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On Some Classes of Mixed Generalized Quasi-Einstein Manifolds

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Abstract. In this paper, we consider generalized quasi-Einstein manifolds. Then we investigate some properties of Ricci-pseudosymmetric and Ricci semi-symmetric mixed generalized quasi-Einstein manifolds. Finally, we get relation mixed generalized quasi Einstein manifold.

Key Words and Phrases: Quasi-Einstein manifold, Generalized quasi-Einstein manifold, Mixed generalized quasi-Einstein manifold, Quasi-conformal curvature tensor.

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1. Introduction

The notion of a quasi-Einstein manifold was introduced by M. C. Chaki in [2]. A non flat n-dimensional Riemannian manifold (M^n, g) is said to be a quasi-Einstein manifold if its Ricci tensor S satisfies

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \quad \forall X, Y \in TM$$

for some non-zero scalars a and $b \neq 0$, where η is a non zero 1-form such that

$$g(X,\xi) = \eta(X), \quad g(\xi,\xi) = \eta(\xi) = 1$$

for the associated vector field ξ. The 1-forms η is called the associated 1-form and the unit vector field ξ is called the generator of the manifold. In [3], U. C. De and G. C. Ghosh introduced the notion of a generalized quasi-Einstein manifolds. A non-flat Riemannian manifold M is called a generalized quasi-Einstein manifold if its Ricci tensor S of type (0,2) is non-zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y),$$

where a, b, c are some non-zero scalars and A, B are two non-zero 1-forms such that

$$g(X,U) = A(X), \quad g(X,V) = B(X), \quad g(U,V) = 0,$$

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i.e. U, V are orthogonal vector fields on M. In [1], A. Bhattacharyya and T. De introduced notion of a mixed quasi-Einstein manifold. A non-flat Riemannian manifold is called a mixed generalized quasi-Einstein manifold if its Ricci tensor S of type (0,2) is non-zero and satisfies the condition

$$S(X,Y) = ag(X;Y) + bK(X)K(Y) + cL(X)L(Y) + +d[K(X)L(Y) + L(X)K(Y)], \quad (1)$$

where a, b, c, d are non-zero scalars,

$$g(X,U) = K(X), \quad g(X,V) = L(X), \quad g(U,V) = 0, \quad (2)$$

K, L being two non-zero 1-forms, and U, V are unit vector fields corresponding to the 1-forms K and L, respectively. We denote this type of manifold by $MG(QE)_n$. If $d = 0$, then this manifold reduces to a $G(QE)_n$. From (1), we get

$$r = na + b + c, \quad (3)$$

where r denotes the scalar curvature of the manifold.

2. Ricci-pseudosymmetric mixed generalized quasi-Einstein manifolds

An n-dimensional semi-Riemannian manifold (M^n, g) is called Ricci-pseudosymmetric [4] if the tensors R.S and $Q(g,S)$ are linearly dependent, where

$$(R(X,Y).S)(Z,W) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W), \quad (4)$$

$$Q(g,S)(Z,W;X,Y) = -S((X \wedge Y)Z,W) - S(Z,(X \wedge Y)W) \quad (5)$$

and

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y \quad (6)$$

for vector fields X, Y, Z, W on M^n, R denotes the curvature tensor of M^n [6]. The condition of Ricci-pseudosymmetricity is equivalent to the relation

$$(R(X,Y).S)(Z,W) = L_s Q(g,S)(Z,W;X,Y), \quad (7)$$

which holds on the set

$$U_s = \{x \in M : S \neq \frac{r}{n} g \ \ at \ \ x\},$$

where L_s is some function on U_s [6]. If $R.S = 0$ then M^n is called Ricci-semisymmetric. Every Ricci-semisymmetric manifold is Ricci-pseudosymmetric but the converse is not true ([4],[6]).

In this section, we prove the following theorem:

Theorem 1. *Let (M^n, g) be an n-dimensional mixed generalized quasi-Einstein manifold. If M^n is Ricci-pseudosymmetric then we have*

$$R(X,Y,U,V) = L_s\{K(Y)L(X) - L(Y)K(X)\}, \tag{8}$$

Proof. Let M^n be Ricci-pseudosymmetric.Then from (4)-(7) we obtain

$$S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = L_s\{g(Y,Z)S(X,W) \tag{9}$$

$$-g(X,Z)S(Y,W) + g(Y,W)S(X,Z) - g(X,W)S(Y,Z)\}.$$

Since M^n is a mixed generalized quasi-Einstein manifold, using the well-known properties of the curvature tensor R we get

$$b[A(R(X,Y)Z)A(W) + A(R(X,Y)W)A(Z)] \tag{10}$$

$$+c[B(R(X,Y)Z)B(W) + B(R(X,Y)W)B(Z)]$$

$$+d[A(R(X,Y)Z)B(W) + B(R(X,Y)Z)A(W)$$

$$+A(R(X,Y)W)B(Z) + B(R(X,Y)W)A(Z)]$$

$$= L_s\{b[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)$$

$$+g(Y,W)A(X)A(Z) - g(X,W)A(Y)A(Z)]$$

$$+c[g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)$$

$$+g(Y,W)B(X)B(Z) - g(X,W)B(Y)B(Z)]$$

$$+d[g(Y,Z)A(X)B(W) + g(Y,Z)B(X)A(W)$$

$$-g(X,Z)A(Y)B(W) - g(X,Z)B(Y)A(W)$$

$$+g(Y,W)A(X)B(Z) + g(Y,W)A(Z)B(X)$$

$$-g(X,W)A(Y)B(Z) - g(X,W)A(Z)B(Y)]\}.$$

Taking $Z = W = U$ in (10) we get
$2d\{R(X,Y,Z,W) - L_s\{B(X)A(Y) - A(X)B(Y)\}\} = 0.$
Since $d \neq 0$, we have
$R(X,Y,Z,W) - L_s\{B(X)A(Y) - A(X)B(Y)\} = 0$. This completes the proof of the theorem.

3. Ricci semi-symmetric mixed generalized quasi-Einstein manifolds

An n-dimensional manifold $(M^n; g)$ is called semi-symmetric [5], if $R(X;Y).S = 0\ \forall X, Y$ where $R(X;Y)$ denotes the curvature operator.
Now, we prove the following theorem:

Theorem 2. *There is no mixed generalized quasi-Einstein manifold satisfying R(X,Y).S = 0.*

Proof. Since $R(X,Y).S)(Z,W) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W)$, the condition $R(X,Y).S = 0$ gives $0 = S(R(X,Y)Z,W) + S(Z,R(X,Y)W)$. Then

$$\begin{aligned} ag(R(X,Y)Z,W) + bK(R(X,Y)Z)K(W) + cL(R(X,Y)Z)L(W) \\ +d[K(R(X,Y)Z)L(W) + L(R(X,Y)Z)K(W)] + ag(R(X,Y)W,Z) \\ +bK(R(X,Y)W)K(Z) + cL(R(X,Y)W)L(Z) \\ +d[K(R(X,Y)W)L(Z) + L(R(X,Y)W)K(Z) = 0. \end{aligned} \tag{11}$$

Putting $Z = W = U$ in (11) we obtain

$$2dL(R(X,Y)U) = 0$$

or,

$$2dg(R(X,Y)U,V) = 0$$

or,

$$2d\acute{R}(X,Y,U,V) = 0,$$

where $\acute{R}$(X,Y,U,V) = g(R(X,Y)U,V). Since $d \neq 0$ and $\acute{R} \neq 0$, this is not possible. This completes the proof of the theorem.

4. Mixed generalized quasi-Einstein manifolds satisfying the condition $\tilde{C}.S = 0$

Let (M^n, g) be a Riemannian manifold, the quasi-conformal curvature tensor be defined as in [7],

$$\begin{aligned} \tilde{C}(X,Y)Z = \lambda R(X,Y)Z + \mu\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ -\frac{r}{n}[\frac{\lambda}{n-1} + 2\mu]\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{12}$$

where Q is the Ricci operator defined by

$$S(X,Y) = g(QX,Y).$$

An n-dimensional Riemannian manifold (M^n, g) $(n > 3)$, is called quasi-conformally flat if $\tilde{C} = 0$. If $\lambda = 1$ and $\mu = -\frac{1}{n-2}$, then quasi-conformal curvature tensor is reduced to conformal curvature tensor [6].
Now we can state the following theorem:

Theorem 3. *Assume that* (M^n, g) $(n > 3)$ *is a mixed generalized quasi-Einstein manifold. From the condition* $\tilde{C}.S = 0$ *holding on* (M^n) *we get:*

$$\lambda R(X,Y,V,U) = [\frac{(2-n)(b+c)\mu}{n} + \frac{(an+b+c)\lambda}{n(n-1)}]\{A(X)B(Y) - B(X)A(Y)\} \quad (13)$$

Proof. From the condition $\tilde{C}.S = 0$ holding on (M^n) we get:

$$S(\tilde{C}(X,Y)Z,W) + S(\tilde{C}(X,Y)W,Z) = 0,$$

for all vector fields X, Y, Z, W on (M^n).
Since (M^n) is a mixed generalized quasi-Einstein manifold and $\tilde{C}(X,Y)W,Z) = \tilde{C}(X,Y)Z,W)$ we obtain

$$\begin{aligned} 0 &= b[A(\tilde{C}(X,Y)Z)A(W) + A(\tilde{C}(X,Y)W)A(Z)] \\ &+c[B(\tilde{C}(X,Y)Z)B(W) + B(\tilde{C}(X,Y)WB(Z)] \\ &+d[A(\tilde{C}(X,Y)Z)B(W) + B(\tilde{C}(X,Y)Z)A(W) \\ &+A(\tilde{C}(X,Y)W)B(Z) + B(\tilde{C}(X,Y)W)A(Z)]. \end{aligned} \quad (14)$$

Taking $Z = W = V$ in (14) we obtain

$$2d[A(\tilde{C}(X,Y)V)] = 0. \quad (15)$$

Since $d \neq 0$, using (15) we have

$$A(\tilde{C}(X,Y)V) = \tilde{C}(X,Y,V,U) = 0. \quad (16)$$

From (12) and (16) we obtain (13). This completes the proof of the theorem.

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