*Azerbaijan Journal of Mathematics*
*V. 4, No 2, 2014, July*
*ISSN 2218-6816*

# Structure of Root Subspaces and Oscillation Properties of Eigenfunctions of One Fourth Order Boundary Value Problem

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**Abstract.** In this paper, we consider the spectral problem for an ordinary differential operator of fourth order with regular boundary conditions. We study the general characteristics of the location of the eigenvalues on the real axis, the structure of the root subspaces and oscillation properties of eigenfunctions of this problem. It is found that the number of zeros of eigenfunctions corresponding to the positive eigenvalues behaves in a usual way.

**Key Words and Phrases**: spectral problem, eigenvalue, oscillation properties of eigenfunctions, Prufer-type transformation

**2010 Mathematics Subject Classifications**: 34B05, 34B09, 34B24

## 1. Introduction

We consider the following boundary-value problem

$$(p(x)y''(x))'' - (q(x)y'(x))' = \lambda\, r(x)y(x), \ \ 0 < x < l, \tag{1}$$

$$\begin{gathered} a) y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \\ b) y(0)\cos\beta + Ty(0)\sin\beta = 0, \\ c) y'(l)\cos\gamma + (py'')(l)\sin\gamma = 0, \\ d) y(l)\cos\delta - Ty(l)\sin\delta = 0, \end{gathered} \tag{2}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $Ty \equiv (py'')' - qy'$, the functions $p(x)$, $r(x)$ are strictly positive and continuous on $[0, l]$, $p(x)$ has an absolutely continuous derivative on $[0, l]$, $q(x)$

 

is a non-negative absolutely continuous function on $[0, l]$, $\alpha, \beta, \gamma, \delta$ are real constants and $\alpha, \beta, \gamma \in [0, \pi/2]$, $\delta \in (\pi/2\,, \pi)$.

Eigenvalues of ordinary differential operators can be uniquely related to the number of zeros of eigenfunctions. The problem of the number of zeros of eigenfunctions (called oscillatory properties) in classical oscillatory topics consists of a package of bound properties (called oscillation spectral properties) – reality, positivity and simplicity of the spectrum, the availability a certain number of zeros and alternation of zeros, the Chebyshev property and Markov chains of eigenfunctions. These properties are the classical in the oscillatory topics started by O.D.Kellog and M.G.Krein. At the beginning of the 20th century [1, 2] O.D. Kellogg introduced a class of kernels (containing Green's function of the Sturm-Liouville problem), later named Kellogg kernels, and showed that the integral operators generated by these kernels have oscillation spectral properties. Surprisingly, these properties for the Sturm-Liouville problem were detected much earlier by Sturm [3], who used qualitative methods (such as comparison theorems) without integral equations. Much later for fourth-order equation O.Davidoglou [4, 5] found only first three of the above oscillation spectral properties in the case of simple boundary conditions. S.N.Janczewsky [6] did it for a very broad class of (regular) boundary conditions. M.G. Krein [7], F.R.Gantmakher and M.G.Krein [8] proved that the class of Kellogg kernels belongs not only to the influence function of the string, but also to the influence function of the rod (beam) as well as to the Green's function for a wide class of two-point boundary value problems for ordinary differential equations of higher order, which later provoked the rapid development of the oscillation theory in the works by A.Yu.Levin and G.D.Stepanov [9, 10] U.Elias [11] J.Przybycin [12], etc. A.Yu.Levin and G.D.Stepanov established oscillation of the spectrum of boundary value problems with strong sign-regularity Green's functions. The transition from the oscillating kernels to the strongly sign-regular kernels, in general, can not be reduced to the oscillating (up to a sign) by shifting. Later A.V. Borovskii and Yu.V.Pokornyi [13] extended Kellogg's results [2] to discontinuous kernels.

Another approach - the application of the method of polar coordinates to establish Sturmian oscillation properties dates back to H. Prufer [14]. In [15] this method was applied to the study of oscillation properties of first-order systems. In [16, 17], D.O Banks and G.E Kurowski, using Prufer-type transformation, studied the oscillation properties for the eigenfunctions and their derivatives for the spectral problem (1), (2) with $\delta \in [0, \pi/2]$.

Problem (1), (2) in the case of $\delta \in (\pi/2,\, \pi)$ was considered in [18], but the oscillation properties of the eigenfunctions were not completely investigated there. In a recent paper [19] J. Ben Amara studied the oscillation properties of eigenfunctions of this problem for $\alpha = \beta = 0$ in the disfocal case. However, methods of those works do not cover the more general case we considered here. Moreover, they are not even applicable in our case.

In this paper, using the Prufer -type transformation and Sturm type comparison theorems, we fully investigate the general characteristics of the location of eigenvalues on the real axis, study the structure of the root subspaces and oscillation properties of the eigenfunctions of the problem (1), (2).

## 2. On the existence and uniqueness of solutions of problem (1), (2a) - (2c)

As in [16, 17], for the analysis of the oscillation properties of eigenfunctions of the boundary-value problem (1), (2) we shall use a Prufer -type transformation of the following form:

$$\begin{cases} y(x) = r(x)\sin\psi(x)\cos\theta(x), \\ y'(x) = r(x)\cos\psi(x)\sin\varphi(x), \\ (py)''(x) = r(x)\cos\psi(x)\cos\varphi(x), \\ Ty(x) = r(x)\sin\psi(x)\sin\theta(x). \end{cases} \tag{3}$$

Equation (1) has the following equivalent formulation in matrix form:

$$U' = MU, \tag{4}$$

where

$$U = \begin{pmatrix} y \\ y' \\ py'' \\ Ty \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/p & 0 \\ 0 & q & 0 & 1 \\ \lambda r & 0 & 0 & 0 \end{pmatrix}.$$

Setting $w(x) = \ ctg\ \psi(x)$ and applying the transformation (3) to (4) we obtain a system of first-order differential equations with respect to the functions $r$,$\psi$,$\theta$,$\varphi$ of the following form:

a)
$$r' = \Big[\sin 2\psi\cos\theta\sin\varphi + (q+1/p)\cos^2\psi\sin 2\varphi + \sin 2\psi\sin\theta\cos\varphi + \\ + (\lambda r/2)\sin^2\psi\sin 2\theta\Big]\, r/2,$$

b)
$$w' = -w^2\cos\theta\sin\varphi + (1/2)\,(q+1/p)\,w\sin 2\varphi + \sin\theta\cos\varphi - (\lambda r/2)\,w\sin 2\theta, \tag{5}$$

c)
$$\theta' = -w\sin\varphi\sin\theta + \lambda r\cos^2\theta,$$

d)
$$\varphi' = (1/p)\cos^2\varphi - q\sin^2\varphi - (1/w)\sin\theta\sin\varphi.$$

Below we will need the following results.

**Lemma A.** *Let $y(x,\lambda)$ be a non-trivial solution of equation (1) for $\lambda > 0$. If $y$, $y'$, $y''$, $Ty$ are non-negative and not all equal to zero for $x = a \in (0,l)$, then they are positive for $x > a$. On the other hand, if $y$, $-y'$, $y''$,and $-Ty$ are non-negative and not all equal to zero for $x = b \in (0,l)$, then they are positive for $x < b$.*

This lemma was proved in [20] for $q(x) \equiv 0$ and in [17] for $q(x) > 0$.

**Lemma B** [16, 17]. *Let $y(x,\lambda)$ be a non-trivial solution of the problem* (1)*, (2a)-(2c) for $\lambda > 0$ and let $\theta(x,\lambda)$,$\varphi(x,\lambda)$and $\psi(x,\lambda)$ be the corresponding functions in* (3)*. Then i) if $\xi$ is a zero of $y$ or $y''$ in the open interval $(0,l)$, then $y'(x)Ty(x) < 0$ in a neigborhood of $\xi$ ; if $\varsigma$ is a zero of $y'$or $Ty$ in the open interval $(0,l)$, then $y(x)y''(x) < 0$ in a neigborhood of $\varsigma$ ; ii) the Jacobian $J[y] = r^3 \sin\psi \cos\psi$ of the transformation of* (3) *does not vanish in $(0,l)$; iii) $\theta(0,\lambda) = \beta - \pi/2$, $\varphi(0,\lambda) = \alpha$, $\varphi(l,\lambda) = k\pi - \gamma$, $k \in N$, where $\alpha = 0$ when $\psi(0,\lambda) = \pi/2$ and $\gamma = 0$ when $\psi(l,\lambda) = \pi/2$. Furthermore, $w(0,\lambda) = ctg\psi(0,\lambda)$ is determined by at least one of the following relations:*

$$a)\ w(0,\ \lambda) = \frac{y'(0,\lambda)\sin\beta}{y(0,\lambda)\sin\alpha}, \qquad b)\ w(0,\ \lambda) = -\frac{(py'')(0,\lambda)\cos\beta}{Ty(0,\lambda)\cos\alpha},$$

$$c)\ w(0,\ \lambda) = \frac{(py'')(0,\lambda)\sin\beta}{y(0,\lambda)\cos\alpha}, \quad d)\ w(0,\ \lambda) = -\frac{y'(0,\lambda)\cos\beta}{Ty(0,\lambda)\sin\alpha}.$$

*iv) $\theta(l,\lambda)$ is a strictly increasing continuous function of $\lambda$.*

We also have the following oscillation theorem.

**Theorem A [16, 17].** *For fixed $\alpha, \beta, \gamma$ the eigenvalues of the problem* (1), (2) *(except for the case $\beta = \delta = \pi/2$) are real and simple, and form a sequence $\{\mu_n(\delta)\}_{n=1}^{\infty}$ such that $0 < \mu_1(\delta) < \mu_2(\delta) < ... < \mu_n(\delta) < ...$ . Moreover, the eigenfunction $\vartheta_n^{(\delta)}(x)$, corresponding to the eigenvalue $\mu_n(\delta)$, has exactly $n-1$ simple zeros in the interval $(0,l)$, and the function $T\vartheta_n^{(\delta)}(x)$ has exastly $n$ zeros in the interval $[0,l]$, zeros of functions $\vartheta_n^{(\delta)}(x)$ and $T\vartheta_n^{(\delta)}(x)$ interleaving.*

**Remark 1.** *In the case $\beta = \delta = \pi/2$ the first eigenvalue of boundary-value problem* (1)*,* (2) *is equal to zero and the corresponding eigenfunction is constant, and the statement of Theorem A is true for $n \geq 2$.*

**Theorem 1.** *For each fixed $\lambda \in \mathbb{C}$ there exists a non-trivial solution $y(x,\lambda)$of the problem* (1)*, (2a)-(2c), which is unique up to a constant coefficient.*

*Proof.* Let $\varphi_k(x,l),\ k = \overline{1,4}$, be solutions of equation (1) normalized for $x = 0$ by the Cauchy conditions

$$\varphi_k^{(s-1)}(0,\ \lambda) = \delta_{ks},\ s = \overline{1,\ 3},\ \ T\varphi_k(0,\ \lambda) = \delta_{k4}, \tag{6}$$

where $\delta_{ks}$ is the Kronecker delta.

We shall seek the function $y(x,\lambda)$ in the following form:

$$y(x,\lambda) = \sum_{k=1}^{4} c_k \varphi_k(x,\lambda), \tag{7}$$

where $c_k,\ k = \overline{1,4}$ are constants.

Suppose $\alpha \neq 0, \beta \neq 0,\ \gamma \neq 0$ in the boundary conditions (2a), (2b), (2c). From (6), (7) and the boundary conditions (2a), (2b) it follows that

$$c_3 = \frac{c_2}{p(0)} ctg\alpha, \;\; c_4 = -c_1 ctg\beta. \tag{8}$$

Taking into account (8), by (7) we obtain

$$y(x,\lambda) = c_1 \left\{\varphi_1(x,\lambda) - \varphi_4(x,\lambda) ctg\beta\right\} + c_2 \left\{\varphi_2(x,\lambda) + \varphi_3(x,\lambda)\frac{ctg\alpha}{p(0)}\right\}. \tag{9}$$

Using (9) and (2c), we get the following relation for the determination of $c_1$ and $c_2$:

$$c_1\alpha^*(\lambda) + c_2\beta^*(\lambda) = 0,$$

where

$$\alpha^*(\lambda) = \left\{\varphi_1'(l,\lambda) ctg\gamma + p(l)\varphi_1''(l,\lambda)\right\} - ctg\beta \left\{\varphi_4'(l,\lambda) ctg\gamma + p(l)\varphi_4''(l,\lambda)\right\}, \tag{10}$$

$$\beta^*(\lambda) = \left\{\varphi_2'(l,\lambda) ctg\gamma + p(l)\varphi_2''(l,\lambda)\right\} + \frac{ctg\alpha}{p(0)} \left\{\varphi_3'(l,\lambda) ctg\gamma + p(l)\varphi_3''(l,\lambda)\right\}. \tag{11}$$

To complete the proof of Theorem 1 in considered case it is sufficient to demonstrate that

$$|\alpha^*(\lambda)| + |\beta^*(\lambda)| > 0. \tag{12}$$

It follows by Lemma A that $\varphi_k'(l,\lambda) > 0$, $\varphi_k''(l,\lambda) > 0$, $k = \overline{1,4}$ for $\lambda > 0$. Hence (12) holds for $\lambda > 0$.

Now let $\lambda \in C\backslash(0,+\infty)$. If (12) fails for such $\lambda$, then the functions

$$\phi_1(x,\lambda) = \varphi_1(x,\lambda) - ctg\beta\varphi_4(x,\lambda) \,\text{and}\, \phi_2(x,\lambda) = \varphi_2(x,\lambda) + \frac{ctg\alpha}{p(0)}\varphi_3(x,\lambda)$$

solve the problem (1), (2a)-(2c). We now define the function $\vartheta(x,\lambda)$ :

$$\vartheta(x,\lambda) = \phi_2(l,\lambda)\phi_1(x,\lambda) - \phi_1(l,\lambda)\phi_2(x,\lambda).$$

Since $\vartheta(l,\lambda) = 0$, then the function $\vartheta(x,\lambda)$ is an eigenfunction of the problem (1), (2) with $\delta = 0$ corresponding to the eigenvalue $\lambda \in C\backslash(0,+\infty)$. However, this contradicts Theorem A. This contradiction proves (12).

The remaining cases are treated similarly. Theorem is proved. ◄

**Remark 2.** *It follows from the proof of Theorem 1 that without loss of generality we can consider each solution $y(x,\lambda)$ of the problem* (1), *(2a)-(2c) for fixed $x \in [0,l]$ as an entire function of $\lambda$ of the following form (for $\alpha,\beta,\gamma \neq 0$):*

$$y(x,\lambda) = \beta^*(\lambda)\phi_1(x,\lambda) - \alpha^*(\lambda)\phi_2(x,\lambda).$$

*In fact, the functions $\varphi_k(x,l)$, $k = \overline{1,4}$ and their derivatives are entire functions of $\lambda$ (see [21],Ch.I), and therefore $y(x,\lambda)$ is also an entire function of $\lambda$ for each fixed $x \in [0,l]$.*

## 3. Main properties of the solution of the problem (1), (2a) - (2c)

Let $y(x,\lambda)$ be a non-trivial solution of the problem (1), (2a)-(2c). Obviously, the eigenvalues $\lambda_n(0)$ and $\lambda_n(\pi/2)$, $n \in \mathrm{N}$, of the boundary-value problem (1), (2) for $\delta = 0$ and $\delta = (\pi/2)$ are zeros of the entire functions $y(l,\lambda)$ and $Ty(l,\lambda)$, respectively. Note that the function $F(\lambda) = \frac{Ty(l,\lambda)}{y(l,\lambda)}$ is well defined for $\lambda \in \mathrm{A} \equiv (\bigcup_{n=1}^{\infty} \mathrm{A}_n) \bigcup (\mathrm{C}\backslash\mathrm{R})$, where $\mathrm{A}_n = (\lambda_{n-1}(0),\ \lambda_n(0)),\ n \in \mathrm{N},\ \lambda_0(0) = -\infty,$ and is a meromorphic function of finite order, and $\lambda_n(\pi/2),\ \lambda_n(0),\ n \in \mathrm{N}$, are zeros and poles of this function, respectively.

**Lemma 1.** *The following relation holds:*

$$\frac{dF(\lambda)}{d\lambda} = \frac{1}{y^2(l,\lambda)} \int_0^l ry^2(x,\lambda)dx,\ \ \lambda \in \mathrm{A}. \tag{13}$$

*Proof.* By (1) we obtain

$$(Ty(x,\mu))' y(x,\lambda) - y(x,\mu)(Ty(x,\lambda))' = (\mu - \lambda)r(x)y(x,\mu)y(x,\lambda).$$

Integrating this equality from 0 to $l$, using integration by parts and assuming (2a)-(2c) we obtain

$$y(l,\lambda)Ty(l,\mu) - y(l,\mu)Ty(l,\lambda) = (\mu - \lambda) \int_0^l r(x)y(x,\mu)y(x,\lambda)dx. \tag{14}$$

For $\mu \in \mathrm{A},\ \ \mu \neq \lambda$, we have

$$\frac{Ty(l,\mu)}{y(l,\mu)} - \frac{Ty(l,\lambda)}{y(l,\lambda)} = (\mu - \lambda)\frac{1}{y(l,\mu)y(l,\lambda)} \int_0^l r(x)y(x,\mu)y(x,\lambda)dx. \tag{15}$$

Dividing both sides of (15) by $\mu - \lambda$ and passing to the limit as $\mu \to \lambda$ we obtain (13). The proof of Lemma 1 is complete. ◄

**Remark 3.** *From property 1 in [17] and formula* (13)*, one has the relations*

$$\lambda_1(\pi/2) < \lambda_1(0) < \lambda_2(\pi/2) < \lambda_2(0) < ... \tag{16}$$

Set $\lambda = \rho^4$ in equation (1) . By Theorem 1 in [21, p.59], in each subdomain $T$ of the complex $\rho$-plane equation (1) has four linearly independent solutions $z_k(x,\rho), k = \overline{1,4}$, which are regular with respect to $\rho$ (for sufficiently large $\rho$) and satisfy the relations

$$z_k^{(s)}(x,\rho) = \left(\rho\omega_k\,(r/p)^{\frac{1}{4}}\right)^s e^{\rho\omega_k X}\left[1 + O\left(1/\rho\right)\right],\ \ k = \overline{1,4}, s = \overline{0,3}, \tag{17}$$

where $\omega_k,\ \ k = \overline{1,4}$, are the distinct fourth roots of unity, and $X = \int_0^x (r/p)^{\frac{1}{4}}\,dt$.

Let $\omega_1 = -i,\ \ \omega_2 = i,\ \omega_3 = -1,\ \omega_4 = 1$, and $h = \int_0^l (r/p)^{\frac{1}{4}}\,dt$. We shall seek the solution $y(x,\lambda)$ in the following form:

$$y(x,\lambda)=\sum_{k=1}^{4}c_k z_k(x,\rho). \tag{18}$$

Taking into account (18) in the boundary conditions (2a) - (2c), we obtain the following system of equations:

$$\begin{cases} \{-i\cos\alpha+\rho h_0\sin\alpha\}\,c_1[1]+\{i\cos\alpha+\rho h_0\sin\alpha\}\,c_2[1]- \\ -\{\cos\alpha+\rho h_0\sin\alpha\}\,c_3[1]+\{\cos\alpha-\rho h_0\sin\alpha\}\,c_4[1]=0, \\ \left\{\cos\beta+i\rho^3 h_{(0)}\sin\beta\right\}c_1[1]+\left\{\cos\beta-i\rho^3 h_{(0)}\sin\beta\right\}c_2[1]+ \\ +\left\{\cos\beta-\rho^3 h_{(0)}\sin\beta\right\}c_3[1]+\left\{\cos\beta+\rho^3 h_{(0)}\sin\beta\right\}c_4[1]=0, \\ \left\{i\cos\gamma+\rho h_{(1)}\sin\gamma\right\}e^{-i\rho h}c_1[1]-\left\{i\cos\gamma-\rho h_{(1)}\sin\gamma\right\}e^{-i\rho h}c_2[1]+ \\ +\left\{\cos\gamma-\rho h_{(1)}\sin\gamma\right\}e^{-\rho h}c_3[1]+\left\{\cos\gamma+\rho h_{(1)}\sin\gamma\right\}e^{\rho h}c_4[1]=0, \end{cases}$$

where

$$h_0=\sqrt[4]{p^3(0)r(0)},\ h_{(0)}=\sqrt[4]{p(0)r^3(0)},\ h_{(1)}=\sqrt[4]{p^3(l)r(l)}$$

and $[1]=1+O(1/\rho)$. Solving this system we get:

1) for $\alpha\in(0,\ \pi/2],\ \beta=0,\ \gamma\in(0,\ \pi/2]$,

$$c_1=[1],\ c_2=-[1],\ c_3=O(1/\rho)\,,\quad c_4=-2ie^{-\rho h}\sin\rho h[1],$$

2) for $\alpha\in(0,\ \pi/2],\ \beta=0,\ \gamma=0$,

$$c_1=[1],\ c_2=-[1],\ c_3=O(1/\rho)\,,\quad c_4=2ie^{-\rho h}\cos\rho h[1],$$

3) for $\alpha\in(0,\ \pi/2],\ \beta\in(0,\ \pi/2],\ \gamma\in(0,\ \pi/2]$,

$$c_1=[1],\ c_2=\frac{i-1}{i+1}[1],\ c_3=\frac{2i}{i+1}[1],\quad c_4=\frac{2i(\cos\rho h-\sin\rho h)}{(i+1)}e^{-\rho h}[1],$$

4) for $\alpha\in(0,\ \pi/2],\ \beta\in(0,\ \pi/2],\ \gamma=0$,

$$c_1=[1],\ c_2=\frac{i-1}{i+1}[1],\ c_3=\frac{2i}{i+1}[1],\quad c_4=\frac{2i}{i+1}(\cos\rho h-\sin\rho h)\ e^{-\rho h}[1],$$

5) for $\alpha=0,\ \beta=0,\ \gamma\in(0,\ \pi/2]$,

$$c_1=[1],\ c_2=\frac{i-1}{i+1}[1],\ c_3=-\frac{2i}{i+1}[1],\quad c_4=\frac{2i}{i+1}(\cos\rho h-\sin\rho h)\,e^{-\rho h}[1],$$

6) for $\alpha=0,\ \beta=0,\ \gamma=0$,

$$c_1=[1],\ c_2=\frac{i-1}{i+1}[1],\ c_3=-\frac{2i}{i+1}[1],\quad c_4=\frac{2i}{i+1}(\cos\rho h+\sin\rho h)\ e^{-\rho h}[1],$$

7) for $\alpha = 0,\ \beta \in (0,\ \pi/2],\ \gamma \in (0,\ \pi/2]$,

$$c_1 = [1],\ c_2 = [1],\ c_3 = O\left(1/\rho\right), \quad c_4 = 2e^{-\rho h}\cos\rho h[1],$$

8) for $\alpha = 0,\ \beta \in (0,\ \pi/2],\ \gamma = 0$,

$$c_1 = [1],\ c_2 = [1],\ c_3 = O\left(1/\rho\right), \quad c_4 = 2ie^{-\rho h}\sin\rho h[1].$$

Given the values obtained for $c_k,\ k = \overline{1,4}$ in (18), for the function $y\left(x, \lambda\right)$ in the above cases 1) - 8), we obtain the corresponding asymptotic formulas:

$$1) y\left(x, \lambda\right) = \left[\sin\rho X + \sin\rho\, h\ e^{\rho(X-h)}\right]\ [1],$$

$$2) y\left(x, \lambda\right) = \left[\sin\rho X + \cos\rho\, h\ e^{\rho(X-h)}\right]\ [1],$$

$$3) y\left(x, \lambda\right) = \left[\sin\rho X - \cos\rho\, X\ - e^{-\rho\, X} + \sqrt{2}\sin\left(\rho\, h - \pi/4\right)\ e^{\rho(X-h)}\right]\ [1],$$

$$4) y\left(x, \lambda\right) = \left[\sin\rho X - \cos\rho X\ + e^{-\rho\, X} - \sqrt{2}\sin\left(\rho\, h + \pi/4\right)\ e^{\rho(X-h)}\right] [1],$$

$$5) y\left(x, \lambda\right) = \left[\sin\rho X - \cos\rho X\ + e^{-\rho\, X} + \sqrt{2}\sin\left(\rho\, h - \pi/4\right)\ e^{\rho(X-h)}\right] [1],$$

$$6) y\left(x, \lambda\right) = \left[\sin\rho X - \cos\rho X\ \ + e^{-\rho\, X} - \sqrt{2}\sin\left(\rho\, h + \pi/4\right)\ e^{\rho(X=h)}\right] [1],$$

$$7) y\left(x, \lambda\right) = \left[\cos\rho X + \cos\rho\, h\ e^{\rho(X-h)}\right]\ [1],$$

$$8) y\left(x, \lambda\right) = \left[\cos\rho X + \sin\rho\, h\ e^{\rho(X-h)}\right]\ [1].$$

For brevity, we introduce the notation $s(\delta_1, \delta_2) = sgn\delta_1 + sgn\delta_2$. Combining these formulas, for $y(x, \lambda)$ we obtain the following asymptotic formula:

$$y\left(x, \lambda\right) = \begin{cases} \left(\sin\left(\rho X + \frac{\pi}{2} sgn\beta\right) - \cos\left(\rho h + \frac{\pi}{2} s(\ \beta,\ \gamma)\right) e^{\rho(X-h)}\right)\ [1], \\ \qquad\qquad\qquad if \quad s(\alpha, \beta) = 1, \\ \left(\sin\rho X - \cos\rho X + (-1)^{sgn\alpha} e^{-\rho X} + (-1)^{1-sgn\gamma} \times \right. \\ \left. \times\sqrt{2}\sin\left(\rho h + \frac{\pi}{4}(-1)^{sgn\gamma}\right)\ e^{\rho(X-h)}\right)\ [1], \quad if \quad s(\alpha, \beta) \neq 1. \end{cases} \tag{19}$$

Similarly, for the function $Ty(x, \lambda)$ we obtain the following asymptotic behavior

$$Ty(x, \lambda) = \begin{cases} -\rho^3 (pr^3)^{\frac{1}{4}} \left[\cos\left(\rho X + \frac{\pi}{2}\ sgn\ \beta\right) + \cos\left(\rho h + \frac{\pi}{2} s(\ \beta,\ \gamma)\right) \times \right. \\ \left. \qquad\qquad \times\ e^{\rho(X-h)}\right] [1], \ if \quad s(\alpha, \beta) = 1, \\ -\rho^3 (pr^3)^{\frac{1}{4}} \left[\cos\rho X + \sin\rho X + (-1)^{sgn\ \alpha} e^{-\rho X} - (-1)^{1-sgn\ \gamma} \times \right. \\ \left. \times\sqrt{2}\sin\left(\rho h + \frac{\pi}{4}(-1)^{sgn\ \gamma}\right)\ e^{\rho(X-h)}\right] [1], \ if \quad s(\alpha, \beta) \neq 1. \end{cases} \tag{20}$$

**Remark 4.** *As an immediate consequence of* (19), *we obtain that the number of zeros of function* $y(x,\lambda)$ *in the interval* $(0,l)$ *tends to* $+\infty$ *as* $\lambda \to \pm\,\infty$.
*Taking into account relations* (19) *and* (20), *we obtain the asymptotic formulas*

$$F(\lambda)=\begin{cases} -\rho^3(p(1)r^3(1))^{\frac{1}{4}}\dfrac{(\sqrt{2})^{1-2sgn\,\gamma}\cos\left(\rho h+\frac{\pi}{2}\,sgn\,\beta+\frac{\pi}{4}\,sgn\,\gamma\right)}{\cos\left(\rho l+\frac{\pi}{2}\,sgn\,\beta+\frac{\pi}{4}\,(1+sgn\,\gamma)\right)}\times \\ \qquad\qquad\times[1]\,,\ if\quad s(\alpha,\beta)=1, \\ -\rho^3(p(1)r^3(1))^{\frac{1}{4}}\dfrac{(\sqrt{2})^{1-2sgn\,\gamma}\cos\left(\rho h-(1-sgn\,\gamma)\frac{\pi}{4}\right)}{\cos\left(\rho h+\frac{\pi}{4}sgn\,\gamma\right)}\times \\ \qquad\qquad\times[1]\,,\ if\quad s(\alpha,\beta)\neq 1. \end{cases} \tag{21}$$

In turn, (21) implies the asymptotic formula

$$F(\lambda)=-\left(\sqrt{2}\right)^{1-2sgn\,\gamma}(p(1)\,r^3(1))^{\frac{1}{4}}\sqrt[4]{|\lambda|^3}\left(1+O\left(1\Big/\sqrt[4]{|\lambda|}\right)\right)\,,\ \ \lambda\to-\infty. \tag{22}$$

From (22) it follows immediately

**Lemma 2.** *The following relation holds:*

$$\lim_{\lambda\to-\infty}F(\lambda)=-\infty\,. \tag{23}$$

**Remark 5.** *From Lemmas 1 and 2 it follows that* $F(\lambda)<0$ *for* $\lambda<0$ *or* $\lambda=0,\ \beta\in[0,\pi/2)$; $F(\lambda)=0$ *for* $\lambda=0,\ \beta=\pi/2$.

Now we study the problem on the number of zeros of function $y(x,\lambda)$.

**Lemma 3.** *Every zero* $x(\lambda)\in(0,\,1)$ *of the equation* $y(x,\lambda)=0$ *is simple and is a* $C^1$ *function of* $\lambda\in R$.

*Proof.* Impossibility of the multiple zero of the equation $y(x,\lambda)=0$ for $\lambda>0$ follows from the statement *i)* of Lemma B.

We suppose now that $\xi\in(0,l)$ and $\lambda\le 0$ with $y(\xi,\lambda)=y'(\xi,\lambda)=0$. Then the function $y(x,\lambda)$ solves the problem (1), (2) for $l=\xi,\ \gamma=\delta=0$, which contradicts the condition $\lambda\le 0$ in view of Theorem A. The rest of the proof concerning the smoothness of $x(\lambda)$ follows from the well-known implicit function theorem. The Lemma 3 is proved. ◀

**Corollary 1.** *As* $\lambda>0$ *(*$\lambda\le 0$*) varies, the solution* $y(x,\lambda)$ *can lose or gain zeros only by these zeros leaving or entering the interval* $[0,\,1]$ *through its endpoint* $x=1\,(x=0)$.

Let $s(\lambda)$ be the number of zeros of the function $y(x,\lambda)$ in the interval $(0,l)$.

**Lemma 4.** *Let* $\lambda>0$. *If* $\lambda\in(\lambda_{n-1}(0),\,\lambda_n(0)]$, $n\in\mathrm{N}$, *then* $s(\lambda)=n-1$.

*Proof.* Let $\lambda > 0$ and $\theta(x,\lambda)$ be the corresponding function in transformation (3). By statement *iii)* of Lemma B we have $\theta(0,\lambda) = \beta - \pi/2$. On the basis of Theorem A we have the equality

$$\theta(l, \lambda_n(0)) = (2n-1)\pi/2, \; n \in \mathrm{N}.$$

As is known [16, 17], if $\lambda > 0$, then the function $\theta(x,\lambda)$, which is strictly increasing, assumes the values $k\pi/2$ for $k = -1, 0, 1, \ldots$ . From these arguments and Lemma B it follows that
$\theta(x,\lambda) \in (-\pi/2, \pi/2]$ for $\lambda \in (0, \lambda_1(0)]$, and

$$\theta(x,\lambda) \in (-\pi/2, (2n-1)\pi/2] \text{for } \lambda \in (\lambda_{n-1}(0),\, \lambda_n(0)], n = 2, 3, \ldots.$$

Hence it follows validity of assertion of the lemma. The Lemma 4 is proved. ◄

By Theorem A we have $y(l,\lambda) \neq 0$ for $\lambda < 0$. Therefore, by Remark 4 and Corollary 1, as $\lambda < 0$ varies the new zeros of the function $y(x,\lambda)$ can enter the interval (0, 1) only through the endpoint $x = 0$. Obviously, if the function $y(x,\lambda)$ acquires a new zero for $\lambda = \lambda^*$, then $y(0,\lambda^*) = y'(0,\lambda^*) = y''(0,\lambda^*)$ for $\beta = 0$ or $y(0,\lambda^*) = Ty(0,\lambda^*)$ for $\beta \in (0,\pi/2]$.

**Lemma 5.** *Let $\lambda_0 < 0$ and either $y''(0,\lambda_0) \neq 0$ for $\beta = 0$, or $Ty''(0,\lambda_0) \neq 0$ for $\beta \in (0,\pi/2)$, or $y(0,\lambda_0) \neq 0$ for $\beta = \pi/2$. Then there exists $\varepsilon > 0$ such that for any $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)/\{\lambda_0\}$ the number of zeros of the function $y(x,\lambda)$ belonging to the interval $(0,l)$ coincides with the number of zeros of the function $y(x,\lambda_0)$ belonging to the interval $(0,l)$, i.e. $s(\lambda) = s(\lambda_0)$ for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)/\{\lambda_0\}$.*

*Proof.* Let $\lambda_0 < 0$ and $y''(0,\lambda_0) \neq 0$ for $\beta = 0$. Then there exists $\delta_0 > 0$ such that the function $y''(x,\lambda_0)$ does not vanish on $[0,\delta_0]$ and the function $y(x,\lambda_0)$ does not vanish on $[l-\delta_0,\, l]$. By Remark 2 there exists $\varepsilon_1 > 0$ such that for any $\lambda \in (\lambda_0 - \varepsilon_1, \lambda_0 + \varepsilon_1)$ , the function $y''(x,\lambda)$ does not vanish on $[\,0,\delta_0]$ and the function $y(x,\lambda)$ does not vanish on $[\,l-\delta_0, l]$. Therefore, for $\lambda \in (\lambda_0 - \varepsilon_1, \lambda_0 + \varepsilon_1)$ , all zeros of the function $y(x,\lambda)$ belonging to the interval $(0,l)$ are concentrated on the interval $(\delta_0,\; l-\delta_0)$ .

By Lemma 3 all zeros of the function $y(x,\lambda_0)$ on the interval $(\delta_0,\; l-\delta_0)$ are simple and there exists $\varepsilon \in (0,\varepsilon_1)$ such that for any $\lambda \in (\lambda_0 - \varepsilon,\; \lambda_0 + \varepsilon)/\{\lambda_0\}$, the number of zeros of the function $y(x,\lambda)$ on the interval $(\delta_0,\; l-\delta_0)$ coincides with that for the function $y(x,\lambda_0)$. Since the functions $y(x,\lambda_0)$ and $y(x,\lambda)$ have no zeros on the half-open intervals $(0,\delta_0]$ and $[\,l-\delta_0, l)$ , according to what was said above, we have $s(\lambda) = s(\lambda_0)$ for $\lambda \in (\lambda_0 - \varepsilon, \lambda + \varepsilon)/\{\lambda_0\}$.

Now let $Ty\,(0,\lambda_0) \neq 0$ for $\beta \in (0,\pi/2)$. By (2b) we have $y\,(0,\lambda_0) \neq 0$. Then there exists $\delta_1 > 0$ such that the function $y\,(x,\lambda_0)$ does not vanish on $[0,\delta_1]\bigcup[l-\delta_1, l]$. By Remark 2 there exists $\varepsilon_2 > 0$ such that for $\lambda \in (\lambda_0 - \varepsilon_2, \lambda_0 + \varepsilon_2)$ the function $y\,(x,\lambda)$ does not vanish on $[0,\delta_1]\bigcup[l-\delta_1, l]$. Consequently, for $\lambda \in (\lambda_0 - \varepsilon_2, \lambda_0 + \varepsilon_2)$ all zeros of the function $y\,(x,\lambda)$ belonging to the interval $(0,l)$ are concentrated on the interval $(\delta_1, l-\delta_1)$. The rest of the proof is similar to the previous case.

The case of $y\,(0,\lambda_0)\neq 0$ when $\beta=\pi/2$ is considered similarly. The Lemma 5 is proved. ◄

**Corollary 2.** *Let $\lambda^{(2)}<\lambda^{(1)}<0$ and $s(\lambda^{(2)})\neq s(\lambda^{(1)})$. Then the interval $(\lambda_2,\,\lambda_1)$ contains an eigenvalue of the problem defined by the equation* (1) *and the boundary conditions $y(0)=y'(0)=y''(0)=0$, (2c) for $\beta=0$, or $y(0)=Ty(0)=0$, (2a), (2c) for $\beta\in(0,\pi/2]$.*

Let $\lambda<0$ and $\mu$ be a real eigenvalue of the equation (1) with boundary condition $y(0)=y'(0)=y''(0)=0$, (2c) for $\beta=0$, or $y(0)=Ty(0)=0$, (2a), (2c) for $\beta\in(0,\pi/2]$. The oscillation index of this eigenvalue $\mu$ is the difference between the number of zeros of the function $y(x,\lambda)$ for $\lambda=\mu-0$ belonging to the interval $(0,l)$ and the number of the same zeros for $\lambda=\mu+0$ (see [22]). From this definition, it directly follows that the number of zeros of the function $y(x,\lambda)$ belonging to the interval $(0,l)$ is equal to the sum of the oscillation indices of all eigenvalues of the equation (1) with the boundary conditions $y(0)=y'(0)=y''(0)=0$, (2c) for $\beta=0$, or $y(0)=Ty(0)=0$, (2a), (2c) for $\beta\in(0,\pi/2]$ belonging to the interval $(\lambda,\,0)$.

Following the scheme of the proof of Theorem 4.1 of [22] we can prove that there exists $\xi<0$ such that lying on the ray $(-\infty,\xi)$ the eigenvalues $\xi_k, k=1,2,...,$ of the equation (1) with the boundary conditions $y(0)=y'(0)=y''(0)=0$, (2c) for $\beta=0$, or $y(0)=Ty(0)=0$, (2a), (2c) for $\beta\in(0,\pi/2]$ enumerated in the decreasing order are simple, admit the asymtotics

$$\xi_k=\begin{cases} -4\left(k+\frac{1}{2}-\frac{1}{4}sgn\gamma\right)^4\left(\frac{\pi}{h}\right)^4+o\,(k^4),\ if\ \beta=0,\\ -4\left(k+\frac{1}{2}-\frac{1}{8}\ \left|(-1)^{sgn\alpha}+(-1)^{sgn\gamma}\right|\,(3-s(\alpha,\gamma))\right)^4\left(\frac{\pi}{h}\right)^4+o\,(k^4),\ if\ \beta\in(0,\pi/2],\end{cases}$$

and have oscillation index 1.

Let $i(\xi_k)$ be the oscillation index of the eigenvalue $\xi_k$. From the above it follows that the formula

$$s(\lambda)=\sum_{\xi_k\in(\lambda,0)} i(\xi_k) \tag{24}$$

holds. Now consider the problem (1), (2) for $\delta\in(\pi/2,\,\pi)$.

**Lemma 6.** *The eigenvalues of the problem* (1), (2) *for $\delta\in(\pi/2,\pi)$ are simple.*

*Proof.* By [16, 17] the positive eigenvalues of this problem are simple. Let $\lambda^*\leq 0$ be an eigenvalue of the problem (1), (2) for $\delta\in(\pi/2,\pi)$, which corresponds to the eigenfunctions of $y^{(1)}(x)$ and $y^{(2)}(x)$. By Theorem A we have $y^{(1)}(l)\neq 0,\ y^{(2)}(l)\neq 0$. We define the function:

$$\vartheta\,(x)=y^{(2)}(l)y^{(1)}(x)-y^{(1)}(l)\ y^{(2)}(x).$$

Since $\vartheta(l)=0$, the function $\vartheta(x)$ is an eigenfunction of the problem (1), (2) with $\delta=0$ corresponding to the eigenvalue $\lambda^*\leq 0$, which contradicts Theorem A. The proof of Lemma 6 is complete. ◄

**Theorem 2.** *For fixed $\alpha, \beta, \gamma$ the eigenvalues of the problem* (1), (2) *for* $\delta \in (\pi/2, \pi)$ *form a sequence* $\{\mu_n(\delta)\}_{n=1}^{\infty}$ *such that* $\mu_1(\delta) < \mu_2(\delta) < ... < \mu_n(\delta) < ...$ *and besides* $\mu_n(\delta) > 0$ *at* $n \geq 2$. *Moreover, in the case* $\beta \in [0, \pi/2)$ *there exists* $\delta_0 \in (\pi/2, \pi)$ *such that if* $\delta \in (\pi/2, \delta_0)$ , *then* $\mu_1(\delta) > 0$ , *if* $\delta = \delta_0$, *then* $\lambda_1(\delta) = 0$ , *if* $\delta \in (\delta_0, \pi)$ , *then* $\lambda_1(\delta) < 0$ , *and in the case* $\beta = \pi/2$ *it holds* $\mu_1(\delta) < 0$. *The eigenfunction* $\vartheta_n^{(\delta)}(x)$, $n \geq 2$, *corresponding to the eigenvalue* $\mu_n(\delta)$ , *has exactly* $n-1$ *simple zeros in the interval* $(0, l)$; *in the case* $\mu_1(\delta) \geq 0$ *the eigenfunction* $\vartheta_1^{(\delta)}(x)$ *has no zeros in the interval* $(0, l)$ ,*and in the case* $\mu_1(\delta) < 0$ *the eigenfunction* $\vartheta_1^{(\delta)}(x)$ *may have any number of zeros in the interval* $(0, l)$, *which are also simple (see* (24)*).*

*Proof.* It is easy to see that the eigenvalues of the problem (1), (2) for $\delta \in (0, \pi)$ are the roots of the equation

$$F(\lambda) = ctg\delta. \tag{25}$$

By Lemma 1 and formula (23) the function $F(\lambda)$ is continuous and increasing in each of the intervals $(\mu_{n-1}(0), \mu_n(0))$ and the relations $\lim\limits_{\lambda \to \lambda_{n-1}(0)+0} F(\lambda) = -\infty$, $\lim\limits_{\lambda \to \lambda_n(0)-0} F(\lambda) = +\infty$ are true. Hence the function $F(\lambda)$ assumes each value in $(-\infty, +\infty)$ at a unique point in the interval $(\mu_{n-1}(0), \mu_n(0))$, $n \in$ N. Hence this interval contains a unique point $\lambda = \mu_n^*$ which is a solution of equation (25), i.e., condition (2d) is satisfied. It means $\mu_n^*$ is an eigenvalue of problem (1), (2) for $\delta \in (0, \pi)$. It is easy to see that $\mu_n^*$ is an $n$-th eigenvalue of this problem, i.e., $\mu_n(\delta) = \lambda_n^*$. By Lemma 1 we have $\mu_n(\delta) \in (\mu_n(\pi/2), \mu_n(0))$ for $\delta \in (0, \pi/2)$ and $\mu_n(\delta) \in (\mu_{n-1}(0), \mu_n(\pi/2))$ for $\delta \in (\pi/2, \pi)$ . By Remark 5 we obtain $F(0) < 0$ for $\beta \in [0, \pi/2)$ and $F(0) = 0$ for $\beta = \pi/2$. Consequently, in the case $\beta \in [0, \pi/2)$ we see that $\mu_1(\delta) > 0$ if $ctg\delta > F(0)$, $\mu_1(\delta) = 0$ if $ctg\delta = F(0)$ and $\mu_1(\delta) < 0$ if $ctg\delta < F(0)$, and in the case $\beta = \pi/2$ we have $\mu_1(\delta) < 0$ (the number $\delta_0$ appearing in the theorem is defined as $\delta_0 = arcctg\ F(0)$).

The last assertion of the theorem follows from Lemma 4 and formula (24). The proof of Theorem 2 is complete. ◄

## References

[1] Kellogg O.D. The Oscillation of Functions of an Orthogonal Set, Amer. J. Math. 1916, v. 38, p. 1-5.

[2] Kellogg O.D. Orthogonal Function Sets Arising from Integral Equations, Amer. J. Math. 1918, v. 40, p. 145-154.

[3] Sturm C. Sur une classe d'equations *a* derivee partielle, J.Math. Pures Appl., 1836, v.1, p. 373-444.

[4] Davidoglou O. Sur l'equation des vibrations transversales des verges elastiques, Ann. Ecole Norm. 1900, v.17, No 3, p. 359-444.

[5] Davidoglou O. Etude de l'equation differentielle $(\theta y'')'' = K\varphi(x)\, y$, Ann. Ecole Norm. 1905, v.22, No 3, p. 539-565.

[6] Janczewsky S.N. Oscillation theorems for the differential boundary value prob-lems of the fourth order, Ann. Math., 1928, v.29, No 2, p. 521-542.

[7] Krein M.G. On non-symmetric Kellogg kernels, Dokl. Akad. Nauk SSSR, 1936, v.1, No 1, p. 3-5.

[8] Gantmakher F.R., Krein M.G. Oscillatory matrices and kernels and small oscillations of mechanical systems, M.-L.: Gostekhizdat, 1950 (in Russian).

[9] Levin A.Yu., Stepanov G.D. One-dimensional boundary value problems with operators not reducing the number of changes of sign. I, Siber. Math.J. 1976, v. 17, No 3, p. 606-625.

[10] Levin A.Yu., Stepanov G.D. One-dimensional boundary value problems with operators not reducing the number of changes of sign. II, Siber. Math.J. 1976, v. 17, No 4, p. 612-625.

[11] Elias U. Eigenvalue problems for the equation $Ly + \lambda p(x)\, y = 0$, J. Diff. Equat., 1978, v.29, p. 28-57.

[12] Przybycin J. Spectral properties of the certain differential operators of the second and fourth order, Univ. Lag. Acta Math., 1985, v.25, p. 269-276.

[13] Borovskikh A.V., Pokornyi Yu.V. Chebyshev-Haar systems in the theory of discontinuous Kellogg kernels, Russian Mathematical Surveys, 1994, v. 49, No 3, p.3-42.

[14] Prufer H. Neue Herleitung der Sturm-Liouvilleschen Reihenencklung stetiger Funktionen, Mathematische Annalen, v. 95, p. 499-518.

[15] Atkinson F.V. Discrete and continuous boundary problems. Academic Press, New York, 1968.

[16] Banks D.O., Kurowski G.J. A Prufer transformation for the equation of the vibrating beam, Trans. Amer. Math. Soc., 1974, v. 199, p. 203-222.

[17] Banks D.O. and G.J.Kurowski . A Prufer transformation for the equation of a vibrating beam subject to axial forces, J. Diff. Equat., 1977, v. 24 , p. 57-74.

[18] Kerimov N.B, Aliyev Z.S. On oscillation properties of the eigenfunctions of a fourth order differential operator, Trans. NAS Azerb., ser. phys.-tech. math. sci., math. mech., 2005, v. 25, No 4, p. 63-76.

[19] Amara J. Ben. Sturm theory for the equation of vibrating beam, J.Math. Anal. Appl., 2009, v. 349, p. 1-9.

[20] Leighton W., Nehari Z. On the oscillation of solutions of self-adjoint linear differential equations of the fourth order, Trans. Amer. Math. Soc., 1958, v. 98, p. 325-377.

[21] Naimark M.A. Linear differential operators, M.: Nauka, 1969; English transl. of $1^{st}$ ed., vols. 1, 2, Frederick Ungar, New York 1967, 1968.

[22] Amara J. Ben, Vladimirov A.A. On oscillation of eigenfunctions of a fourth order problem with spectral parameter in the boundary condition, J. Math. Sciences, 2008, v. 150, No 5, p. 2317-2325.

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Received 8 April 2014
Accepted 20 May 2014