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Limit behaviour of sample modes for large number of independent random variables

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Abstract. Although the estimators of mode on finite independent random variables are widely used in various engineering fields, the feature regarding limit behaviour of sample modes for large number of independent random variables is not known except some special cases. Here we discuss the methodology for estimating the limit behaviour of sample modes for independent random variables and induce the generic form of it, which reveals the relation between the local tendency of probability density function around the mode and the entire behaviour of sample modes. The actual limit behaviour of sample modes in some examples are estimated by using that form and their properties are discussed.

Key Words and Phrases: limit mode behaviour, mode, limit distribution, central limit theorem, multimodal

2010 Mathematics Subject Classifications: Central limit and other weak theorems (60F05)

1. Introduction

In statistics, as the measures of central tendency we often use mean, median and mode. The central limit theorem is widely used for estimating the limit of mean and variance of identically distributed random variables. The central limit theory for mean is extended to describe the nature of specific sum of random variables in various mathematical frameworks such as martingale [2], convex bodies [7], and Lacunary trigonometric series [5]. Regarding the behavior of sample modes, various methods such as the kernel density estimation [9][8], Grenander's direct estimator [6], and for Brownian motion [4] are known. Those methods are mainly to estimate the mode itself for the given finite data, while few limit behaviours of sample modes are known except that under particularly specified conditions [1].

Since sometimes mode gives better measure of central tendency than mean especially when the distribution is multimodal, it is thought useful in such case to consider the limit behaviour of sample modes to describe the feature of independent random variables.

Prior to discuss the limit behaviour of sample modes for independent random variables, we assume that the probability density function of those random variables is continuous and may be multimodal, however at least it must have unique global maximum. As discussed later, the shape of the probability density function around the global maximum gives the entire feature of limit behaviour of sample modes. We deal with two examples having 1) smooth, and 2) cusp-like peaks at the global maximum of the probability density functions. In the first case it is shown that the limit behaviour of sample modes does not follow Gauss's normal distribution, but is expressed as

$$f(x) = a \cdot exp(-bx^4) \quad (a, b : constants).$$

Additionally the treatment for the case in which the sample space is finite is discussed.

2. Relation of deviation from global maximum and limit behaviour of sample modes

As mentioned in Introduction, we assume that the random variable X dealt with here has the continuous probability density $f(x)$ which has unique global maximum but may be multimodal.

Firstly we take equidistant intervals divided at $x_0, x_1, ..., x_T$ on the sample space. The probability of random variable lying in the interval which includes the global maximum x_M is

$$Pr(x_i \le X < x_{i+1}) = \int_{x_i}^{x_{i+1}} f(x)\, dx, \tag{1}$$

where $[x_i, x_{i+1})$ is the interval including x_M. Let d the width of interval. Then from the first mean value theorem we know there exists θ such that

$$\int_{x_i}^{x_{i+1}} f(x)\, dx = f(x_M + \theta)d, \tag{2}$$

where $x_i \le x_M + \theta < x_{i+1}$. We denote $x_M + \theta$ as $\bar{x}_M$. After repeating the trial n times, the probability P_y of the event that y random variables lie in $[x_i, x_{i+1})$ follows the binomial distribution as

$$P_y = {}_nC_y (f(\bar{x}_M)d)^y (1 - f(\bar{x}_M)d)^{n-y}. \tag{3}$$

According to the central limit theorem, the above binomial distribution converges to the following normal distribution $p_i(y)$ when $n \to \infty$.

$$P_y \to p_i(y) = N(nf(\bar{x}_M)d, nf(\bar{x}_M)d(1 - f(\bar{x}_M)d)), \tag{4}$$

where i is the index of $[x_i, x_{i+1})$ in which the global maximum x_M is included. On the above normal distribution, both mean and variance are infinite when $n \to \infty$. Transforming y to $n\cdot d\cdot y$ and renormalizing, we get the following normal distribution

$$n\cdot d\cdot p_i(n\cdot d\cdot y) = N(f(\bar{x}_M), \frac{f(\bar{x}_M)(1 - f(\bar{x}_M)d)}{nd}. \tag{5}$$

We denote $n\cdot d\cdot p_i(n\cdot d\cdot y)$ as $N_i(y)$. We can regard it as the probability density function for y on the interval $[x_i, x_{i+1})$. It has a finite mean and the variance converges to zero when $n \to \infty$. Therefore intuitively it is expected that the probability of the event that the number of random variables lying in the interval which doesn't include the mode x_M is greater than that of the interval which includes x_M converges to zero when $n \to \infty$.

As the global maximum of $f(x)$ is assumed unique, the following inequality is satisfied by taking sufficiently small interval d.

$$Pr(x_j \le X < x_{j+1}) < Pr(x_i \le X < x_{i+1}), \tag{6}$$

where $i \neq j$ and $[x_i, x_{i+1})$ is the interval including x_M. Similar to (5), $n\cdot d\cdot p_j(n\cdot d\cdot y)$ corresponding to the interval $[x_j, x_{j+1})$ is expressed as

$$n\cdot d\cdot p_j(n\cdot d\cdot y) = N(f(\bar{x}_j), \frac{f(\bar{x}_j)(1 - f(\bar{x}_j)d)}{nd}), \tag{7}$$

where $\bar{x}_j$ $(x_j \le \bar{x}_j < x_{j+1})$ satisfies the following equation

$$\int_{x_j}^{x_{j+1}} f(x)\, dx = f(\bar{x}_j)d. \tag{8}$$

Setting δ as

$$\delta = f(\bar{x}_M) - f(\bar{x}_j), \tag{9}$$

the equation (7) is rewritten as

$$n\cdot d\cdot p_j(n\cdot d\cdot y) = N(f(\bar{x}_M) - \delta, \frac{(f(\bar{x}_M) - \delta)(1 - (f(\bar{x}_M) - \delta)d)}{nd}). \tag{10}$$

It should be noted that δ is always positive since $f(\bar{x}_M)$ is the global maximum. We denote $n\cdot d\cdot p_j(n\cdot d\cdot y)$ as $N_j(y)$. When n is finite, both distributions $N_i(y)$

and $N_j(y)$ are normal distributions and not zero on $(-\infty, \infty)$. Therefore in some cases the number of random variables lying in the interval $[x_j, x_{j+1})$ after n trials may be greater than that of the interval $[x_i, x_{i+1})$ after n trials. We denote that probability as P_{ij}. When $n \to \infty$, P_{ij} converges as

$$\begin{aligned} P_{ij} &\to \int_{-\infty}^{\infty} dx\ N_i(x) \int_{x}^{\infty} dy\ N_j(y) \\ &= \int_{-\infty}^{\infty} dx\ N_i(x)\{1 - \int_{-\infty}^{x} dy\ N_j(y)\} \\ &= 1 - \int_{-\infty}^{\infty} dx\ N_i(x) \int_{-\infty}^{x} dy\ N_j(y). \end{aligned} \tag{11}$$

Since $N_i(x)$ and $\int_{-\infty}^{x} dy\ N_j(y)$ converge to the delta function $\delta(x - \bar{x}_M)$ and step function $\chi_{[\bar{x}_M-\delta,\infty)}(x)$ respectively when $n \to \infty$, P_{ij} converges to zero as long as δ is positive finite constant.

Now we consider the δ as changing along with n so as to converge to zero when $n \to \infty$. Actually δ depends on the width d of interval, therefore firstly we set d as

$$d = n^{-\rho}, \tag{12}$$

where ρ is a constant and $0 < \rho < 1$. Namely the width of interval shrinks as the number of trials increases. Although d is the variable of n, d is constant during the n-time trials. When $n \to \infty$, d converges to zero and $[x_j, x_{j+1})$ converges to the point x_j. In that case the above probability P_{ij} is regarded as the rate of probability that $X = x_j$ becomes the mode to that of $X = x_M$. We set x_j and new variable $s = x_j - x_M$ so as to satisfy the following expression for δ.

$$\delta = \kappa(s) n^{-\frac{1}{2}(1-\rho)} + o(n^{-\frac{1}{2}(1-\rho)-\nu}), \tag{13}$$

where $\kappa(s)$ is a continuous function and ν is a positive constant. Then the following theorem is satisfied.

Theorem 1. *When the width of equidistance interval shrinks along with the number of trials n as described in (12) and δ defined in (9) is expressed as (13), the limit probability density $p_M(x)$ that $x_M + x$ is the mode when $n \to \infty$ is expressed as*

$$p_M(x) = \frac{1}{C}\ exp(-\frac{\kappa(x)^2}{2f(x_M)}), \tag{14}$$

where $f(x)$ is the probability density function, $f(x_M)$ is its global maximum and C is expressed as

$$C = \int_{-\infty}^{\infty} dx\ exp(-\frac{\kappa(x)^2}{2f(x_M)}). \tag{15}$$

Proof. When $n \to \infty$, converged P_{ij} (11) is expressed with (5) and (10) as

$$\begin{aligned} P_{ij} &\to 1 - \int_{-\infty}^{\infty} dx \, \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma_i})^2} \int_{-\infty}^{x} dy \, \frac{1}{\sqrt{2\pi}\sigma_j} e^{-\frac{1}{2}(\frac{y+\delta-\mu}{\sigma_j})^2} \\ &= 1 - \int_{-\infty}^{\infty} dx \, \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2}(\frac{x-\mu-\delta}{\sigma_i})^2} \int_{-\infty}^{x} dy \, \frac{1}{\sqrt{2\pi}\sigma_j} e^{-\frac{1}{2}(\frac{y-\mu}{\sigma_j})^2}, \end{aligned} \tag{16}$$

where μ and σ_i are the mean and standard deviation for $N_i(y)$, σ_j is the standard deviation for $N_j(y)$ and actually they are expressed as

$$\mu = f(\bar{x}_M) \tag{17}$$

$$\sigma_i = \sqrt{\frac{f(\bar{x}_M)(1 - f(\bar{x}_M)d)}{nd}} \tag{18}$$

$$\sigma_j = \sqrt{\frac{(f(\bar{x}_M) - \delta)(1 - (f(\bar{x}_M) - \delta)d)}{nd}}. \tag{19}$$

Although both σ_i and σ_j converge to zero when $n \to \infty$, here we deal with them individually for the sake of convenience on the calculation. When $\sigma_j \to 0$, P_{ij} converges to the following expressions.

$$\begin{aligned} P_{ij} &\to 1 - \int_{-\infty}^{\infty} dx \, \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2}(\frac{x-\mu-\delta}{\sigma_i})^2} \, \chi_{[\mu,\infty)}(x) \\ &= 1 - \int_{\mu}^{\infty} dx \, \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2}(\frac{x-\mu-\delta}{\sigma_i})^2} \\ &= \int_{-\infty}^{\mu} dx \, \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2}(\frac{x-\mu-\delta}{\sigma_i})^2} \\ &= \int_{-\infty}^{\delta} dx \, \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2}(\frac{x}{\sigma_i})^2} \\ &= \int_{-\infty}^{\frac{\delta}{\sigma_i}} dx \, \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \end{aligned} \tag{20}$$

$\frac{\delta}{\sigma_i}$ is expressed with (13) and (18) as

$$\begin{aligned} \frac{\delta}{\sigma_i} &= \frac{\kappa(s)n^{-\frac{1}{2}(1-\rho)} + o(n^{-\frac{1}{2}(1-\rho)-\nu})}{\sqrt{\frac{f(\bar{x}_M)(1-f(\bar{x}_M)d)}{nd}}} \\ &= \frac{\kappa(s)n^{-\frac{1}{2}(1-\rho)} + o(n^{-\frac{1}{2}(1-\rho)-\nu})}{\sqrt{\frac{f(\bar{x}_M)(1-f(\bar{x}_M)d)}{n^{1-\rho}}}} \\ &= \frac{\kappa(s)n^{-\frac{1}{2}(1-\rho)} + o(n^{-\frac{1}{2}(1-\rho)-\nu})}{n^{-\frac{1}{2}(1-\rho)}\sqrt{f(\bar{x}_M)(1 - f(\bar{x}_M)d)}} \end{aligned}$$

$$= \frac{\kappa(s) + o(n^{-\nu})}{\sqrt{f(\bar{x}_M)(1 - f(\bar{x}_M)d)}}. \tag{21}$$

Since $f(\bar{x}_M) \to f(x_M)$ and $d = n^{-\rho} \to 0$ when $n \to \infty$, P_{ij} expressed in (20) converges to the following expression when $\sigma_i \to 0$ (i.e. $n \to \infty$).

$$P_{ij} \to \int_{-\infty}^{\frac{\kappa(s)}{\sqrt{f(x_M)}}} dx \, \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}. \tag{22}$$

Hence the limit of derivative of P_{ij} is expressed as

$$\frac{dP_{ij}}{dx} \to \frac{1}{\sqrt{2\pi}} \, exp(-\frac{\kappa(s)^2}{2f(x_M)}). \tag{23}$$

As mentioned earlier, the limit of P_{ij} is the rate of probability that $f(x_j)$ becomes the mode to that of $f(x_M)$. Since $s = x_j - x_M$, the limit probability density $p_M(x)$ that $x_M + x$ becomes the mode is obtained by normalizing (23) as

$$p_M(x) = \frac{exp(-\frac{\kappa(x)^2}{2f(x_M)})}{\int_{-\infty}^{\infty} dx \, exp(-\frac{\kappa(x)^2}{2f(x_M)})}. \blacktriangleleft \tag{24}$$

3. Example of limit behaviour of sample modes (1)

The above theorem indicates that the limit probability density function for sample modes $p_M(x)$ is determined by $f(x_M)$ and $\kappa(x)$, i.e. the global maximum of probability density function $f(x)$ and the local deviation around it. Namely the entire form of $p_M(x)$ is determined by the local feature of $f(x)$ around the global maximum. We discuss on the following two examples how the local feature of $f(x)$ determines the form of $p_M(x)$ practically. Firstly we consider the case that the probability density function $f(x)$ has continuous second derivative $f''(x)$ around the global maximum $f(x_M)$. In this case $f(x)$ is expanded around x_M as

$$f(x) = f(x_M) + \frac{1}{2} f''(x_M)(x - x_M)^2 + o((x - x_M)^{2+\nu}), \tag{25}$$

where $f''(x_M)$ is negative. Then the following corollary is satisfied.

Corollary 1. *Let* $\hat{X} = n^{\frac{1}{4}(1-\rho)}(X - x_M)$. *When the probability density function* $f(x)$ *has the form expressed as (25) around* x_M*, the limit probability density* $p_M(\hat{x})$ *that* $\hat{x}$ *is the mode when* $n \to \infty$ *is expressed as*

$$p_M(\hat{x}) = \frac{1}{C}\, exp(-\frac{f''(x_M)^2}{8f(x_M)}\, \hat{x}^4), \tag{26}$$

where C *is expressed as*

$$C = \int_{-\infty}^{\infty} dx\; exp(-\frac{f''(x_M)^2}{8f(x_M)}\, \hat{x}^4). \tag{27}$$

Proof. From the premiss of the corollary, $f(x_M + \hat{x})$ is expanded around $\hat{x} = 0$ as

$$\begin{aligned} f(x_M + \hat{x}) &= f(x_M) + \frac{1}{2} f''(x_M)\, \hat{x}^2 n^{-\frac{1}{2}(1-\rho)} + o(n^{-\frac{2+\nu}{4}(1-\rho)}) \\ &= f(x_M) + \frac{1}{2} f''(x_M)\, \hat{x}^2 n^{-\frac{1}{2}(1-\rho)} + o(n^{-\frac{1}{2}(1-\rho)-\frac{\nu}{4}(1-\rho)}), \end{aligned} \tag{28}$$

therefore δ is expressed with $s = \hat{x}_j$ as

$$\delta = -\frac{1}{2} f''(x_M)\, s^2 n^{-\frac{1}{2}(1-\rho)} + o(n^{-\frac{1}{2}(1-\rho)-\frac{\nu}{4}(1-\rho)}). \tag{29}$$

This form coincides with that of δ expressed in (13). Substituting $-\frac{1}{2} f''(x_M)\, \hat{x}^2$ in (14) and (15) for $\kappa(x)$, we obtain $p_M(\hat{x})$ as

$$p_M(\hat{x}) = \frac{1}{C}\, exp(-\frac{f''(x_M)^2}{8f(x_M)}\, \hat{x}^4), \tag{30}$$

and C as

$$C = \int_{-\infty}^{\infty} dx\; exp(-\frac{f''(x_M)^2}{8f(x_M)}\, \hat{x}^4). \tag{31}$$

As seen in the figure 1, the limit probability density function of sample modes induced from the above corollary has blunter peak than that of normal distribution. In return, it has much slighter tails. Also it is noteworthy that the function does not depend on ρ, viz. as far as ρ satisfies the condition: $0 < \rho < 1$, the probability density function of sample modes converges to a unique function. This property is useful for estimating the limit probability density of mode on the trials empirically since we can choose arbitrary ρ within $(0, 1)$ for that purpose.

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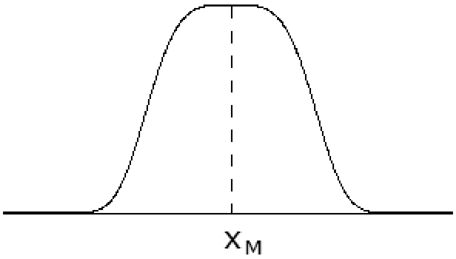

Figure 1: Limit probability density function of mode (1)

4. Example of limit distribution of sample modes (2)

Next we consider the case that the probability density function $f(x)$ around x_M forms a cusp, viz. $f'(x)$ is not continuous and has two different values at x_M as

$$\lim_{\Delta x \to 0} f'(x_M + |\Delta x|) = -\alpha \tag{32}$$

$$\lim_{\Delta x \to 0} f'(x_M - |\Delta x|) = \alpha, \tag{33}$$

where α is a positive constant. In this case $f(x)$ is expanded around x_M as

$$f(x) = f(x_M) - \alpha(x - x_M) + o((x - x_M)^{1+\nu}) \quad (x \geq x_M) \tag{34}$$

$$= f(x_M) + \alpha(x - x_M) + o((x - x_M)^{1+\nu}) \quad (x < x_M). \tag{35}$$

Then the following corollary is satisfied.

Corollary 2. *Let $\hat{X} = n^{\frac{1}{2}(1-\rho)}(X - x_M)$. When the probability density function $f(x)$ has the form expressed as (34) and (35) around x_M, the limit probability density $p_M(\hat{x})$ that $\hat{x}$ is the mode when $n \to \infty$ is expressed as*

$$p_M(\hat{x}) = \frac{\alpha}{\sqrt{2\pi f(x_M)}} \, exp(-\frac{\alpha^2}{2f(x_M)} \, \hat{x}^2). \tag{36}$$

Proof. From the premiss of the corollary, $f(x_M + \hat{x})$ is expanded around $\hat{x} = 0$ as

$$f(x_M + \hat{x}) = f(x_M) - \alpha \hat{x} n^{-\frac{1}{2}(1-\rho)} + o(n^{-\frac{1}{2}(1-\rho) - \frac{\nu}{2}(1-\rho)}) \quad (\hat{x} \geq 0)$$

$$= \quad f(x_M) + \alpha\hat{x}n^{-\frac{1}{2}(1-\rho)} + o(n^{-\frac{1}{2}(1-\rho)-\frac{\nu}{2}(1-\rho)}) \quad (\hat{x} < 0), \quad (37)$$

therefore δ is expressed with $s = \hat{x}_j$ as

$$\delta = \alpha|s|n^{-\frac{1}{2}(1-\rho)} + o(n^{-\frac{1}{2}(1-\rho)-\frac{\nu}{2}(1-\rho)}). \quad (38)$$

This form coincides with that of δ expressed in (13). Substituting $\alpha|\hat{x}|$ in (14) and (15) for $\kappa(x)$, we get $p_M(\hat{x})$ as

$$p_M(\hat{x}) = \frac{\alpha}{\sqrt{2\pi f(x_M)}}\, exp(-\frac{\alpha^2}{2f(x_M)}\,\hat{x}^2). \quad (39)$$

In this case the limit probability density function of sample modes follows the normal distribution. As well as the previous case, the function is independent from ρ. ◀

5. Implementation of equidistance intervals

To estimate the limit probability density function of sample modes based on the above theorem and corollaries, the sample space must be divided with equidistance intervals properly.

When the sample space has finite range such as $[a, b)$, the space is simply divided with $T = \lfloor n^{-\rho} \rfloor$ equidistance intervals where $\lfloor\ \rfloor$ is floor function as

$$[x_0(= a), x_1),\ [x_1, x_2),\ ...,\ [x_{T-1}, x_T(= b)). \quad (40)$$

After normalizing the sample space, we set the width of interval d as $n^{-\rho}$.

When the range of the sample space is infinite, we must cut off the tail(s) of that range in order to provide a finite range of the sample subspace as seen in the figure 2. Most of random variables must lie in that finite range made by cutting off the tail(s). The equidistance intervals mentioned in the earlier case is applied to that finite range, and normalization is carried out on it. Since the mode is insensitive to outliers in general,[3] the modes obtained by using the above theorems on the sample subspaces in which the tails are cut off properly but with the different manners coincide each other.

6. Applications to engineering fields

The significant difference between the limit behaviours of sample means and sample modes for independent random variables is in their forms. Namely the former has Gaussian curve, while the latter has not only Gaussian curve, but also

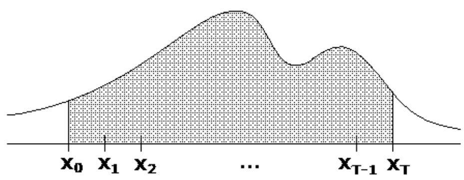

Figure 2: Cut off the tails of distribution

other forms of distribution. Moreover the latter behaviour does not depend on the variance of sample modes as seen in the form of (14). It implies that when the convergence of sample means is slow, measuring sample modes along with the methodology described above is useful for removing background noise on the signal processing since generally it needs quick filtering the received signals under a certain frequency band width.

In engineering fields the mode is sometimes more appropriate estimator than mean for acquired data. For example, when the histogram for the data shows two peaks and their mean is in the valley between those peaks, measuring mean is generally meaningless. In this case, making the mode, or higher peak between them represent the data is more reasonable for engineers, and the theorem and its corollaries mentioned above provides quantitative methodology to evaluate the behaviour of those sample modes.

7. Summary

The methodology to obtain the limit behaviour of sample modes on the independent random variables which has continuous probability density is discussed. Firstly the sample space is divided with equidistance intervals and the frequency on each interval is estimated by using a normal distribution as the limit of binomial distribution, and the probability of the event that a specified interval becomes the mode is discussed based on that normal distribution.

Next we discuss the generic relation of the deviation from global maximum of probability density function with the limit behaviour of sample modes when the interval converges to zero. This generic relation indicates the form of limit behaviour of sample modes is determined by the local feature of probability density function around its global maximum. We survey two specific limit behaviours of sample modes derived from the probability density functions having 1) smooth and 2) cusp-like peak around the global maximum.

The above methodology provides quantitative evaluations for the behaviour of sample modes in various engineering fields. Further, since the behaviour of sample modes is independent from the variance of distribution, applying that methodology under some conditions shows much faster convergence, which is a useful feature for filtering signals.

Additionally the method for implementing equidistance intervals on the sample space is discussed. When the sample space consists of infinite range, that range must be modified by cutting off the tails properly in order to apply the results mentioned above. We discuss how to cut off the tails of infinite range of sample space to obtain the sample subspace having finite range.

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