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On the Basis Properties of the Systems in the Intuitionistic Fuzzy Metric Space

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Abstract. Fuzzy metric space is considered. The concepts of fuzzy completeness, fuzzy minimality, fuzzy biorthogonality, fuzzy basicity and fuzzy space of coefficients are introduced. Strong completeness of fuzzy space of coefficients with regard to fuzzy metric and strong basicity of canonical system in this space are proved. Strong basicity criterion in fuzzy metric space is presented in terms of coefficient operator.

Key Words and Phrases: Fuzzy basicity, fuzzy completeness, fuzzy minimality, fuzzy biorthogonality, fuzzy space of coefficients

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1. Introduction

The concept of the space of coefficients belongs to the theory of bases. As is known, every basis in a Banach space has a Banach space of coefficients which is isomorphic to an initial one (see, e.g., [9;18]). Every nondegenerate system (to be defined later) in a Banach space generates the corresponding Banach space of coefficients with canonical basis (see, e.g., [2;9]). Therefore, space of coefficients plays an important role in the study of approximative properties of systems. It has very important applications in various fields of science, such as solid body physics, molecular physics, multiple production of particles, aviation, medicine, biology, data compression, etc (see, e.g., [4;7] and references therein). All these applications are closely related to wavelet analysis, and there arose a great interest in them lately [see, e.g., 4]. It is well known that many topological spaces are nonnormable. Therefore, the study of various properties of the space of coefficients in topological spaces is of special scientific interest. Applications in various branches of mathematics and natural sciences have lately induced a strong interest toward the study of different research problems in terms of fuzzy structures. A large number of research works is appearing these days which deal with the concept of fuzzy set-numbers, and the fuzzification of many classical theories has also been made. The concept of Schauder basis in *intuitionistic fuzzy normed space* and some results related to this concept have recently been studied in [5;12;13;16;19]. These works introduced the concepts of strongly and strongly intuitionistic fuzzy (Schauder) bases in *intuitionistic fuzzy Banach spaces* (IFBS in short). Some of their properties are revealed. The concepts of *strongly* and *weakly intuitionistic fuzzy approximation properties* (*sif-AP* and *wif-AP* in short, respectively) are also introduced in these works. It is proved that if the *intuitionistic fuzzy space* has a *sif-basis*, then it has a *sif-AP*. All the results in these works are obtained on condition that IFBS admits equivalent topology using the family of norms generated by *t-norm* and *t-conorm* (we will define them later).

In our work, we define the basic concepts of classical basis theory in *intuitionistic fuzzy metric spaces* (IFMS in short). Concept of *strongly fuzzy space of coefficients* is introduced. *Strong completeness of these spaces with regard to fuzzy metric* and *strong basicity of canonical system in them are proved. Strong basicity criterion in fuzzy metric space is presented in terms of coefficient operator.*

In Section 2, we recall some notations and concepts. In Section 3, we state our main results. We first define the *fuzzy space of coefficients* and then introduce the corresponding *fuzzy metrics.* We prove that for nondegenerate system the corresponding *fuzzy space of coefficients* is *strongly fuzzy complete.* Moreover, we show that the canonical system forms a *strong basis* for this space. It should be noted that similar results were earlier obtained in [3] for IFBS.

2. Some preliminary notations and concepts

We will use the standard notation: $\mathbb{N}$ will denote the set of all positive integers, $\mathbb{R}$ will be the set of all real numbers, $\mathbb{C}$ will be the set of complex numbers and K will denote a field of scalars ($K \equiv \mathbb{R}$, or $K \equiv \mathbb{C}$), $\mathbb{R}_+ \equiv (0, +\infty)$. We state some concepts and facts from IFMS theory to be used later.

Definition 1. *Let X be a linear space over a field K. Functions $\mu\,;\,\nu : X^2 \times \mathbb{R} \to [0\,,\,1]$ are called fuzzy metrics on X if the following conditions hold:*

1. $\mu(x;y;t) = 0\,,\ \forall t \le 0\,, \forall x,y \in X$;
2. $\mu(x;y;t) = 1\,,\ \forall t > 0 \Rightarrow\ x = y$;
3. $\mu(x;y;t) = \mu(y;x;t)\ ,\ \forall x,y \in X, \forall t \in \mathbb{R}$;
4. $\mu(x;z;t+s) \ge \min\{\mu(x;y;t)\,;\,\mu(y;z;s)\}\,,\ \forall x,z,y \in X\,,\ \forall t,s \in \mathbb{R}$;
5. $\mu(x;y;\cdot) : \mathbb{R} \to [0,1]$ *is a non-decreasing function of t for $\forall x,y \in X$ and* $\lim\limits_{t\to\infty} \mu(x;y;t) = 1$, $\forall x,y \in X$;
6. $\nu(x;y;t) = 1\,,\ \forall t \le 0\,,\ \forall x,y \in X$;
7. $\nu(x;y,t) = 0,\ \forall t > 0 \Rightarrow\ x = y$;
8. $\nu(x;y;t) = \nu(y;x;t)\ ,\ \forall x,y \in X, \forall t \in \mathbb{R}$;

9. $\nu(x;z;t+s) \le \max\{\nu(x;y;t);\, \nu(y;z;s)\}$, $\forall x,z,y \in X$, $\forall t,s \in \mathbb{R}$;

10. $\nu(x;y;\cdot) : \mathbb{R} \to [0,1]$ *is a non-increasing function of* t *for* $\forall x,y \in X$ *and* $\lim_{t\to\infty} \nu(x;y;t) = 0$, $\forall x,y \in X$*;*

11. $\mu(x;y;t) + \nu(x;y;t) \le 1$, $\forall x,y \in X$, $\forall t \in \mathbb{R}$.

Then the triplet $(X;\mu;\nu)$ *is called an intuitionistic fuzzy metric space (IFMS in short)*

Definition 2. *Let* $(X;\mu;\nu)$ *be a fuzzy metric space and let* $\{x_n\}_{n\in\mathbb{N}} \subset X$ *be some sequence. Then this sequence is said to be strongly intuitionistic fuzzy convergent to* $x \in X$ *(denoted by* $x_n \overset{s}{\to} x$, $n \to \infty$ *or* $s\text{-}\lim_{n\to\infty} x_n = x$ *in short) if and only if for* $\forall \varepsilon > 0$, $\exists n_0 = n_0(\varepsilon)$: $\mu(x_n;x;t) \ge 1-\varepsilon$, $\nu(x_n;x;t) \le \varepsilon$, $\forall n \ge n_0$, $\forall t \in \mathbb{R}$.

Definition 3. *Let* $(X;\mu;\nu)$ *be a fuzzy metric space and let* $\{x_n\}_{n\in\mathbb{N}} \subset X$ *be some sequence. Then sequence is said to be strongly Cauchy sequence if* $\lim_{n,m\to\infty} \mu(x_n;x_m;t) = 1$, $\lim_{n,m\to\infty} \nu(x_n;x_m;t) = 0$, *uniformly in* $\forall t \in \mathbb{R}$.

If every strongly Cauchy sequence converges (strongly) in X*, then* $(X;\mu;\nu)$ *is said to be strongly complete fuzzy metric space.*

More details on these concepts can be found in [1;5;6;8;10;11;13-15;17;19].

Let $(X;\mu;\nu)$ be an IFMS, and let $M \subset X$ be some set. By $L[M]$ we denote the linear span of M in X. The strongly intuitionistic fuzzy convergent closure of $L[M]$ will be denoted by $\overline{L_s[M]}$. If X is complete with respect to the strongly intuitionistic fuzzy convergence, then we will call it intuitionistic fuzzy strongly complete metric space (IFM_sS or X_s in short). Let X be an IFM_sS . We denote by X_s^* the linear space of linear and strongly continuous in IFM_sS functionals over the same field K.

Now we define the corresponding concepts of basis theory for IFMS. Let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be some system.

Definition 4. *System* $\{x_n\}_{n\in\mathbb{N}}$ *is called s-complete in* X_s*, if* $\overline{L_s\left[\{x_n\}_{n\in\mathbb{N}}\right]} \equiv X_s$.

Definition 5. *System* $\{x_n^*\}_{n\in\mathbb{N}} \subset X_s^*$ *is called s-biorthogonal to the system* $\{x_n\}_{n\in\mathbb{N}}$*, if* $x_n^*(x_k) = \delta_{nk}$, $\forall n,k \in \mathbb{N}$*, where* δ_{nk} *is the Kronecker symbol.*

Definition 6. *System* $\{x_n\}_{n\in\mathbb{N}} \subset X_s$ *is called s-linearly independent in* X*, if* $\sum_{n=1}^{\infty} \lambda_n x_n = 0$ *in* X_s *implies* $\lambda_n = 0$, $\forall n \in \mathbb{N}$.

Definition 7. *System* $\{x_n\}_{n\in\mathbb{N}} \subset X_s$ *is called an s-basis for* X_s *if* $\forall x \in X$, $\exists! \{\lambda_n\}_{n\in\mathbb{N}} \subset K : \sum_{n=1}^{\infty} \lambda_n x_n = x$ *in* X_s .

We will also need the following concept.

Definition 8. *System* $\{x_n\}_{n\in\mathbb{N}} \subset X$ *is called nondegenerate , if* $x_n \ne 0$, $\forall n \in \mathbb{N}$.

To obtain our main results we will use the following conditions on $IFMS$.

α) *linear operations of addition and multiplication by a scalar in* IFM_sS *are strongly continuous in* X, *i.e. from* $\lambda_n \to \lambda$, $n \to \infty$, *in* $\mathbb{C}$ *and from* $x_n \overset{s}{\to} x$, $y_n \overset{s}{\to} y$, $n \to \infty$, *in* X_s *it follows that* $\lambda_n x_n \overset{s}{\to} \lambda x$, $x_n + y_n \overset{s}{\to} x + y$, $n \to \infty$, *in* X_s.

β) *let* $\tau_{\mu,\nu}$ *be a topology for* X_s, *generated by a pair of* (μ, ν). *We will assume that the boundednesses of a set in the spaces* X_s *and* IFM_sS $(X; \mu; \nu)$ *are equivalent to each other with respect to the topology* $\tau_{\mu,\nu}$, *i.e. these concepts are the same in spaces* $(X; \tau_{\mu,\nu})$ *and* $(X; \mu; \nu)$.

3. Main results

3.1. Space of coefficients

Let $(X; \mu; \nu)$ be some IFM_sS, with conditions α), β), and $\{x_n\}_{n\in\mathbb{N}} \subset X$ be some nondegenerate system. Assume that

$$\mathscr{K}_{\bar{x}}^s \equiv \left\{ \{\lambda_n\}_{n\in\mathbb{N}} \subset C : \sum_{n=1}^{\infty} \lambda_n x_n \ \ converges\ in \ \ X_s \right\}.$$

It is not difficult to see that $\mathscr{K}_{\bar{x}}^s$ is a linear space with regard to component-specific summation and component-specific multiplication by a scalar. Take $\forall \bar{\lambda}, \bar{\mu} \in \mathscr{K}_{\bar{x}}^s$, $\bar{\lambda} \equiv \{\lambda_n\}_{n\in\mathbb{N}}$, $\bar{\mu} \equiv \{\mu_n\}_{n\in\mathbb{N}}$ and assume

$$\mu_{\mathscr{K}_{\bar{x}}^s} \left(\bar{\lambda}; \bar{\mu}; t\right) = \inf_m \mu \left(\sum_{n=1}^{m} \lambda_n x_n; \sum_{n=1}^{m} \mu_n x_n; t \right),$$

$$\nu_{\mathscr{K}_{\bar{x}}^s} \left(\bar{\lambda}; \bar{\mu}; t\right) = \sup_m \nu \left(\sum_{n=1}^{m} \lambda_n x_n; \sum_{n=1}^{m} \mu_n x_n; t \right).$$

Let's show that $\mu_{\mathscr{K}_{\bar{x}}^s}$ and $\nu_{\mathscr{K}_{\bar{x}}^s}$ satisfy the conditions 1)-11). At first let's consider $\mu_{\mathscr{K}_{\bar{x}}^s}$.

1) It is clear that $\mu_{\mathscr{K}_{\bar{x}}^s} \left(\bar{\lambda}; \bar{\mu}; t\right) = 0$, $\forall t \leq 0$.

2) Let $\mu_{\mathscr{K}_{\bar{x}}^s} \left(\bar{\lambda}; \bar{\mu}; t\right) = 1$, $\forall t > 0$. Hence, $\mu_{\mathscr{K}_{\bar{x}}^s} \left(\sum_{n=1}^{m} \lambda_n x_n; \sum_{n=1}^{m} \mu_n x_n; t\right) = 1$, $\forall m \in \mathbb{N}$, $\forall t > 0$. Suppose that the system $\{x_n\}_{n\in\mathbb{N}}$ is nondegenerate. It follows from the above-stated relations that for $m = 1$ we have $\mu (\lambda_1 x_1; \mu_1 x_1; t) = 1$, $\forall t > 0$. Hence, $\lambda_1 x_1 = \mu_1 x_1 \Rightarrow \lambda_1 = \mu_1$. Continuing this process, we come to the conclusion that$\lambda_n = \mu_n$, $\forall n \in \mathbb{N}$, i.e. $\bar{\lambda} = \bar{\mu}$.

3) It is clear that $\mu_{\mathscr{K}_{\bar{x}}^s} \left(\bar{\lambda}; \bar{\mu}; t\right) = \mu_{\mathscr{K}_{\bar{x}}^s} \left(\bar{\mu}; \bar{\lambda}; t\right)$, $\forall t \in \mathbb{R}$.

4) Let $\bar{\lambda}, \bar{\mu}, \bar{\nu} \in \mathscr{K}_{\bar{x}}^s$ and $s, t \in \mathbb{R}$. We have

$$\mu_{\mathscr{K}_{\bar{x}}^s} \left(\bar{\lambda}; \bar{\mu}; t + s\right) = \inf_m \mu \left(\sum_{n=1}^{m} \lambda_n x_n; \sum_{n=1}^{m} \mu_n x_n; t + s \right) \geq$$

$$\geq \inf_m \min \left\{ \mu \left(\sum_{n=1}^{m} \lambda_n x_n; \sum_{n=1}^{m} \nu_n x_n; t \right); \mu \left(\sum_{n=1}^{m} \nu_n x_n; \sum_{n=1}^{m} \mu_n x_n; s \right) \right\} =$$

$$= \min\left\{\inf_m \mu\left(\sum_{n=1}^m \lambda_n x_n; \sum_{n=1}^m \nu_n x_n; t\right); \inf_m \mu\left(\sum_{n=1}^m \nu_n x_n; \sum_{n=1}^m \mu_n x_n; s\right)\right\} =$$

$$= \min\left\{\mu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda};\bar{\nu};t\right); \mu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\nu};\bar{\mu};s\right)\right\}.$$

5) Let's show that $\mu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda};\bar{\mu};\cdot\right) : R \to [0,1]$ is a non-decreasing function of t for $\forall\bar{\lambda},\bar{\mu} \in \mathscr{K}_{\bar{x}}^s$ and $\lim_{t\to\infty} \mu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda};\bar{\mu};t\right) = 1$, $\forall\bar{\lambda},\bar{\mu} \in \mathscr{K}_{\bar{x}}^s$. As $\mu(x;y;\cdot)$ is a non-decreasing function on $\mathbb{R}$, it is not difficult to see that $\mu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda};\bar{\mu};\cdot\right)$ has the same property. Let us show that $\lim_{t\to\infty} \mu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda};\bar{\mu};t\right) = 1$. Take $\forall\varepsilon > 0$. Let $S_m^{(1)} = \sum_{n=1}^m \lambda_n x_n$, $S_m^{(2)} = \sum_{n=1}^m \mu_n x_n$ and $s\text{-}\lim_{m\to\infty} S_m^{(k)} = S^{(k)} \in X_s, k = 1,2$. It is clear that $\exists t_0 > 0 : \mu\left(S^{(1)};S^{(2)};t_0\right) \geq 1-\varepsilon$. Then it follows from the definition of s-lim that $\exists m_0 = m_0(\varepsilon;t_0) \in \mathbb{N} : \mu\left(S_m^{(k)};S^{(k)};t_0\right) \geq 1-\varepsilon$, $\forall m \geq m_0$, $k = 1,2$. Property 4) implies

$$\mu\left(S_m^{(1)};S_m^{(2)};3t_0\right) \geq \min\left\{\mu\left(S_m^{(1)};S^{(1)};t_0\right); \mu\left(S^{(1)};S_m^{(2)};2t_0\right)\right\},$$

$$\mu\left(S^{(1)};S_m^{(2)};2t_0\right) \geq \min\left\{\mu\left(S^{(1)};S^{(2)};t_0\right); \mu\left(S^{(2)};S_m^{(2)};t_0\right)\right\}.$$

Thus

$$\mu\left(S_m^{(1)};S_m^{(2)};3t_0\right) \geq \min\left\{\mu\left(S_m^{(1)};S^{(1)};t_0\right); \mu\left(S^{(1)};S^{(2)};t_0\right); \mu\left(S^{(2)};S_m^{(2)};t_0\right)\right\}.$$

As a result we obtain

$$\mu\left(S_m^{(1)};S_m^{(2)};3t_0\right) \geq 1-\varepsilon, \forall m \geq m_0. \tag{1}$$

As $\mu(x;y;\cdot)$ is a non-decreasing function of t , it follows from (1) that

$$\left(S_m^{(1)};S_m^{(2)};3t_0\right) \geq 1-\varepsilon, \forall m \geq m_0, \forall t \geq 3t_0. \tag{2}$$

We have

$$\mu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda};\bar{\mu};t\right) = \min\left\{\mu\left(S_k^{(1)};S_k^{(2)};t\right), k = \overline{1,m_0-1}; \inf_{m\geq m_0} \mu\left(S_m^{(1)};S_m^{(2)};t\right)\right\}. \tag{3}$$

As $\lim_{t\to\infty} \mu\left(S_k^{(1)};S_k^{(2)};t\right) = 1$ for $\forall k \in \mathbb{N}$, we have $\exists t_k(\varepsilon); \forall t \geq t_k(\varepsilon)$: $\mu\left(S_k^{(1)};S_k^{(2)};t\right) \geq 1-\varepsilon$, $k = \overline{1,m_0-1}$. Let $t_\varepsilon^0 = \max_{1\leq k\leq m_0-1} t_k(\varepsilon)$. Then it is clear that

$$\mu\left(S_k^{(1)};S_k^{(2)};t\right) \geq 1-\varepsilon, \forall t \geq t_\varepsilon^0. \tag{4}$$

It follows from (2) that

$$\inf_{m\geq m_0} \mu\left(S_m^{(1)};S_m^{(2)};t\right) \geq 1-\varepsilon, \ \forall t \geq 3t_0.$$

Let $t_\varepsilon = \max\left\{3t_0;\, t_\varepsilon^0\right\}$. Hence we obtain from (3) and (4) that

$$\mu_{\mathscr{K}^s_{\bar{x}}}\left(\bar{\lambda};\bar{\mu};\, t\right) \geq 1-\varepsilon, \forall t \geq t_\varepsilon.$$

Thus $\lim\limits_{t\to\infty} \mu_{\mathscr{K}^s_{\bar{x}}}\left(\bar{\lambda};\bar{\mu};\, t\right) = 1$, $\forall \bar{\lambda},\bar{\mu} \in \mathscr{K}^s_{\bar{x}}$.

6) As $\nu\left(x;y;\, t\right) = 1$, $\forall t \leq 0$, $\forall x,y \in X$, it is clear that $\nu_{\mathscr{K}^s_{\bar{x}}}\left(\bar{\lambda};\bar{\mu};\, t\right) = 1$, $\forall t \leq 0$, $\forall \bar{\lambda},\bar{\mu} \in \mathscr{K}^s_{\bar{x}}$.

7) Assume that $\nu_{\mathscr{K}^s_{\bar{x}}}\left(\bar{\lambda};\bar{\mu};\, t\right) = \backslash 0$, $\forall t > 0$ .Then $\nu\left(\sum_{n=1}^m \lambda_n x_n; \sum_{n=1}^m \mu_n x_n; t\right) = 0, \forall t > 0$, $\forall m \in \mathbb{N}$.For $m=1$ we have $\nu\left(\lambda_1 x_1; \mu_1 x_1; t\right) = 0$, $\forall t > 0 \Rightarrow \lambda_1 x_1 = \mu_1 x_1 \Rightarrow \lambda_1 = \mu_1$, if the system $\{x_n\}_{n\in\mathbb{N}}$ is nondegenerate. Continuing this way, we get $\lambda_n = \mu_n$, $\forall n \in \mathbb{N} \Rightarrow \bar{\lambda} = \bar{\mu}$.

8) Fulfillment of the condition $\nu_{\mathscr{K}^s_{\bar{x}}}\left(\bar{\lambda};\bar{\mu};\, t\right) = \nu_{\mathscr{K}^s_{\bar{x}}}\left(\bar{\mu};\, \bar{\lambda};t\right)$ is obvious.

9) Let $\bar{\lambda},\bar{\mu},\bar{\nu} \in \mathscr{K}^s_{\bar{x}}$ ($\bar{\lambda} \equiv \{\lambda_n\}_{n\in\mathbb{N}}$,$\bar{\mu} \equiv \{\mu_n\}_{n\in\mathbb{N}}$, $\bar{\nu} \equiv \{\nu_n\}_{n\in\mathbb{N}}$) and $s,t \in \mathbb{R}$. We have

$$\nu_{\mathscr{K}^s_{\bar{x}}}\left(\bar{\lambda};\bar{\mu};t+s\right) = \sup_m \nu\left(\sum_{n=1}^m \lambda_n x_n; \sum_{n=1}^m \mu_n x_n; s+t\right) \leq$$

$$\leq \sup_m \max\left\{\nu\left(\sum_{n=1}^m \lambda_n x_n; \sum_{n=1}^m \nu_n x_n; s\right); \nu\left(\sum_{n=1}^m \nu_n x_n; \sum_{n=1}^m \mu_n x_n; t\right)\right\} =$$

$$= \max\left\{\sup_m \nu\left(\sum_{n=1}^m \lambda_n x_n; \sum_{n=1}^m \nu_n x_n; s\right); \sup_m \nu\left(\sum_{n=1}^m \nu_n x_n; \sum_{n=1}^m \mu_n x_n; t\right)\right\} =$$

$$= \max\left\{\nu_{\mathscr{K}^s_{\bar{x}}}\left(\bar{\lambda};\bar{\nu};s\right); \nu_{\mathscr{K}^s_{\bar{x}}}\left(\bar{\nu};\bar{\mu};t\right)\right\}.$$

10) It follows from property 10) that $\nu_{\mathscr{K}^s_{\bar{x}}}\left(\bar{\lambda};\bar{\mu};\,\cdot\right)$ is a non-increasing function on $\mathbb{R}$. Let us show that $\lim\limits_{t\to\infty} \nu_{\mathscr{K}^s_{\bar{x}}}\left(\bar{\lambda};\bar{\mu};\, t\right) = 0$. Take $\forall \varepsilon > 0$. Let $S_m^{(1)} = \sum_{n=1}^m \lambda_n x_n$, $S_m^{(2)} = \sum_{n=1}^m \mu_n x_n$and $s\text{-}\lim\limits_{m\to\infty} S_m^{(k)} = S^{(k)} \in X_s, k=1,2$. It is clear that $\exists t_0 > 0$: $\nu\left(S^{(1)}; S^{(2)};\, t_0\right) \leq \varepsilon$. Then it follows from the definition of s-lim that $\exists m_0 = m_0\left(\varepsilon; t_0\right) \in \mathbb{N} : \nu\left(S_m^{(k)}; S^{(k)};\, t_0\right) \leq \varepsilon$, $\forall m \geq m_0$, $k=1,2$. Property 9) implies

$$\nu\left(S_m^{(1)}; S_m^{(2)};\, 3t_0\right) \leq \max\left\{\nu\left(S_m^{(1)}; S^{(1)};\, t_0\right); \nu\left(S^{(1)}; S_m^{(2)};\, 2t_0\right)\right\},$$

$$\nu\left(S^{(1)}; S_m^{(2)};\, 2t_0\right) \leq \max\left\{\nu\left(S^{(1)}; S^{(2)};\, t_0\right); \mu\left(S^{(2)}; S_m^{(2)};\, t_0\right)\right\}.$$

Thus

$$\nu\left(S_m^{(1)}; S_m^{(2)};\, 3t_0\right) \leq \max\left\{\nu\left(S_m^{(1)}; S^{(1)};\, t_0\right); \nu\left(S^{(1)}; S^{(2)};\, t_0\right); \nu\left(S^{(2)}; S_m^{(2)};\, t_0\right)\right\}.$$

As a result we obtain

$$\nu\left(S_m^{(1)}; S_m^{(2)};\, 3t_0\right) \leq \varepsilon, \forall m \geq m_0. \tag{5}$$

As $\nu(x; y; \cdot)$ is a non-increasing function of t, it follows from (5) that

$$\nu\left(S_m^{(1)}; S_m^{(2)}; t\right) \le \varepsilon, \forall m \ge m_0, \forall t \ge 3t_0. \tag{6}$$

We have

$$\nu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda}; \bar{\mu}; t\right) = \max\left\{\nu\left(S_k^{(1)}; S_k^{(2)}; t\right), k = \overline{1, m_0 - 1}; \sup_{m \ge m_0} \nu\left(S_m^{(1)}; S_m^{(2)}; t\right)\right\}. \tag{7}$$

As $\lim_{t\to\infty} \nu\left(S_k^{(1)}; S_k^{(2)}; t\right) = 0$ for $\forall k \in \mathbb{N}$, we have $\exists t_k(\varepsilon); \forall t \ge t_k(\varepsilon)$: $\nu\left(S_k^{(1)}; S_k^{(2)}; t\right) \le \varepsilon$, $k = \overline{1, m_0 - 1}$. Let $t_\varepsilon^0 = \max_{1 \le k \le m_0 - 1} t_k(\varepsilon)$. Then it is clear that

$$\nu\left(S_k^{(1)}; S_k^{(2)}; t\right) \le \varepsilon, \forall t \ge t_\varepsilon^0. \tag{8}$$

It follows from (6) that

$$\sup_{m \ge m_0} \nu\left(S_m^{(1)}; S_m^{(2)}; t\right) \le \varepsilon, \ \forall t \ge 3t_0.$$

Let $t_\varepsilon = \max\left\{3t_0; t_\varepsilon^0\right\}$. Hence we obtain from (7) and (8) that

$$\nu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda}; \bar{\mu}; t\right) \le \varepsilon, \forall t \ge t_\varepsilon.$$

Thus $\lim_{t\to\infty} \nu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda}; \bar{\mu}; t\right) = 0$, $\forall \bar{\lambda}, \bar{\mu} \in \mathscr{K}_{\bar{x}}^s$.

11) We have

$$\mu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda}; \bar{\mu}; t\right) + \nu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda}; \bar{\mu}; t\right) = \inf_m \mu\left(\sum_{n=1}^m \lambda_n x_n; \sum_{n=1}^m \mu_n x_n; t\right) +$$

$$+ \sup_m \nu\left(\sum_{n=1}^m \lambda_n x_n; \sum_{n=1}^m \mu_n x_n; t\right) \le \sup_m \left[\mu\left(\sum_{n=1}^m \lambda_n x_n; \sum_{n=1}^m \mu_n x_n; t\right)\right.$$

$$\left. + \nu\left(\sum_{n=1}^m \lambda_n x_n; \sum_{n=1}^m \mu_n x_n; t\right)\right] \le 1, \forall \bar{\lambda}, \bar{\mu} \in \mathscr{K}_{\bar{x}}^s, \forall t \in \mathbb{R}.$$

Thus, we have proved the validity of the following

Theorem 1. *Let $(X; \mu; \nu)$ be a strongly fuzzy metric space and let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a nondegenerate system. Then the space of coefficients $\left(\mathscr{K}_{\bar{x}}^s; \mu_{\mathscr{K}_{\bar{x}}^s}; \nu_{\mathscr{K}_{\bar{x}}^s}\right)$ is also strongly fuzzy metric space.*

3.2. Completeness of the space of coefficients.

In the sequel, we will assume that $(X;\mu;\nu)$ is strongly complete *IFMS.* Let us show that $\left(\mathscr{K}_{\bar{x}}^{s};\mu_{\mathscr{K}_{\bar{x}}^{s}};\nu_{\mathscr{K}_{\bar{x}}^{s}}\right)$ is also a strongly fuzzy complete metric space.

First we prove the following

Lemma 1. *Let $x_0 \neq 0$, $x_0 \in X$, and let $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be some sequence. If $s\text{-}\lim_{n\to\infty}(\lambda_n x_0) = 0$, i.e. $\forall\varepsilon>0$, $\exists n_0 = n_0(\varepsilon)$: $\mu(\lambda_n x_0;0;\,t) > 1-\varepsilon$, $\nu(\lambda_n x_0;0;\,t) < \varepsilon$, $\forall t\in\mathbb{R}_+$, $\forall n\geq n_0$; then $\lambda_n \to 0$, $n\to\infty$.*

Indeed, assume that the relation $\lim_{n\to\infty}\lambda_n = 0$ is not true. Suppose that $\{\lambda_n\}_{n\in\mathbb{N}}$ has a bounded subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$. Then $\exists\lambda_0\in C:\ \lambda_{n_k}\to\lambda_0,\ k\to\infty$. We have $\lambda_{n_k}x_0 \overset{s}{\to} \lambda_0 x_0,\ k\to\infty$, and hence $\lambda_0 = 0$, since *s-convergent* sequence has a unique limit. Assume that the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ has an unbounded subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$: $\lambda_{n_k}\to\infty,\ k\to\infty$. Consequently, $\lambda_{n_k}^{-1}\to 0,\ k\to\infty$. We have $\lambda_{n_k}^{-1}\cdot\lambda_{n_k}x = x\neq 0, \forall k\in\mathbb{N}$. On the other hand, $\lim_{k\to\infty}\left(\lambda_{n_k}^{-1}\lambda_{n_k}x\right) = \lim_{k\to\infty}\lambda_{n_k}^{-1}\lim_{k\to\infty}(\lambda_{n_k}x) = 0$. So, we came upon a contradiction which proves the lemma.

Take *s-fundamental* sequence $\left\{\bar{\lambda}_n\right\}_{n\in\mathbb{N}}\subset\mathscr{K}_{\bar{x}}^{s}$, $\lambda_n \equiv \left\{\lambda_k^{(n)}\right\}_{k\in\mathbb{N}}$. Then $\lim_{n,m\to\infty}\mu_{\mathscr{K}_{\bar{x}}^{s}}\left(\bar{\lambda}_n;\bar{\lambda}_m;\,t\right) = 1$ and $\lim_{n,m\to\infty}\nu_{\mathscr{K}_{\bar{x}}^{s}}\left(\bar{\lambda}_n;\bar{\lambda}_m;\,t\right) = 0$ uniformly in $t\in\mathbb{R}$, i.e.

$$\left.\begin{array}{l}\lim\limits_{n,m\to\infty}\inf\limits_{r}\mu\left(\sum_{k=1}^{r}\lambda_k^{(n)}x_k;\sum_{k=1}^{r}\lambda_k^{(m)}x_k\,;t\right) = 1\,,\\ \lim\limits_{n,m\to\infty}\sup\limits_{r}\nu\left(\sum_{k=1}^{r}\lambda_k^{(n)}x_k;\sum_{k=1}^{r}\lambda_k^{(m)}x_k\,;t\right) = 0\,,\end{array}\right\}\tag{9}$$

uniformly in $t\in\mathbb{R}$. In the sequel, we will assume that the functions μ and ν are shift invariant, i.e. the following condition holds:

12) $\mu(x;y;t) = \mu(x-z;y-z;t)\,,\nu(x;y;t) = \nu(x-z;y-z;t)\,,\forall x,y,z\in X,\ \forall t\in\mathbb{R}$. Taking into account the conditions 3) and 8), we immediately obtain that

$$\mu(x;0;t) = \mu(-x;0;t)\ ,\ \ \nu(x;0;t) = \nu(-x;0;t)\ ,\ \forall x\in X, \forall t\in\mathbb{R}.$$

It is absolutely clear that the functions $\mu_{\mathscr{K}_{\bar{x}}^{s}}$ and $\nu_{\mathscr{K}_{\bar{x}}^{s}}$ also satisfy these conditions. Thus

$$\mu(x;y;t) = \mu(-x;-y;t)\,,\nu(x;y;t) = \nu(-x;-y;t)\,,\forall x,y\in X,\ \forall t\in\mathbb{R}.\tag{10}$$

It follows directly from (9) that $\mu\left(\lambda_1^{(n)}x_1;\lambda_1^{(m)}x_1;\,t\right)\to 1$, $\nu\left(\lambda_1^{(n)}x_1;\lambda_1^{(m)}x_1;\,t\right)\to 0$, $n,m\to\infty$, uniformly in $t\in\mathbb{R}$. Consider

$$\mu\left(\lambda_2^{(n)}x_2;\lambda_2^{(m)}x_2;\,t\right)\geq\min\left\{\mu\left(\lambda_2^{(n)}x_2;\lambda_2^{(m)}x_2+\lambda_1^{(m)}x_1-\lambda_1^{(n)}x_1;\,t\right);\right.$$

$$\left.\mu\left(\lambda_2^{(m)}x_2+\lambda_1^{(m)}x_1-\lambda_1^{(n)}x_1;\lambda_2^{(m)}x_2;\,t\right)\right\} =$$

$$\min\left\{\mu\left(\lambda_1^{(n)}x_1+\lambda_2^{(n)}x_{2_0};\lambda_1^{(m)}x_1+\lambda_2^{(m)}x_2;\,t\right);\right.$$

$$\mu\left(-\lambda_1^{(n)}x_1; -\lambda_1^{(m)}x_1;\, t\right)\right\}.$$

Taking into consideration the relations (9) and (10), from here we have $\mu\left(\lambda_2^{(n)}x_2; \lambda_2^{(m)}x_2;\, t\right) \to 1$, $n, m \to \infty$, uniformly in $t \in \mathbb{R}$. Similarly we obtain that $\nu\left(\lambda_2^{(n)}x_2; \lambda_2^{(m)}x_2;\, t\right) \to 0$, $n, m \to \infty$, uniformly in $t \in \mathbb{R}$. Continuing this reasoning, we get $\mu\left(\lambda_k^{(n)}x_k; \lambda_k^{(m)}x_k;\, t\right) \to 1$, $\nu\left(\lambda_k^{(n)}x_k; \lambda_k^{(m)}x_{k_0};\, t\right) \to 0$, $n, m \to \infty$, uniformly in $t \in \mathbb{R}$, for each fixed $k \in \mathbb{N}$, i.e. $s\text{-}\lim_{n,m\to\infty}\left(\lambda_k^{(n)} - \lambda_k^{(m)}\right)x_k = 0$, $\forall k \in \mathbb{N}$. By Lemma 1, from here it follows that the sequence $\left\{\lambda_k^{(n)}\right\}_{n\in\mathbb{N}}$ is fundamental for $\forall k \in \mathbb{N}$. Let $\lambda_k^{(n)} \to \lambda_k$, $n \to \infty$. Assume that $\bar{\lambda} \equiv \langle\lambda_k\rangle_{k\in\mathbb{N}}$ and let us show that $\lim_{n\to\infty}\mu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda}_n; \bar{\lambda};\, t\right) = 1$ and $\lim_{n\to\infty}\nu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda}_n; \bar{\lambda};\, t\right) = 0$ uniformly in $t \in \mathbb{R}$. Let us establish it with respect to $\mu_{\mathscr{K}_{\bar{x}}^s}$. Take $\forall\varepsilon > 0$. It is clear that $\exists n_0$, $\forall n \geq n_0$, $\forall p \in \mathbb{N}$:

$$\mu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda}_n; \bar{\lambda}_{n+p};\, t\right) > 1 - \varepsilon, \forall t \in \mathbb{R}.$$

Consequently

$$\inf_r \mu\left(\sum_{k=1}^r \lambda_k^{(n)}x_k; \sum_{k=1}^r \lambda_k^{(n+p)}x_k;\, t\right) > 1 - \varepsilon, \forall n \geq n_0\, , \forall p \in \mathbb{N}, \forall t \in \mathbb{R}_+. \tag{11}$$

Now we need the following condition:

13) *From* $\lambda_n \to \lambda$, $n \to \infty$ *it follows that* $s\text{-}\lim_{n\to\infty}(\lambda_n x) = \lambda x$, *i.e.* $\lim_{n\to\infty}\mu(\lambda_n x; \lambda x; t) = 1$, $\lim_{n\to\infty}\nu(\lambda_n x; \lambda x; t) = 0$, *uniformly in* $t \in \mathbb{R}$, $\forall x \in X$.

From here it directly follows that, if $\lambda_n^{(k)} \to \lambda^{(k)}$, $n \to \infty$, $\forall k = \overline{1, r}$, then

$$\lim_{n\to\infty}\mu\left(\sum_{k=1}^r \lambda_n^{(k)}x_k; y; t\right) = \mu\left(\sum_{k=1}^r \lambda^{(k)}x_k; y; t\right),$$

$$\lim_{n\to\infty}\nu\left(\sum_{k=1}^r \lambda_n^{(k)}x_k; y; t\right) = \nu\left(\sum_{k=1}^r \lambda^{(k)}x_k; y; t\right), \forall\{x_1; ...; x_r; y\} \subset X, \forall t \in \mathbb{R}.$$

Indeed, without loss of generality we consider the case $r = 2$. It suffices to conduct to lead the proof with respect to ν because, this scheme is applied to $\tilde{\mu} = 1 - \mu$. Let $\lambda_n \to \lambda$, $\mu_n \to \mu$, $n \to \infty$. By definition

$$\nu(x; y; t) \leq \max\{\nu(x; z; t); \nu(y; z; t)\} \leq \nu(x; z; t) +$$

$$\nu(y; z; t)\ , \ \forall x, y, z \in X\ , \forall t \in \mathbb{R}.$$

Hence

$$\nu(x; y; t) - \nu(x; z; t) \leq \nu(y; z; t).$$

Similarly we obtain

$$\nu(x;z;t)-\nu(x;y;t)\le\nu(y;z;t).$$

Thus

$$|\nu(x;y;t)-\nu(x;z;t)|\le\nu(y;z;t). \tag{12}$$

Taking here $y=\lambda_n a$, $z=\lambda a$, we obtain $\lim\limits_{n\to\infty}\nu(x;\lambda_n a;t)=\nu(x;\lambda a;t)$, uniformly in $t\in\mathbb{R}$ and for $\forall x,a\in X$. On the other hand, we have

$$\nu(\lambda_n x+\mu_n y;\lambda x+\mu y;t)\le\nu(\lambda_n x+\mu_n y;\mu_n y+\lambda x;t)+\nu(\mu_n y+\lambda x;\lambda x+\mu y;t).$$

Taking into account the property 12) we have

$$\nu(\lambda_n x+\mu_n y;\lambda x+\mu y;t)\le\nu(\lambda_n x;\lambda x;t)+\nu(\mu_n y;\mu y;t).$$

Consequently, $s\text{-}\lim\limits_{n\to\infty}(\lambda_n x+\mu_n y)=\lambda x+\mu y$, $\forall x,y\in X$. From here it directly follows that, if $\lambda_n^{(k)}\to\lambda^{(k)}$, $n\to\infty$, $\forall k=\overline{1,r}$, then $s\text{-}\lim\limits_{n\to\infty}\left(\sum_{k=1}^{r}\lambda_n^{(k)}x_k\right)=\sum_{k=1}^{r}\lambda^{(k)}x_k$, $\forall\{x_k\}_1^r\subset X$. If in (12) we assume $y=\sum_{k=1}^{r}\lambda_n^{(k)}x_k$ and $z=\sum_{k=1}^{r}\lambda^{(k)}x_k$, then we have $\lim\limits_{n\to\infty}\nu\left(x;\sum_{k=1}^{r}\lambda_n^{(k)}x_k;t\right)=\nu\left(x;\sum_{k=1}^{r}\lambda^{(k)}x_k;t\right)$, uniformly in $t\in\mathbb{R}$ and $\forall\{x;x_1;...;x_r\}\subset X$. The similar results are also true with respect to μ. Then, passing to the limit in the inequality (11) as $p\to\infty$ we get

$$\inf_r\mu\left(\sum_{k=1}^{r}\lambda_k^{(n)}x_k;\sum_{k=1}^{r}\lambda_k x_k;\,t\right)\ge 1-\varepsilon,\forall n\ge n_0\,,\ \forall t\in\mathbb{R}_+. \tag{13}$$

In the same way we obtain that $\exists m_0\in\mathbb{N}$:

$$\sup_r\nu\left(\sum_{k=1}^{r}\lambda_k^{(n)}x_k;\sum_{k=1}^{r}\lambda_k x_k;\,t\right)\le\varepsilon,\forall n\ge m_0\,,\ \forall t\in\mathbb{R}_+. \tag{14}$$

We have

$$\mu\left(\sum_{k=r}^{r+p}\lambda_k^{(n)}x_k;\sum_{k=r}^{r+p}\lambda_k x_k;\,t\right)=\mu\left(\sum_{k=r}^{r+p}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;0;t\right)\ge$$

$$\ge\min\left\{\mu\left(\sum_{k=r}^{r+p}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;-\sum_{k=1}^{r-1}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;\frac{t}{2}\right);\right.$$

$$\left.\mu\left(-\sum_{k=1}^{r-1}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;0;\frac{t}{2}\right)\right\}=$$

$$=\min\left\{\mu\left(\sum_{k=1}^{r+p}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;0;\frac{t}{2}\right);\ \mu\left(\sum_{k=1}^{r-1}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;0;\frac{t}{2}\right)\right\}.$$

Taking into account the inequality (13) we obtain

$$\mu\left(\sum_{k=r}^{r+p}\lambda_k^{(n)}x_k;\sum_{k=r}^{r+p}\lambda_k x_k;\,t\right)\geq 1-\varepsilon,\forall n\geq n_0\,,\ \forall r,p\in\mathbb{N},\forall t\in\mathbb{R}_+. \tag{15}$$

As $\bar{\lambda}_n\in\mathscr{K}_{\bar{x}}^s$ it is clear that $\exists m_0^{(n)}$:

$$\mu\left(\sum_{k=m}^{m+p}\lambda_k^{(n)}x_k;0;t\right)>1-\varepsilon,\forall m\geq m_0^{(n)},\ \forall p\in\mathbb{N},\forall t\in\mathbb{R}_+. \tag{16}$$

We have

$$\mu\left(\sum_{k=m}^{m+p}\lambda_k x_k;0;t\right)\geq\min\left\{\mu\left(\sum_{k=m}^{m+p}\lambda_k x_k;\sum_{k=m}^{m+p}\lambda_k^{(n)}x_k;t\right);\mu\left(\sum_{k=m}^{m+p}\lambda_k^{(n)}x_k;0;t\right)\right\}.$$

Taking into account the relations (15) and (16) from here we obtain

$$\mu\left(\sum_{k=m}^{m+p}\lambda_k x_k;0;t\right)\geq 1-\varepsilon,\forall m\geq m_0^{(n)},\ \forall p\in\mathbb{N},\forall t\in\mathbb{R}_+.$$

In the same way we establish that $\exists m_1\in\mathbb{N}:\nu\left(\sum_{k=m}^{m+p}\lambda_k x_k;0;t\right)\leq\varepsilon,\forall m\geq m_1$, $\forall p\in\mathbb{N}$, $\forall t\in\mathbb{R}_+$. It follows that the series $\sum_{k=1}^{\infty}\lambda_k x_k$ is strongly fuzzy convergent in X, i.e. if X is strongly complete, then $\exists\, s\text{-}\lim_{m\to\infty}\sum_{k=1}^{m}\lambda_k x_k$. Consequently, $\bar{\lambda}\in\mathscr{K}_{\bar{x}}^s$, and the relations (13),(14) imply that $\lim_{n\to\infty}\mu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda}_n;\bar{\lambda};t\right)=1$, $\lim_{n\to\infty}\nu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda}_n;\bar{\lambda};t\right)=0$ uniformly in $\forall t\in\mathbb{R}$. As a result we obtain that the space $\left(\mathscr{K}_{\bar{x}}^s;\mu_{\mathscr{K}_{\bar{x}}^s};\nu_{\mathscr{K}_{\bar{x}}^s}\right)$ is strongly fuzzy complete. Thus, we have proved the following

Theorem 2. *Let $(X;\mu;\nu)$ be a fuzzy strongly complete metric space with conditions $\alpha)$, $\beta)$, 12) and 13) . If $\{x_n\}_{n\in\mathbb{N}}\subset X$ is a nondegenerate system, then the space of coefficients $\left(\mathscr{K}_{\bar{x}}^s;\mu_{\mathscr{K}_{\bar{x}}^s};\nu_{\mathscr{K}_{\bar{x}}^s}\right)$ is a strongly fuzzy complete metric space.*

Consider operator $T:\mathscr{K}_{\bar{x}}^s\to X$ defined by

$$T\bar{\lambda}=\sum_{n=1}^{\infty}\lambda_n x_n,\bar{\lambda}\equiv\{\lambda_n\}_{n\in\mathbb{N}}\in\mathscr{K}_{\bar{x}}^s.$$

Let $s\text{-}\lim_{n\to\infty}\bar{\lambda}_n=\bar{\lambda}$ in $\mathscr{K}_{\bar{x}}^s$, where $\bar{\lambda}_n\equiv\left\{\lambda_k^{(n)}\right\}_{k\in\mathbb{N}}\in\mathscr{K}_{\bar{x}}^s$. We have

$$\mu\left(T\bar{\lambda}_n;T\bar{\lambda}\,;\,t\right)=\mu\left(\sum_{k=1}^{\infty}\left(\lambda_k^{(n)}-\lambda_k\right)\,x_k\,;\,0;t\right)\geq$$

$$\inf_m\mu\left(\sum_{k=1}^{m}\left(\lambda_k^{(n)}-\lambda_k\right)\,x_k\,;0;\,t\right)=\mu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda}_n;\bar{\lambda}\,;\,t\right).$$

It follows directly that $s\text{-}\lim\limits_{n\to\infty} T\,\bar{\lambda}_n = T\bar{\lambda}$, i.e. the operator T is strongly fuzzy continuous. Let $\bar{\lambda} \in KerT$, i.e. $T\bar{\lambda} = 0 \Rightarrow \sum_{n=1}^{\infty} \lambda_n x_n = 0$, where $\bar{\lambda} \equiv \{\lambda_n\}_{n\in\mathbb{N}} \in \mathscr{K}_{\bar{x}}^s$. It is clear that if the system $\{x_n\}_{n\in\mathbb{N}}$ is *s-linearly independent*, then $\lambda_n = 0$, $\forall n \in \mathbb{N}$, and, as a result, $KerT = \{0\}$. In this case $\exists T^{-1} : X \supset ImT \to \mathscr{K}_{\bar{x}}^s$.

Denote by $\{\bar{e}_n\}_{n\in\mathbb{N}} \subset \mathscr{K}_{\bar{x}}^s$ a canonical system in $\mathscr{K}_{\bar{x}}^s$, where $\bar{e}_n = \{\delta_{nk}\}_{k\in\mathbb{N}} \in \mathscr{K}_{\bar{x}}^s$. Obviously, $T\bar{e}_n = x_n$, $\forall n \in \mathbb{N}$. Let us prove that $\{\bar{e}_n\}_{n\in\mathbb{N}}$ forms an *s-basis* for $\mathscr{K}_{\bar{x}}^s$. Take $\forall\bar{\lambda} \equiv \{\lambda_n\}_{n\in\mathbb{N}} \in \mathscr{K}_{\bar{x}}^s$ and show that the series $\sum_{n=1}^{\infty} \lambda_n \bar{e}_n$ is strongly fuzzy convergent in $\mathscr{K}_{\bar{x}}^s$. In fact, the existence of $s\text{-}\lim\limits_{m\to\infty} \sum_{n=1}^{m} \lambda_n x_n$ in X_s implies that $\forall\varepsilon > 0$, $\exists m_0 \in \mathbb{N}$:

$$\mu\left(\sum_{n=m}^{m+p} \lambda_n x_n; 0; t\right) > 1 - \varepsilon, \forall m \geq m_0\,, \forall p \in \mathbb{N}, \forall t \in \mathbb{R}_+.$$

We have

$$\mu_{\mathscr{K}_{\bar{x}}^s}\left(\sum_{n=m}^{m+p} \lambda_n \bar{e}_n; 0; t\right) = \inf_r \left(\sum_{n=m}^{r} \lambda_n x_n; 0; t\right) \geq 1 - \varepsilon, \forall m \geq m_0\,, \forall p \in \mathbb{N}, \forall t \in \mathbb{R}_+.$$

It follows that the series $\sum_{n=1}^{\infty} \lambda_n \bar{e}_n$ is strongly fuzzy convergent in $\mathscr{K}_{\bar{x}}^s$. Moreover

$$\mu_{\mathscr{K}_{\bar{x}}^s}\left(\bar{\lambda} - \sum_{n=1}^{m} \lambda_n \bar{e}_n; 0; t\right) = \mu_{\mathscr{K}_{\bar{x}}^s}\left(\{\ ...; 0;\ \lambda_{m+1}; ...\}\ ; 0;\ t\right) =$$

$$= \inf_r \mu\left(\sum_{n=m+1}^{r} \lambda_n x_n; 0; t\right) \geq 1 - \varepsilon, \forall m \geq m_0\,, \forall t \in \mathbb{R}_+.$$

Consequently, $s\text{-}\lim\limits_{m\to\infty} \sum_{n=1}^{m} \lambda_n \bar{e}_n = \bar{\lambda}$, i.e. $\bar{\lambda} \overset{s}{=} \sum_{n=1}^{m} \lambda_n \bar{e}_n$. Consider the functionals $e_n^*\left(\bar{\lambda}\right) = \lambda_n$, $\forall n \in \mathbb{N}$. Let us show that they are *s-continuous*. Let $s\text{-}\lim\limits_{n\to\infty} \bar{\lambda}_n = \bar{\lambda}$, where $\bar{\lambda}_n \equiv \left\{\lambda_k^{(n)}\right\}_{k\in\mathbb{N}} \in \mathscr{K}_{\bar{x}}^s$. As established in the proof of Theorem 2, we have $\lambda_k^{(n)} \to \lambda_k$ as $n \to \infty$, i.e. $e_k^*\left(\bar{\lambda}_n\right) \to e_k^*\left(\bar{\lambda}\right)$ as $n \to \infty$ for $\forall k \in \mathbb{N}$. Thus, e_k^* is *s- continuous* in $\mathscr{K}_{\bar{x}}^s$ for $\forall k \in \mathbb{N}$. On the other hand, it is easy to see that $e_n^*\left(\bar{e}_k\right) = \delta_{nk}$, $\forall n, k \in \mathbb{N}$, i.e. $\left\{e_n^*\right\}_{n\in\mathbb{N}}$ is *s-biorthogonal* to $\{\bar{e}_n\}_{n\in\mathbb{N}}$. As a result we obtain that the system $\{\bar{e}_n\}_{n\in\mathbb{N}}$ forms an *s-basis* for $\mathscr{K}_{\bar{x}}^s$. So we get the validity of the following

Theorem 3. *Let $(X; \mu; \nu)$ be a fuzzy strongly complete metric space with conditions $\alpha)$, $\beta)$, 12) and 13). Let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a nondegenerate system. Then the corresponding space of coefficients $\left(\mathscr{K}_{\bar{x}}^s; \mu_{\mathscr{K}_{\bar{x}}^s}; \nu_{\mathscr{K}_{\bar{x}}^s}\right)$ is strongly fuzzy complete with canonical s-basis $\{\bar{e}_n\}_{n\in\mathbb{N}}$.*

Suppose that the system $\{x_n\}_{n\in\mathbb{N}}$ is *s-linearly independent* and ImT is closed. Then it is easily seen that $\{x_n\}_{n\in\mathbb{N}}$ forms an *s-basis* for ImT and, in case of its *s-completeness* in X_s, it forms an *s-basis* for X_s. In this case, $\mathscr{K}_{\bar{x}}^s$ and X_s are isomorphic, and T is an isomorphism between them. The opposite of it is also true, i.e. if the above-defined operator T is an isomorphism between $\mathscr{K}_{\bar{x}}^s$ and X_s, then the system $\{x_n\}_{n\in\mathbb{N}}$ forms an *s-basis* for X_s. We will call T a coefficient operator. Thus, the following theorem holds.

Theorem 4. *Let $(X;\mu;\nu)$ be a fuzzy strongly complete metric space with conditions $\alpha)$, $\beta)$, 12) and 13). Let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a nondegenerate system, $\left(\mathscr{K}_{\bar{x}}^s;\mu_{\mathscr{K}_{\bar{x}}^s};\nu_{\mathscr{K}_{\bar{x}}^s}\right)$ be a corresponding strongly fuzzy complete normed space and $T{:}\mathscr{K}_{\bar{x}}^s \rightarrow X_s$ be a corresponding coefficient operator. System $\{x_n\}_{n\in\mathbb{N}}$ forms an s-basis for X_s if and only if the operator T is an isomorphism between $\mathscr{K}_{\bar{x}}^s$ and X_s.*

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