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Anti-Periodic Solutions for a Class of Fourth-Order Nonlinear Differential Equations with Multiple Deviating Arguments

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Abstract. This paper deals with a class of fourth-order nonlinear differential equations with multiple deviating arguments, and some sufficient conditions are established for the existence and exponential stability of anti-periodic solutions of the equation.

Key Words and Phrases: fourth order non-linear differential equation, exponential stability, anti-periodic solution, multiple deviating arguments.
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1. Introduction

Consider the following fourth order nonlinear differential equation with multiple deviating arguments

$$x^{(4)}(t)+a(t)x^{(3)}(t)+b(t)x^{(2)}(t)+c(t)x^{(1)}(t)+g_0(t,x(t))+\sum_{i=1}^{n}g_i(t,x(t-\tau_i(t))=p(t), \quad (1)$$

where $a, b, c, p, \tau_i(t) \geq 0$ $(i = 1, 2...n)$ are continuous functions on $R = (-\infty, \infty)$, and $g_i(i = 0, 1, 2...n)$ are continuous functions on R^2.

$y(t) = \dfrac{dx(t)}{dt} + d_1x(t)$, $z(t) = \dfrac{dy(t)}{dt} + d_2y(t)$ and $w(t) = \dfrac{dz(t)}{dt} + d_3z(t)$, where d_1, d_2 and d_3 are some constants. Then we can transform (1) into the following system

$$\begin{aligned}
\frac{dx(t)}{dt} &= -d_1x(t) + y(t),\\
\frac{dy(t)}{dt} &= -d_2y(t) + z(t),\\
\frac{dz(t)}{dt} &= -d_3z(t) + w(t),\\
\frac{dw(t)}{dt} &= -(a(t) - d_1 - d_2 - d_3)w(t)
\end{aligned}$$

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$$
\begin{aligned}
&+(-(d_1+d_2-a(t))(d_1+d_2+d_3)+(d_1d_2-b(t))-d_3^2)z(t)\\
&+((d_1+d_2-a(t))(d_1^2+d_1d_2+d_2^2)-(d_1d_2-b(t))(d_1+d_2)-c(t))y(t)\\
&+((a(t)-d_1-d_2)d_1^3+(d_1d_2-b(t))d_1^2+d_1c(t))x(t))-g_0(t,x(t))\\
&-\sum_{i=1}^{n} g_i(t,x(t-\tau_i(t))+p(t). \qquad (2)
\end{aligned}
$$

In applied science some practical problems are associated with higher-order nonlinear differential equations, such as nonlinear oscillations [1]–[4], electronic theory [5], biological model and other models [6, 7]. Just as above, in the past few decades, the study for higher order differential equations has been paid attention to by many scholars. Many results relative to the stability, boundedness of solutions, and existence of periodic and anti-periodic solutions for higher-order nonlinear differential equations have been obtained (see [8]–[16] and references therein).

Besides, the authors in [17] used the Leray-Schauder degree theory to establish some new results on the existence and uniqueness of anti-periodic solutions for a kind of nonlinear second order Rayleigh equations with delays of the form

$$x'' + f(t, x'(t)) + g(t, x(t-\tau(t)) = e(t).$$

The authors in [18] and the author in [19] used the Leray Schauder degree theory to establish new results on the existence and uniqueness of anti-periodic solutions for a class of nonlinear nth-order differential equations with delays of the forms

$$x^{(n)} + f(t, x^{(n-1)}(t)) + g(t, x(t-\tau(t)) = e(t),$$

and

$$x^{(n)} + f(t, x^{(n-1)}(t)) + g_1(t, x(t-\tau_1(t)) + g_2(t, x(t-\tau_2(t)) = e(t),$$

respectively.

The authors in [20] pointed out the existence of anti-periodic solutions to a class of fourth-order nonlinear differential equations with a deviating argument in Remark 3.1 while considering the existence of anti-periodic solutions for a class of fourth-order nonlinear differential equations with variable coefficients in the form of

$$u'''(t) - a(t)u'''(t) - b(t)u''(t) - c(t)u'(t) - g(t, u(t)) = e(t),$$

by applying the method of coincidence degree.

On the other hand, the authors in [21] established some sufficient conditions on the existence and exponential stability of anti-periodic solutions for a class of Cohen–Grossberg neutral networks(CGNNs) with time-varying delays.

The authors in [22] considered the Liénard-type systems with multiple varying time delays by establishing some sufficient conditions for the existence and exponential stability of the almost periodic solutions.

Especially, the authors in [23] obtained some results as in [21] and [22] for a class of third-order nonlinear differential equations with a deviating argument of the form

$$x'''(t) + a(t)x''(t) + b(t)x'(t) + g_1(t, x(t)) + g_2(t, x(t - \tau(t)) = p(t),$$

by using Lyapunov functional method and differential inequality technique.

As mentioned above, there are few studies related to the existence of anti-periodic solutions for a class of fourth order nonlinear differential equations with delay so far. Hence, in this study we extend the results of [23] to the fourth order nonlinear differential equations with multiple deviating arguments by not using the methods of [17]–[20]. Moreover, it is well known that the existence of anti-periodic solutions plays a key role in characterizing the behavior of nonlinear differential equations (see [24]–[28]). Thus, it is worthwhile to continue to investigate the existence and stability of anti-periodic solutions of Eq. (1).

A primary purpose of this paper is to study the problem of anti-periodic solutions of Eq. (1). We will establish some sufficient conditions for the existence and exponential stability of the anti-periodic solutions of Eq. (1). Incidentally, we do not deal with exact solution, which is hard to be determined, of the Eq. (1). Therefore, we shall concentrate on some qualitative behaviors of the solution $(x(t), y(t), z(t), w(t))$ of the system (2), exclusively.

The paper is organized as follows. After giving some basic definitions and assumptions in Section 2, we establish some preliminary results which are important in the proofs of our main results, in Section 3. Based on the preparations in Section 3, we state and prove our main results in Section 4. Moreover, an illustrative example is given in Section 5.

2. Definitions and Assumptions

Throughout this paper, it will be assumed that there exists a constant $T > 0$ such that

$$\begin{aligned} p(t+T) &= -p(t),\ \sum_{i=0}^{n} g_i(t+T, u) = -\sum_{i=0}^{n} g_i(t, -u),\ a(t+T) = a(t), \\ b(t+T) &= b(t),\ c(t+T) = c(t),\ \tau(t+T) = \tau(t),\ \forall t, u \in R. \end{aligned} \tag{3}$$

We suppose that there exists a constant L^+ such that $L^+ > \sup_{t \in R} |p(t)|$.

We assume that $h = \max_{1 \le i \le n} \left\{ \sup_{t \in R} \tau_i(t) \right\} \ge 0$. Let $C([-h, 0], R)$ denote the space of continuous functions $\varphi : [-h, 0] \to R$ with the supremum norm $\|.\|$.

Definition 1. *([29]) Suppose $h \ge 0$ is a given number, $R = (-\infty, \infty)$, R^n is an $n-$dimensional linear vector space over the reals with the norm $\|.\|$, $C([a, b], R^n)$ is the Banach space of continuous functions mapping the interval $[a, b]$ into R^n with the topology of uniform convergence. If $[a, b] = [-h, 0]$, we let $\mathbf{C} = C([-h, 0], R^n)$ and define the norm of an element Φ in $\mathbf{C}$ by $\|\Phi\| = \sup_{-h \le s \le 0} \Phi(s)$. If $\sigma \in R$, $T \ge 0$ and $\mathbf{x} \in C([\sigma - h, \sigma + T], R^n)$, then for any $t \in [\sigma, \sigma + T]$, we let $\mathbf{x}_t \in \mathbf{C}$ be defined by $\mathbf{x}_t(s) = \mathbf{x}(t + s)$, $-h \le s \le 0$. If*

D is a subset of $R \times \mathbf{C}$, $f : D \to R^n$ is a given function and "$\cdot$" represents the right-hand derivative, we say that the relation

$$\dot{\mathbf{x}}(t) = f(t, \mathbf{x}_t), \tag{4}$$

is a retarded(deviated) functional differential equation on D. A function $\mathbf{x}$ is said to be a solution of (4) on $[\sigma - h, \sigma + T]$, if $\sigma \in R$, $T > 0$ such that $\mathbf{x} \in C([\sigma - h, \sigma + T], R^n)$, $(t, \mathbf{x}_t) \in D$ and $\mathbf{x}(t)$ satisfies the equation (4).

We assume that $h = \max\limits_{1 \le i \le n} \left\{ \sup\limits_{t \in R} \tau_i(t) \right\} \ge 0$. Let $C([-h, 0], R)$ denote the space of continuous functions $\varphi : [-h, 0] \to R$ with the supremum norm $\|.\|$. It is known from [29, 30, 31] that there exists a solution of (2) on an interval $[0, T)$ satisfying the initial condition and satisfying (1) on $[0, T)$ for $g_i (i = 0, 1, 2, ..., n), \varphi, a, b, c, p$ and $\tau_i(t)(i = 1, 2, ..., n)$ which are continuous, and for a given continuous initial function $\varphi \in C([-h, 0], R)$ and a vector $(y_0, z_0, w_0) \in R^3$. If the solution remains bounded, then $T = +\infty$. We denote such a solution by

$$(x(t), y(t), z(t), w(t)) =$$
$$(x(t, \varphi, y_0, z_0, w_0), y(t, \varphi, y_0, z_0, w_0), z(t, \varphi, y_0, z_0, w_0), w(t, \varphi, y_0, z_0, w_0)),$$

where $y(s) = y(0)$, $z(s) = z(0)$ and $w(s) = w(0)$ for all $s \in [-h, 0]$. Then it follows that $(x(t), y(t), z(t), w(t))$ can be defined on $[-h, +\infty)$.

Definition 2. *Let $u(t) : R \to R$ be continuous in t. Then $u(t)$ is said to be T-anti-periodic on R, if $u(t + T) = -u(t)$ for all $t \in R$.*

Definition 3. *Let $Z^*(t) = (x^*(t), y^*(t), z^*(t), w^*(t))$ be a T-anti- periodic solution of system (2) with initial value $(\varphi^*(t), y_0^*, z_0^*, w_0^*) \in C([-h, 0], R) \times R^3$. If there exist constants $\lambda > 0$ and $M > 1$ such that*

$$\begin{aligned} & \max\{|x(t) - x^*(t)|, |y(t) - y^*(t)|, |z(t) - z^*(t)|, |w(t) - w^*(t)|\} \\ \le \; & M \max\{\|\varphi(t) - \varphi^*(t)\|, |y_0 - y_0^*|, |z_0 - z_0^*|, |w_0 - w_0^*|\} e^{-\lambda t}, \forall t > 0, \end{aligned}$$

for every solution $Z(t) = (x(t), y(t), z(t), w(t))$ of system (2) with any initial value $(\varphi(t), y_0, z_0, w_0) \in C([-h, 0], R) \times R^3$, where $\|\varphi(t) - \varphi^*(t)\| = \sup\limits_{t \in [-h, 0)} |\varphi(t) - \varphi^*(t)|$, then $Z^*(t)$ is said to be globally exponentially stable.

In this work, we also assume that the following conditions hold:

There exist constants $d_1 > 1$, $d_2 > 1$, $d_3 > 1$, $d_4 > 0$ and nonnegative constants $L_i (i = 0, 1, 2, ..., n)$ such that

i) $\left| ((a(t) - d_1 - d_2) d_1^3 + (d_1 d_2 - b(t)) d_1^2 + d_1 c(t)) u) - g_0(t, u) \right),$
$- \left((a(t) - d_1 - d_2) d_1^3 + (d_1 d_2 - b(t)) d_1^2 + d_1 c(t)) v) - g_0(t, v) \right) \right| \le L_0 |u - v|$ for all $t, u, v \in R$,

ii) $|g_i(t, u) - g_i(t, v)| \le L_i |u - v|$ for all $t, u, v \in R$ and $i = 1, 2, ..., n$,

iii) $d_4 = \inf\limits_{t \in R} (a(t) - d_1 - d_2 - d_3)$

$$-(\sup_{t\in R}\left|-(d_1+d_2-a(t))(d_1+d_2+d_3)+(d_1d_2-b(t))-d_3^2\right|$$
$$+\sup_{t\in R}\left|(d_1+d_2-a(t))(d_1^2+d_1d_2+d_2^2)-(d_1d_2-b(t))(d_1+d_2)-c(t)\right|)>\sum_{i=0}^{n}L_i.$$

3. Preliminary Results

The following lemmas will be useful to prove our main results.

Lemma 1. *Let $(i-iii)$ hold. Suppose that $(\widetilde{x}(t),\widetilde{y}(t),\widetilde{z}(t),\widetilde{w}(t))$ is a solution of system (2) with initial conditions $\widetilde{x}(s)=\widetilde{\varphi}(s)$, $\widetilde{y}(0)=y_0$, $\widetilde{z}(0)=z_0$, $\widetilde{w}(0)=w_0$,*

$$\max\{|\widetilde{\varphi}(s)|,|y_0|,|z_0|,|w_0|\}<\frac{L^+}{\eta},\ s\in[-h,0], \tag{5}$$

where $\eta=\min\left\{d_1-1,d_2-1,d_3-1,\ d_4-\sum_{i=0}^{n}L_i\right\}$. Then

$$\max\{|\widetilde{x}(t)|,|\widetilde{y}(t)|,|\widetilde{z}(t)|,|\widetilde{w}(t)|\}<\frac{L^+}{\eta}\ \textit{for all } t\geq 0. \tag{6}$$

Proof. Assume, by contrapositive, that (6) does not hold. Then, one of the following cases must occur.

Case 1: There exists $t_1>0$ such that

$$\max\{|\widetilde{x}(t_1)|,|\widetilde{y}(t_1)|,|\widetilde{z}(t_1)|,|\widetilde{w}(t_1)|\}=|\widetilde{x}(t_1)|=\frac{L^+}{\eta},$$

and

$$\max\{|\widetilde{x}(t)|,|\widetilde{y}(t)|,|\widetilde{z}(t)|,|\widetilde{w}(t)|\}\ <\ \frac{L^+}{\eta}, \tag{7}$$

where $t\in[-h,t_1)$.

Case 2: There exists $t_2>0$ such that

$$\max\{|\widetilde{x}(t_2)|,|\widetilde{y}(t_2)|,|\widetilde{z}(t_2)|,|\widetilde{w}(t_2)|\}=|\widetilde{y}(t_2)|=\frac{L^+}{\eta},$$

and

$$\max\{|\widetilde{x}(t)|,|\widetilde{y}(t)|,|\widetilde{z}(t)|,|\widetilde{w}(t)|\}\ <\ \frac{L^+}{\eta}, \tag{8}$$

where $t\in[-h,t_2)$.

Case 3: There exists $t_3>0$ such that

$$\max\{|\widetilde{x}(t_3)|,|\widetilde{y}(t_3)|,|\widetilde{z}(t_3)|,|\widetilde{w}(t_3)|\}=|\widetilde{z}(t_3)|=\frac{L^+}{\eta},$$

and

$$\max\{|\widetilde{x}(t)|, |\widetilde{y}(t)|, |\widetilde{z}(t)|, |\widetilde{w}(t)|\} < \frac{L^+}{\eta}, \tag{9}$$

where $t \in [-h, t_3)$.

Case 4: There exists $t_4 > 0$ such that

$$\max\{|\widetilde{x}(t_4)|, |\widetilde{y}(t_4)|, |\widetilde{z}(t_4)|, |\widetilde{w}(t_4)|\} = |\widetilde{w}(t_4)| = \frac{L^+}{\eta},$$

and

$$\max\{|\widetilde{x}(t)|, |\widetilde{y}(t)|, |\widetilde{z}(t)|, |\widetilde{w}(t)|\} < \frac{L^+}{\eta}, \tag{10}$$

where $t \in [-h, t_4)$.

If **Case 1** holds, calculating the upper right derivative of $|\widetilde{x}(t)|$, together with $(i-iii)$, (2) and (7) imply that

$$\begin{aligned} 0 &\leq D^+(|\widetilde{x}(t_1)|) = sgn(\widetilde{x}(t_1))\{-d_1\widetilde{x}(t_1) + \widetilde{y}(t_1)\} \\ &\leq -d_1|\widetilde{x}(t_1)| + |\widetilde{y}(t_1)| \leq -(d_1 - 1)\frac{L^+}{\eta} < 0, \end{aligned}$$

which is a contradiction and implies that (6) holds.

If **Case 2** holds, calculating the upper right derivative of $|\widetilde{y}(t)|$, together with $(i-iii)$, (2) and (8) imply that

$$\begin{aligned} 0 &\leq D^+(|\widetilde{y}(t_2)|) = sgn(\widetilde{y}(t_2))\{-d_2\widetilde{y}(t_2) + \widetilde{z}(t_2)\} \\ &\leq -d_2|\widetilde{y}(t_2)| + |\widetilde{z}(t_2)| \leq -(d_2 - 1)\frac{L^+}{\eta} < 0, \end{aligned}$$

which is a contradiction and implies that (6) holds.

If **Case 3** holds, calculating the upper right derivative of $|\widetilde{z}(t)|$, together with $(i-iii)$, (2) and (9) imply that

$$\begin{aligned} 0 &\leq D^+(|\widetilde{z}(t_3)|) = sgn(\widetilde{z}(t_3))\{-d_3\widetilde{z}(t_3) + \widetilde{w}(t_3)\} \\ &\leq -d_3|\widetilde{z}(t_3)| + |\widetilde{w}(t_3)| \leq -(d_3 - 1)\frac{L^+}{\eta} < 0, \end{aligned}$$

which is a contradiction and implies that (6) holds.

If **Case 4** holds, calculating the upper right derivative of $|\widetilde{w}(t)|$, together with $(i-iii)$, (2) and (10) imply that

$$\begin{aligned} 0 \leq\ & D^+(|\widetilde{w}(t_4)|) = sgn(\widetilde{w}(t_4))\{-(a(t_4) - d_1 - d_2 - d_3)\widetilde{w}(t_4) \\ & +(-(d_1 + d_2 - a(t_4))(d_1 + d_2 + d_3) + (d_1d_2 - b(t_4)) - d_3^2)\widetilde{z}(t_4) \\ & +((d_1 + d_2 - a(t_4))(d_1^2 + d_1d_2 + d_2^2) - (d_1d_2 - b(t_4))(d_1 + d_2) - c(t_4))\widetilde{y}(t_4) \end{aligned}$$

$$+((a(t_4)-d_1-d_2)d_1^3+(d_1d_2-b(t_4))d_1^2+d_1c(t_4))\widetilde{x}(t_4))-g_0(t_4,\widetilde{x}(t_4))$$
$$-\sum_{i=1}^{n}g_i(t_4,\widetilde{x}(t_4-\tau_i(t_4))+p(t_4)\}$$

$$\begin{aligned}\leq\ & -\inf_{t\in R}\,(a(t_4)-d_1-d_2-d_3)\,|\widetilde{w}(t_4)| \\ & +\sup_{t\in R}\left|-(d_1+d_2-a(t_4))(d_1+d_2+d_3)+(d_1d_2-b(t_4))-d_3^2\right||\widetilde{z}(t_4)| \\ & +\sup_{t\in R}\left|(d_1+d_2-a(t_4))(d_1^2+d_1d_2+d_2^2)-(d_1d_2-b(t_4))(d_1+d_2)-c(t_4)\right||\widetilde{y}(t_4)| \\ & +L_0\,|\widetilde{x}(t_4)|+\sum_{i=1}^{n}L_i\,|\widetilde{x}(t_4-\tau_i(t_4))|+|p(t_4)| \\ \leq\ & -(d_4-\sum_{i=0}^{n}L_i)\frac{L^+}{\eta}+|p(t_4)|<0,\end{aligned}$$

which is a contradiction and implies that (6) holds. The proof is now complete.

Remark 1. *In view of the boundedness of this solution, from the theory of functional differential equations in [27], it follows that* $(\widetilde{x}(t),\widetilde{y}(t),\widetilde{z}(t),\widetilde{w}(t))$ *can be defined on* $[0,\infty)$.

Lemma 2. *Let* $(i-iii)$ *hold. Moreover, assume that* $Z^*(t)=(x^*(t),y^*(t),z^*(t),w^*(t))$ *is a solution of system (2) with initial value* $(\varphi^*(t),y_0^*,z_0^*,w_0^*)\in C\left([-h,0],R\right)\times R^3$. *Then, there exist constants* $\lambda>0$ *and* $M>1$ *such that*

$$\max\{|x(t)-x^*(t)|\,,|y(t)-y^*(t)|\,,|z(t)-z^*(t)|\,,|w(t)-w^*(t)|\}$$
$$\leq M\max\{\|\varphi(t)-\varphi^*(t)\|\,,|y_0-y_0^*|\,,|z_0-z_0^*|\,,|w_0-w_0^*|\}e^{-\lambda t},$$

for all $t>0$, *for every solution* $Z(t)=(x(t),y(t),z(t),w(t))$ *of system (2) with any initial value* $(\varphi(t),y_0,z_0,w_0)\in C\left([-h,0],R\right)\times R^3$.

Proof. Since $\min\left\{d_1-1,d_2-1,d_3-1,\ d_4-\sum_{i=0}^{m}L_i\right\}>0$, it follows that there exists constant γ such that

$$\gamma=\min\left\{d_1-1-\lambda,d_2-1-\lambda,d_3-1-\lambda,\ d_4-L_0-\sum_{i=1}^{n}L_ie^{\lambda h}-\lambda\right\}>0. \tag{11}$$

Let $Z^*(t)=(x^*(t),y^*(t),z^*(t),w^*(t))$ be a solution of system (2) with initial value $(\varphi^*(t),y_0^*,z_0^*,w_0^*)\in C\left([-h,0],R\right)\times R^3$ and $Z(t)=(x(t),y(t),z(t),w(t))$ be an arbitrary solution of system (2) with any initial value $(\varphi(t),y_0,z_0,w_0)\in C\left([-h,0],R\right)\times R^3$. Set $\overline{u}_1(t)=x(t)-x^*(t)$, $\overline{u}_2(t)=y(t)-y^*(t)$, $\overline{u}_3(t)=z(t)-z^*(t)$ and $\overline{u}_4(t)=w(t)-w^*(t)$. Then

$$\frac{d\overline{u}_1(t)}{dt}=-d_1\overline{u}_1(t)+\overline{u}_2(t),$$

$$\begin{aligned}\frac{d\overline{u}_2(t)}{dt} &= -d_2\overline{u}_2(t)+\overline{u}_3(t),\\ \frac{d\overline{u}_3(t)}{dt} &= -d_3\overline{u}_3(t)+\overline{u}_4(t),\end{aligned}$$

$$\begin{aligned}\frac{d\overline{u}_4(t)}{dt} &= -(a(t)-d_1-d_2-d_3)w(t)\\ &+(-(d_1+d_2-a(t))(d_1+d_2+d_3)+(d_1d_2-b(t))-d_3^2)z(t)\\ &+((d_1+d_2-a(t))(d_1^2+d_1d_2+d_2^2)-(d_1d_2-b(t))(d_1+d_2)-c(t))y(t)\\ &+((a(t)-d_1-d_2)d_1^3+(d_1d_2-b(t))d_1^2+d_1c(t))x(t))-g_0(t,x(t))\\ &-\sum_{i=1}^{n} g_i(t,x(t-\tau_i(t))-(-(a(t)-d_1-d_2-d_3)w^*(t)\\ &+(-(d_1+d_2-a(t))(d_1+d_2+d_3)+(d_1d_2-b(t))-d_3^2)z^*(t)\\ &+((d_1+d_2-a(t))(d_1^2+d_1d_2+d_2^2)-(d_1d_2-b(t))(d_1+d_2)-c(t))y^*(t)\\ &+((a(t)-d_1-d_2)d_1^3+(d_1d_2-b(t))d_1^2+d_1c(t))x^*(t))-g_0(t,x^*(t))\\ &-\sum_{i=1}^{n} g_i(t,x^*(t-\tau_i(t)))). \qquad (12)\end{aligned}$$

We consider the Lyapunov functional

$$V_1(t)=|\overline{u}_1(t)|\,e^{\lambda t},\ V_2(t)=|\overline{u}_2(t)|\,e^{\lambda t},\ V_3(t)=|\overline{u}_3(t)|\,e^{\lambda t},\ V_4(t)=|\overline{u}_4(t)|\,e^{\lambda t}. \qquad (13)$$

Calculating the upper right derivative of $V_i(t)$ $(i=1,2,3,4)$ along the solution $(\overline{u}_1(t), \overline{u}_2(t),\overline{u}_3(t),\overline{u}_4(t))$ of system (12) with the initial value $(\varphi(t)-\varphi^*(t),y_0-y_0^*,z_0-z_0^*,w_0-w_0^*)$, we have

$$\begin{aligned}D^+(V_1(t)) &= \lambda e^{\lambda t}\,|\overline{u}_1(t)|+e^{\lambda t}sgn(\overline{u}_1(t))\{-d_1\overline{u}_1(t)+\overline{u}_2(t)\}\\ &\le e^{\lambda t}\left\{(\lambda-d_1)\,|\overline{u}_1(t)|+|\overline{u}_2(t)|\right\}, \qquad (14)\end{aligned}$$

$$\begin{aligned}D^+(V_2(t)) &= \lambda e^{\lambda t}\,|\overline{u}_2(t)|+e^{\lambda t}sgn(\overline{u}_2(t))\{-d_2\overline{u}_2(t)+\overline{u}_3(t)\}\\ &\le e^{\lambda t}\left\{(\lambda-d_2)\,|\overline{u}_2(t)|+|\overline{u}_3(t)|\right\}, \qquad (15)\end{aligned}$$

$$\begin{aligned}D^+(V_3(t)) &= \lambda e^{\lambda t}\,|\overline{u}_3(t)|+e^{\lambda t}sgn(\overline{u}_3(t))\{-d_3\overline{u}_3(t)+\overline{u}_4(t)\}\\ &\le e^{\lambda t}\left\{(\lambda-d_3)\,|\overline{u}_3(t)|+|\overline{u}_4(t)|\right\}, \qquad (16)\end{aligned}$$

and

$$\begin{aligned}&D^+(V_4(t))=\lambda e^{\lambda t}\,|\overline{u}_4(t)|+e^{\lambda t}sgn(\overline{u}_4(t))\{-(a(t)-d_1-d_2-d_3)w(t)\\ &+(-(d_1+d_2-a(t))(d_1+d_2+d_3)+(d_1d_2-b(t))-d_3^2)z(t)\end{aligned}$$

$$
\begin{aligned}
&+((d_1+d_2-a(t))(d_1^2+d_1d_2+d_2^2)-(d_1d_2-b(t))(d_1+d_2)-c(t))y(t)\\
&+((a(t)-d_1-d_2)d_1^3+(d_1d_2-b(t))d_1^2+d_1c(t))x(t))-g_0(t,x(t))\\
&-\sum_{i=1}^{n} g_i(t,x(t-\tau_i(t))-(-(a(t)-d_1-d_2-d_3)w^*(t)\\
&+(-(d_1+d_2-a(t))(d_1+d_2+d_3)+(d_1d_2-b(t))-d_3^2)z^*(t)\\
&+((d_1+d_2-a(t))(d_1^2+d_1d_2+d_2^2)-(d_1d_2-b(t))(d_1+d_2)-c(t))y^*(t)\\
&+((a(t)-d_1-d_2)d_1^3+(d_1d_2-b(t))d_1^2+d_1c(t))x^*(t))-g_0(t,x^*(t))\\
&-\sum_{i=1}^{n} g_i(t,x^*(t-\tau_i(t)))\}\le e^{\lambda t}\{(\lambda-\inf_{t\in R}(a(t)-d_1-d_2-d_3))\left|\overline{u}_4(t)\right|\\
&+\sup_{t\in R}\left|-(d_1+d_2-a(t))(d_1+d_2+d_3)+(d_1d_2-b(t))-d_3^2\right|\left|\overline{u}_3(t)\right|\\
&+\sup_{t\in R}\left|(d_1+d_2-a(t))(d_1^2+d_1d_2+d_2^2)-(d_1d_2-b(t))(d_1+d_2)-c(t)\right|\left|\overline{u}_2(t)\right|\\
&\qquad\qquad +L_0\left|\overline{u}_1(t)\right|+\sum_{i=1}^{n}L_i\left|\overline{u}_1(t-\tau_i(t))\right|\}. \qquad (17)
\end{aligned}
$$

Let $M>1$ denote an arbitrary real number and set

$$\theta=\max\{\|\varphi-\varphi^*\|,|y_0-y_0^*|,|z_0-z_0^*|,|w_0-w_0^*|\}>0.$$

It follows from (13) that $V_1(t)=|\overline{u}_1(t)|e^{\lambda t}<M\theta,\ V_2(t)=|\overline{u}_2(t)|e^{\lambda t}<M\theta,\ V_3(t)=|\overline{u}_3(t)|e^{\lambda t}<M\theta$ and $V_4(t)=|\overline{u}_4(t)|e^{\lambda t}<M\theta$ for all $t\in[-h,0]$.

We claim that

$$
\begin{aligned}
V_1(t) &= |\overline{u}_1(t)|e^{\lambda t}<M\theta,\ V_2(t)=|\overline{u}_2(t)|e^{\lambda t}<M\theta,\\
V_3(t) &= |\overline{u}_3(t)|e^{\lambda t}<M\theta \text{ and } V_4(t)=|\overline{u}_4(t)|e^{\lambda t}<M\theta, \qquad (18)
\end{aligned}
$$

for all $t>0$. Contrarily, one of the following cases must occur:

Case I: There exists $T_1>0$ such that

$$V_1(T_1)=M\theta \text{ and } V_i(t)<M\theta \text{ for all } t\in[-h,T_1),\ i=1,2,3,4. \qquad (19)$$

Case II: There exists $T_2>0$ such that

$$V_2(T_2)=M\theta \text{ and } V_i(t)<M\theta \text{ for all } t\in[-h,T_2),\ i=1,2,3,4. \qquad (20)$$

Case III: There exists $T_3>0$ such that

$$V_3(T_3)=M\theta \text{ and } V_i(t)<M\theta \text{ for all } t\in[-h,T_3),\ i=1,2,3,4. \qquad (21)$$

Case IV: There exists $T_4>0$ such that

$$V_4(T_4)=M\theta \text{ and } V_i(t)<M\theta \text{ for all } t\in[-h,T_4),\ i=1,2,3,4. \qquad (22)$$

If **Case I** holds, together with $(i-iii)$, (14) and (19) imply that

$$0 \le D^{+}(V_1(T_1)) \le e^{\lambda T_1}\{(\lambda - d_1)|\overline{u}_1(T_1)| + |\overline{u}_2(T_1)|\} \le (\lambda - (d_1 - 1))M\theta.$$

Thus, $0 \le \lambda - (d_1 - 1)$, which contradicts (11). Hence, (18) holds.

If **Case II** holds, together with $(i-iii)$, (15) and (20) imply that

$$0 \le D^{+}(V_2(T_2)) \le e^{\lambda T_2}\{(\lambda - d_2)|\overline{u}_2(T_2)| + |\overline{u}_3(T_2)|\} \le (\lambda - (d_2 - 1))M\theta.$$

Thus, $0 \le \lambda - (d_2 - 1)$, which contradicts (11). Hence, (18) holds.

If **Case III** holds, together with $(i-iii)$, (16) and (21) imply that

$$0 \le D^{+}(V_3(T_3)) \le e^{\lambda T_3}\{(\lambda - d_3)|\overline{u}_3(T_3)| + |\overline{u}_4(T_3)|\} \le (\lambda - (d_3 - 1))M\theta.$$

Thus, $0 \le \lambda - (d_1 - 1)$, which contradicts (11). Hence, (18) holds.

If **Case IV** holds, together with $(i-iii)$, (17) and (22) imply that

$$\begin{aligned} 0 \le\ & D^{+}(V_4(T_4)) \le (\lambda - \inf_{t\in R}(a(t) - d_1 - d_2 - d_3))|\overline{u}_4(T_4)| e^{\lambda T_4} \\ & + \sup_{t\in R}\left|-(d_1 + d_2 - a(t))(d_1 + d_2 + d_3) + (d_1 d_2 - b(t)) - d_3^2\right| |\overline{u}_3(T_4)| e^{\lambda T_4} \\ & + \sup_{t\in R}\left|(d_1 + d_2 - a(t))(d_1^2 + d_1 d_2 + d_2^2) - (d_1 d_2 - b(t))(d_1 + d_2) - c(t)\right| |\overline{u}_2(T_4)| e^{\lambda T_4} \\ & + L_0 |\overline{u}_1(T_4)| e^{\lambda T_4} + \sum_{i=1}^{n} L_i |\overline{u}_1(T_4 - \tau_i(T_4))| e^{\lambda(T_4 - \tau_i(T_4))} e^{\lambda \tau_i(T_4)} \\ \le\ & (\lambda + L_0 + \sum_{i=1}^{n} L_i e^{\lambda h} - d_4)M\theta. \end{aligned}$$

Thus, $0 \le \lambda + L_0 + \sum_{i=1}^{n} L_i e^{\lambda h} - d_4$, which contradicts (11). Hence, (18) holds. It follows that

$$\begin{aligned} & \max\{|x(t) - x^*(t)|, |y(t) - y^*(t)|, |z(t) - z^*(t)|, |w(t) - w^*(t)|\} \\ \le\ & M \max\{\|\varphi(t) - \varphi^*(t)\|, |y_0 - y_0^*|, |z_0 - z_0^*|, |w_0 - w_0^*|\} e^{-\lambda t} \end{aligned}$$

for all $t > 0$. This completes the proof of the Lemma.

Remark 2. *If $Z^*(t) = (x^*(t), y^*(t), z^*(t), w^*(t))$ is the T-anti-periodic solution of system (2), then it follows from Lemma 2 and Definition 3 that $Z^*(t)$ is globally exponentially stable.*

4. Main Results

In this section, we establish some results for the existence, uniqueness and exponential stability of the T-anti-periodic solution of (2).

Theorem 1. *Suppose that $(i-iii)$ are satisfied. Then system (2) has exactly one T-anti-periodic solution $Z^*(t)=(x^*(t),y^*(t),z^*(t),w^*(t))$. Moreover, $Z^*(t)$ is globally exponentially stable.*

Proof. Let $v(t)=(v_1(t),v_2(t),v_3(t),v_4(t))=(x(t),y(t),z(t),w(t))$ be a solution of system (2) with initial conditions (5). By Lemma 1, the solution $(x(t),y(t),z(t),w(t))$ is bounded and (6) holds. From (3), for any natural number k, we obtain

$$\begin{aligned}((-1)^{k+1}x(t+(k+1)T))' &= (-1)^{k+1}x'(t+(k+1)T)\\ &= (-1)^{k+1}[-d_1x(t+(k+1)T)+y(t+(k+1)T)]\\ &= -d_1(-1)^{k+1}x(t+(k+1)T)+(-1)^{k+1}y(t+(k+1)T),\end{aligned} \tag{23}$$

$$\begin{aligned}((-1)^{k+1}y(t+(k+1)T))' &= (-1)^{k+1}y'(t+(k+1)T)\\ &= (-1)^{k+1}[-d_2y(t+(k+1)T)+z(t+(k+1)T)]\\ &= -d_2(-1)^{k+1}y(t+(k+1)T)+(-1)^{k+1}z(t+(k+1)T),\end{aligned} \tag{24}$$

$$\begin{aligned}((-1)^{k+1}z(t+(k+1)T))' &= (-1)^{k+1}z'(t+(k+1)T)\\ &= (-1)^{k+1}[-d_3z(t+(k+1)T)+w(t+(k+1)T)]\\ &= -d_3(-1)^{k+1}z(t+(k+1)T)+(-1)^{k+1}w(t+(k+1)T)\end{aligned} \tag{25}$$

and

$$\begin{aligned}
& ((-1)^{k+1}w(t+(k+1)T))' = (-1)^{k+1}w'(t+(k+1)T)\\
= & (-1)^{k+1}[-(a(t+(k+1)T)-d_1-d_2-d_3)w(t+(k+1)T)\\
& +(-(d_1+d_2-a(t+(k+1)T))(d_1+d_2+d_3)\\
& +(d_1d_2-b(t+(k+1)T))-d_3^2)z(t+(k+1)T)\\
& +((d_1+d_2-a(t+(k+1)T))(d_1^2+d_1d_2+d_2^2)\\
& -(d_1d_2-b(t+(k+1)T))(d_1+d_2)-c(t+(k+1)T))y(t+(k+1)T)\\
& +((a(t+(k+1)T)-d_1-d_2)d_1^3+(d_1d_2-b(t+(k+1)T))d_1^2\\
& +d_1c(t+(k+1)T))x(t+(k+1)T))\\
& -g_0(t+(k+1)T,x(t+(k+1)T))\\
& -\sum_{i=1}^{n}g_i(t+(k+1)T,x(t+(k+1)T-\tau_i(t+(k+1)T))+p(t+(k+1)T)]
\end{aligned}$$

$$\begin{aligned}
= & -(a(t)-d_1-d_2-d_3)(-1)^{k+1}w(t+(k+1)T)\\
& +(-(d_1+d_2-a(t))(d_1+d_2+d_3)\\
& +(d_1d_2-b(t))-d_3^2)(-1)^{k+1}z(t+(k+1)T)\\
& +((d_1+d_2-a(t))(d_1^2+d_1d_2+d_2^2)
\end{aligned}$$

$$\begin{aligned}
&-(d_1d_2-b(t))(d_1+d_2)-c(t))(-1)^{k+1}y(t+(k+1)T)\\
&+((a(t)-d_1-d_2)d_1^3+(d_1d_2-b(t))d_1^2+d_1c(t))(-1)^{k+1}x(t+(k+1)T)\\
&-g_0(t,(-1)^{k+1}x(t+(k+1)T))\\
&-\sum_{i=1}^{n}g_i(t,(-1)^{k+1}x(t+(k+1)T-\tau_i(t)))+p(t).
\end{aligned}\tag{26}$$

Thus, for any natural number k, $(-1)^{k+1}v(t+(k+1)T)$ are the solutions of system (2) on R. Then, by Lemma 2, there exists a constant $M>0$ such that

$$\begin{aligned}
&\left|(-1)^{k+1}v_i(t+(k+1)T)-(-1)^{k}v_i(t+kT)\right|\\
\le\ & Me^{-\lambda(t+kT)}\sup_{-h\le s\le 0}\max_{1\le i\le 4}|v_i(s+T)+v_i(s)|\\
\le\ & 2e^{-\lambda(t+kT)}M\frac{L^+}{\eta}\text{ for all } t+kT>0,\ i=1,2,3,4.
\end{aligned}\tag{27}$$

Hence, for any natural m, we obtain

$$(-1)^{m+1}v_i(t+(m+1)T))=v_i(t)+\sum_{k=0}^{m}[(-1)^{k+1}v_i(t+(k+1)T)-(-1)^{k}v_i(t+kT)],\tag{28}$$

where $i=1,2,3,4$.

In view of (27), we can choose a sufficiently large constant $N>0$ and a positive constant α such that

$$\left|(-1)^{k+1}v_i(t+(k+1)T)-(-1)^{k}v_i(t+kT)\right|\le\alpha(e^{-\lambda T})^k\tag{29}$$

for all $k>N$, $i=1,2,3,4$ on any compact set of R. It follows from (28) and (29) that $(-1)^m v_i(t+mT)$ uniformly converges to a continuous function $Z^*(t)=(x^*(t),y^*(t),z^*(t),w^*(t))^T$ on any compact set of R.

Now, we will show that $Z^*(t)$ is T-anti-periodic solution of system (2). First, $Z^*(t)$ is T-anti-periodic, since $Z^*(t+T)=\lim_{m\to\infty}(-1)^m v(t+T+mT)=-\lim_{(m+1)\to\infty}(-1)^{m+1}v(t+(m+1)T)=-Z^*(t)$. Next, we prove that $Z^*(t)$ is a solution of (1). In fact, together with the continuity of the right side of (2), the relations (23), (24), (25) and (26) imply that $(-1)^{m+1}v(t+(m+1)T)'$ uniformly converges to a continuous function on any compact set of R. Thus, letting $m\to\infty$, we obtain

$$\begin{aligned}
\frac{dx^*(t)}{dt}&=-d_1x^*(t)+y^*(t),\\
\frac{dy^*(t)}{dt}&=-d_2y^*(t)+z^*(t),\\
\frac{dz^*(t)}{dt}&=-d_3z^*(t)+w^*(t),
\end{aligned}$$

$$\begin{aligned}\frac{dw^*(t)}{dt} &= -(a(t)-d_1-d_2-d_3)w^*(t)\\ &+(-(d_1+d_2-a(t))(d_1+d_2+d_3)+(d_1d_2-b(t))-d_3^2)z^*(t)\\ &+((d_1+d_2-a(t))(d_1^2+d_1d_2+d_2^2)-(d_1d_2-b(t))(d_1+d_2)-c(t))y^*(t)\\ &+((a(t)-d_1-d_2)d_1^3+(d_1d_2-b(t))d_1^2+d_1c(t))x^*(t))-g_0(t,x^*(t))\\ &-\sum_{i=1}^{n}g_i(t,x^*(t-\tau_i(t))+p(t).\end{aligned}$$

Therefore, $Z^*(t)$ is a unique T-anti-periodic solution of (2). Further, by Lemma 2 and Remark 2, we can prove that $Z^*(t)$ is globally exponentially stable. This completes the proof.

Remark 3. *The authors in [21] and [23] establish some sufficient conditions on the existence and exponential stability of anti-periodic solutions for a class of Cohen–Grossberg neutral networks(CGNNs) with time-varying delays and for a class of third-order nonlinear differential equations with a deviating argument, respectively, by using Lyapunov functional method and differential inequality technique as in Lemma 2 and Theorem 1. On the other hand, we only find that the authors stress the existence of anti-periodic solutions to a class of fourth-order nonlinear differential equations with a deviating argument by applying the method of coincidence degree in Remark 3.1 in [20]. In this work, for the first time some sufficient conditions for the existence and exponential stability of the anti-periodic solutions of Eq. (1) by extending the results of [23] to a class of fourth-order nonlinear differential equations with multiple deviating arguments. Thus, our results are new and complement to previously known results.*

5. An Example

Consider the following fourth-order non-linear differential equation

$$x^{(4)}(t)+(14-\frac{1}{1+|\sin t|})x^{(3)}(t)+(54-\frac{4}{1+|\sin t|})x^{(2)}(t)+(79-\frac{1is}{1+|\sin t|})x^{(1)}(t)$$

$$\begin{aligned}&+(37-\frac{3}{1+|\sin t|})x(t)+\sin x(t-|\sin t|)+\cos x(t-2\,|\sin t|)\\ &\quad+\cos t\sin x(t-e^{|\sin t|})+\sin t\cos x(t-e^{2|\sin t|})=\cos t. \qquad (30)\end{aligned}$$

Setting $y(t)=\frac{dx(t)}{dt}+2x(t)$ and $z(t)=\frac{dy(t)}{dt}+2y(t)$, we can transform (30) into the following system

$$\frac{dx(t)}{dt}=-2x(t)+y(t),$$

$$\frac{dy(t)}{dt} = -2y(t) + z(t),$$
$$\frac{dz(t)}{dt} = -2z(t) + w(t),$$

$$\begin{aligned}\frac{dw(t)}{dt} &= -(8 - \frac{1}{1+|\sin t|})w(t) + (2 - \frac{2}{1+|\sin t|})z(t) + (1 - \frac{1}{1+|\sin t|})y(t) \\ &(1 - \frac{1}{1+|\sin t|})x(t) - \sin x(t - |\sin t|) - \cos x(t - 2\,|\sin t|) \\ &- \cos t \sin x(t - e^{|\sin t|}) - \sin t \cos x(t - e^{2|\sin t|}) + \cos t. \qquad (31)\end{aligned}$$

Then we can satisfy the assumptions $(i - iii)$:
(i) $\left|1 - \frac{1}{1+|\sin t|}\right| \le L_0\,|u - v|$ for all $t, u, v \in R$,
(ii) $|g_1(t,u) - g_1(t,v)| = |\sin u - \sin v| \le L_1\,|u - v|\,,$
$|g_2(t,u) - g_2(t,v)| = |\cos u - \cos v| \le L_2\,|u - v|\,,$
$|g_3(t,u) - g_3(t,v)| = |\cos t \sin u - \cos t \sin v| \le L_3\,|u - v|\,,$
$|g_4(t,u) - g_4(t,v)| = |\sin t \cos u - \sin t \cos v| \le L_4\,|u - v|$ for all $t, u, v \in R$,
(iii) $d_4 = \inf\limits_{t\in R}(8 - \frac{1}{1+|\sin t|}) - (\sup\limits_{t\in R}\left|2 - \frac{2}{1+|\sin t|}\right| + \sup\limits_{t\in R}\left|1 - \frac{1}{1+|\sin t|}\right|) = \frac{11}{2} > L_0 + L_1 + L_2 + L_3 + L_4$, where $L_0 = L_1 = L_2 = L_3 = L_4 = 1$.

This implies that the system (31) has exactly one π-anti-periodic solution which is globally exponentially stable.

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