|x_{n+m-1}-p\|+\frac{2M}{sL}\gamma_{n+m} \\
&\leq \frac{1}{L}\Big[\frac{1}{L}\|x_{n+m-2}-p\|+\frac{2M}{sL}\gamma_{n+m-1}\Big] \\
&\quad +\frac{2M}{sL}\gamma_{n+m} \\
&\leq \frac{1}{L^2}\|x_{n+m-2}-p\|+\frac{2M}{sL^2}\gamma_{n+m-1} \\
&\quad +\frac{2M}{sL}\gamma_{n+m} \\
&\leq \frac{1}{L^2}\|x_{n+m-2}-p\|+\frac{2M}{sL^2}\Big[\gamma_{n+m-1}+\gamma_{n+m}\Big] \\
&\leq \cdots \\
&\leq \cdots \\
&\leq \frac{1}{L^m}\|x_n-p\|+\frac{2M}{sL^m}\Big[\gamma_{n+m}+\gamma_{n+m-1} \\
&\quad +\cdots+\gamma_{n+1}\Big] \\
&\leq \frac{1}{L^m}\|x_n-p\|+\frac{2M}{sL^m}\sum_{j=n+1}^{n+m}\gamma_j, \qquad (11)
\end{aligned}
$$

for all $p\in\mathcal{F}$ and $m,n\in\mathbb{N}$. Since $\lim_{n\to\infty} d(x_n,\mathcal{F})=0$, for each $\varepsilon>0$, there exists a natural number n_1 such that for $n\geq n_1$:

$$
d(x_n,\mathcal{F})<\frac{L^m\varepsilon}{4} \quad \text{and} \quad \sum_{j=n_1+1}^{n+m}\gamma_j\leq\frac{sL^m\varepsilon}{8M}. \qquad (12)
$$

Thus, there exists a point $q \in \mathcal{F}$ such that $d(x_{n_1}, q) < \frac{L^m \varepsilon}{4}$. It follows from (11) that for all $n \geq n_1$ and $m \geq 1$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - q\| + \|x_n - q\| \\
&\leq \frac{1}{L^m}\|x_{n_1} - q\| + \frac{2M}{sL^m}\sum_{j=n_1+1}^{n+m}\gamma_j \\
&\quad + \frac{1}{L^m}\|x_{n_1} - q\| + \frac{2M}{sL^m}\sum_{j=n_1+1}^{n+m}\gamma_j \\
&< \frac{1}{L^m}.\frac{L^m\varepsilon}{4} + \frac{2M}{sL^m}.\frac{sL^m\varepsilon}{8M} \\
&\quad + \frac{1}{L^m}.\frac{L^m\varepsilon}{4} + \frac{2M}{sL^m}.\frac{sL^m\varepsilon}{8M} \\
&= \varepsilon. \qquad (13)
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Because K is a nonempty closed convex subset of E, so there exists a $p \in K$, such that $x_n \to p$ as $n \to \infty$. Finally, we prove $p \in \mathcal{F}$.

Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$ and $x_n \to p$ as $n \to \infty$, thus $d(p, \mathcal{F}) = 0$. So, for any $\varepsilon_1 > 0$ there exists $p_1 \in \mathcal{F}$, such that $\|p_1 - p\| < \varepsilon_1$. Then we have

$$\begin{aligned}
\|T_i p - p\| &\leq \|T_i p - p_1\| + \|p_1 - p\| \\
&\leq L\|p - p_1\| + \|p_1 - p\| \\
&= (1 + L)\|p_1 - p\| \\
&< (1 + L)\varepsilon_1.
\end{aligned}$$

But by the arbitraryness of ε_1 we know that $\|T_i p - p\| = 0$, for all $i = 1, 2, \ldots, N$, i.e., $p \in \mathcal{F}$. Thus, $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i \in \mathcal{N}\}$. This completes the proof.◀

Theorem 2. *Let E be a real arbitrary Banach space, K be a nonempty closed convex subset of E. Let $T_i\colon K \to K$, $i \in \{1, 2, \ldots, N\} = \mathcal{N}$ be N uniformly quasi-Lipschitzian mappings. Let $\{x_n\}$ be the sequence defined by (6) with the restriction $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\{\beta_n\} \subset (s, 1 - s)$ for some $s \in (0, \frac{1}{2})$. If $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i \in \mathcal{N}\}$ if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to p.*

Proof. The proof of Theorem 1 follows from Lemma 1 and Theorem 1 This completes the proof.

We prove a lemma which plays an important role in establishing strong convergence of the implicit iteration process with errors in a uniformly convex Banach space.◀

Lemma 3. *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E. Let $T_i\colon K \to K$, $i \in \{1, 2, \dots, N\} = \mathcal{N}$ be N uniformly quasi-Lipschitzian mappings. Let $\{x_n\}$ be the sequence defined by (6) with the restriction $\sum_{n=1}^{\infty} \gamma_n < \infty$. If $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ and $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$. Then $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$ for all $l \in \mathcal{N}$.*

Proof. Set $\sigma_n = \left\|T_n^k x_n - x_{n-1}\right\|$, $n = (k-1)N + i$, $i \in \mathcal{N}$. As in the proof of Theorem 1, $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in \mathcal{F}$, so $\{x_n - q, T_i^k x_n - q\}$ is a bounded set. Hence, we can obtain a closed ball $B_r(0) \supset \{x_n - q, T_i^k x_n - q\}$ for some $r > 0$. By Lemma 1.5 and the scheme (6) we get

$$\begin{aligned}
\|x_n - q\|^2 &= \left\|\alpha_n(x_{n-1} - q) + (1-\alpha_n)(T_i^k x_n - q) + \gamma_n(u_n - T_i^k x_n)\right\|^2 \\
&\le \left\|\alpha_n(x_{n-1} - q) + (1-\alpha_n)(T_i^k x_n - q)\right\|^2 + \gamma_n K \quad \text{for some } K > 0 \\
&\le \alpha_n \|x_{n-1} - q\|^2 + (1-\alpha_n)\left\|T_i^k x_n - q\right\|^2 \\
&\quad -W_2(\alpha_n) g(\sigma_n) + \gamma_n K \\
&\le \alpha_n \|x_{n-1} - q\|^2 + (1-\alpha_n) L^2 \|x_n - q\|^2 \\
&\quad -W_2(\alpha_n) g(\sigma_n) + \gamma_n K \\
&\le \alpha_n \|x_{n-1} - q\|^2 + (1-\alpha_n L^2) \|x_n - q\|^2 \\
&\quad -W_2(\alpha_n) g(\sigma_n) + \gamma_n K. \qquad (14)
\end{aligned}$$

Thus, from (14) and (8) we have

$$\begin{aligned}
\|x_n - q\|^2 &\le \frac{1}{L^2}\|x_{n-1} - q\|^2 - \frac{(1-\alpha_n)}{L^2} g(\sigma_n) \\
&\quad + \frac{2\gamma_n}{sL^2} K. \qquad (15)
\end{aligned}$$

Therefore, as in Theorem 1, it can be shown that $\lim_{n\to\infty} \|x_n - q\|^2 = d$ exists. From (15) it follows that

$$\begin{aligned}
\Big(\frac{1-\alpha_n}{L^2}\Big) g(\sigma_n) &\le \frac{1}{L^2}\|x_{n-1} - q\|^2 - \|x_n - q\|^2 + \frac{2\gamma_n}{sL^2} K \\
&\le \frac{1}{L^2}[\|x_{n-1} - q\|^2 - \|x_n - q\|^2] + \frac{2\gamma_n}{sL^2} K. \qquad (16)
\end{aligned}$$

From $(1-\alpha_n) \geq (1-s/2)$ we have

$$\Big(\frac{2-s}{2L^2}\Big)g(\sigma_n) \;\leq\; \frac{1}{L^2}[\|x_{n-1}-q\|^2 - \|x_n-q\|^2] + \frac{2\gamma_n}{sL^2}K. \tag{17}$$

Let m be a positive integer such that $m \geq n$. Then

$$\begin{aligned}\sum_{n=1}^{m} g(\sigma_n) &\leq \Big(\frac{2}{2-s}\Big)\big[\,\|x_0-q\|^2 - \|x_m-q\|^2\,\big] \\ &\quad + \frac{4K}{s(2-s)}\sum_{n=1}^{m}\gamma_n \\ &\leq \Big(\frac{2}{2-s}\Big)\|x_0-q\|^2 + \frac{4K}{s(2-s)}\sum_{n=1}^{m}\gamma_n. \end{aligned}\tag{18}$$

When $m \to \infty$ in (18), we have that $\lim_{n\to\infty} g(\sigma_n) = 0$. Since g is strictly increasing and continuous with $g(0) = 0$, it follows that $\lim_{n\to\infty}\sigma_n = 0$. Hence,

$$\begin{aligned}\|x_n - x_{n-1}\| &\leq \beta_n \left\|T_n^k x_n - x_{n-1}\right\| + \gamma_n\|u_n - x_{n-1}\| \\ &\leq (1-\alpha_n)\left\|T_n^k x_n - x_{n-1}\right\| + \gamma_n Q, \text{ for some } Q>0 \\ &\leq (1-s/2)\left\|T_n^k x_n - x_{n-1}\right\| + \gamma_n Q, \end{aligned}\tag{19}$$

which implies that $\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0$. That is, $\lim_{n\to\infty}\|x_n - x_{n+l}\| = 0$ for all $l > 2N$. For $n > N$ we have

$$\begin{aligned}\|x_{n-1} - T_n x_n\| &\leq \left\|x_{n-1} - T_n^k x_n\right\| + \left\|T_n^k x_n - T_n x_n\right\| \\ &\leq \sigma_n + L\left\|T_n^{k-1}x_n - x_n\right\| \\ &\leq \sigma_n + L\Big[\left\|T_n^{k-1}x_n - T_{n-N}^{k-1}x_{n-N}\right\| \\ &\quad + \left\|T_{n-N}^{k}x_{n-N} - x_{(n-N)-1}\right\|\Big] \\ &\quad + L\left\|x_{(n-N)-1} - x_n\right\|. \end{aligned}\tag{20}$$

By $n \equiv (n-N)(mod\ N)$ we get $T_n = T_{n-N}$. Now the above inequality becomes

$$\begin{aligned}\|x_{n-1} - T_n x_n\| &\leq \sigma_n + L^2\|x_n - x_{n-N}\| + L\sigma_{n-N} \\ &\quad + L\left\|x_{(n-N)-1} - x_n\right\|, \end{aligned}\tag{21}$$

which yields that $\lim_{n\to\infty}\|x_{n-1} - T_n x_n\| = 0$. Since

$$\|x_n - T_n x_n\| \;\leq\; \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|, \tag{22}$$

so we have that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{23}$$

Hence, for all $l \in \mathcal{N}$ we have

$$\begin{aligned} \|x_n - T_{n+l}x_n\| &\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| \\ &\quad + \|T_{n+l}x_{n+l} - T_{n+l}x_n\| \\ &\leq (1+L)\|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_n\|, \end{aligned} \tag{24}$$

which implies that

$$\lim_{n\to\infty} \|x_n - T_{n+l}x_n\| = 0 \quad \forall\ l \in \mathcal{N}. \tag{25}$$

Thus

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0 \quad \forall\ l \in \mathcal{N}. \tag{26}$$

Now we are in a position to prove our strong convergence theorems.◄

Theorem 3. *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E. Let $T_i\colon K \to K$, $i \in \{1, 2, \ldots, N\} = \mathcal{N}$ be N uniformly quasi-Lipschitzian mappings. Let $\{x_n\}$ be the sequence defined by (6) with the restriction $\sum_{n=1}^{\infty} \gamma_n < \infty$. If $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ and $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$. If at least one member T in $\{T_i : i \in I\}$ is semi-compact, then the implicitly defined sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in \mathcal{N}\}$.*

Proof. By Lemma 3 it follows that

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0 \quad \forall\ l \in \mathcal{N}. \tag{27}$$

Without any loss of generality assume that T_1 is semi-compact. Therefore, by (27) it follows that $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$. Since T_1 is semi-compact, therefore there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x^* \in K$. Now consider

$$\|x^* - T_l x^*\| = \lim_{n_j\to\infty} \left\|x_{n_j} - T_l x_{n_j}\right\| = 0 \quad \forall\ l \in I. \tag{28}$$

This proves that $x^* \in \mathcal{F}$. As $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in \mathcal{F}$, therefore $\{x_n\}$ converges to $x^* \in \mathcal{F}$, and hence the result.◄

Definition 3. *(condition $(*)$)(see [1]) The family $\{T_i : i \in \mathcal{N}\}$ of N-self mappings on a subset K of a normed space E satisfies condition $(*)$ if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $\frac{1}{N}\sum_{i=1}^{N} \|x - T_i x\| \geq f(d(x, \mathcal{F}))$ for all $x \in K$, where $d(x, \mathcal{F}) = \inf\{\|x - p\| : p \in \mathcal{F}\}$.*

Note that condition $(*)$ defined above reduces to the condition (A) [11] if we choose $T_i = T$ (say) for all $i \in \mathcal{N}$.

Finally, an application of the convergence criteria established in Theorem 1 is given below to obtain yet another strong convergence result in our setting.

Theorem 4. *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E. Let $T_i : K \to K$, $i \in \{1, 2, \ldots, N\} = \mathcal{N}$ be N uniformly quasi-Lipschitzian mappings such that $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ and satisfy the condition $(*)$. Let $\{x_n\}$ be the sequence defined by (6) with the restriction $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\beta_n\} \subset (s, 1 - s)$ for some $s \in (0, \frac{1}{2})$. Then the iterative sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in \mathcal{N}\}$.*

Proof. As in the proof of Theorem 3, (27) holds. Taking lim inf on both sides of condition $(*)$ and using (27), we have that $\liminf_{n\to\infty} f(d(x_n, \mathcal{F})) = 0$. Since f is a nondecreasing function with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0.\infty)$, it follows that $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Now by Theorem 1, $x_n \to p \in \mathcal{F}$. This shows that $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in \mathcal{N}\}$.◄

Remark 2. *Theorem 1 extends and improves corresponding results of [4, 5, 8, 9, 13]. Especially Theorem 1 extends and improves Theorem 1 and 2 in [5], Theorem 1 in [4] and Theorem 3.2 in [9] in the following ways:*

(1) The asymptotically quasi-nonexpansive mapping in [4], [5] and [9] is replaced by finite family of uniformly quasi-Lipschitzian mappings.

(2) The usual Ishikawa iteration scheme in [4], the usual modified Ishikawa iteration scheme with errors in [5] and the usual modified Ishikawa iteration scheme with errors for two mappings are extended to the implicit iteration scheme with errors for a finite family of mappings.

Remark 3. *Theorem 2 extends and improves Theorem 3 in [5] and Theorem 2.3 extends and improves Theorem 3 in [4] in the following aspects:*

(1) The asymptotically quasi-nonexpansive mapping in [4] and [5] is replaced by finite family of uniformly quasi-Lipschitzian mappings.

(2) The usual Ishikawa iteration scheme in [4] and the usual modified Ishikawa iteration scheme with errors in [5] are extended to the implicit iteration scheme with errors for a finite family of mappings.

Remark 4. *Theorem 1 also extends the corresponding result of [8] to the case of implicit iteration scheme with errors for a finite family of mappings and also it extends the corresponding result of [13] to the case of more general class of asymptotically nonexpansive mappings and implicit iteration scheme with errors for a finite family of mappings considered in this paper.*

Remark 5. *Theorem 3 extends and improves Theorem 3.3 due to Sun [10] to the case of more general class of asymptotically quasi-nonexpansive mapping and implicit iteration process with errors and without the boundedness of K which in turn generalizes Theorem 2 by Wittmann [12] from Hilbert spaces to uniformly convex Banach spaces.*

Remark 6. *Our results also extend the corresponding results of Ud-din and Khan [1] to the case of more general class of asymptotically quasi-nonexpansive mappings considered in this paper.*

Example 1. *Let E be the real line with the usual norm $|.|$ and $K = [0,1]$. Define $T\colon K \to K$ by*

$$T(x) = sin\, x, \quad x \in [0,1],$$

for $x \in K$. Obviously $T(0) = 0$, that is, 0 is a fixed point of T, that is, $F(T) = \{0\}$. Now we check that T is uniformly quasi-Lipschitzian mapping. In fact, if $x \in [0,1]$ and $p = 0 \in [0,1]$, then

$$|T(x) - p| = |T(x) - 0| = |sin\, x - 0| = |sin\, x| \leq |x| = |x - 0| = |x - p|,$$

that is

$$|T(x) - p| \leq |x - p|.$$

Thus, T is quasi-nonexpansive. It follows that T is asymptotically quasi-nonexpansive with the constant sequence $\{k_n\} = \{1\}$ for each $n \geq 1$ and hence it is uniformly quasi-Lipschitzian with constant $L = 1 > 0$. Hence, an asymptotically quasi-nonexpansive mapping is uniformly quasi-Lipschitzian mapping with $L = \sup_{n\geq 1}\{k_n\} < +\infty$. But the converse does not hold in general.

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