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On Quarter-Symmetric Metric Connection in a Lorentzian Para-Sasakian Manifold

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Abstract. In this paper we obtain results on symmetric and concircular symmetric Lorentzian para-Sasakian (briefly LP-Sasakian) manifolds with respect to quarter-symmetric metric connection and Riemannian connection.

Key Words and Phrases: Lorentzian para-Sasakian manifold, quarter-symmetric metric connection, concircular curvature tensor.

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1. Introduction

The idea of semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten [5]. Further, Hayden [7], introduced the idea of metric connection with torsion on a Riemannian manifold. In [25], Yano studied some curvature conditions for semi-symmetric connections in Riemannian manifolds. In [6], Golab defined and studied quarter-symmetric connection in a differentiable manifold with affine connection. After that various properties of quarter-symmetric metric connection have been studied by many geometers like Rastogi ([16], [17]), Mishra and Pandey [12], Yano and Imai [26], De and Sengupta [4], Pradeep Kumar, Venkatesha and Bagewadi ([14], [15]), and many others.

M. M. Tripathi [19] studied the semi-symmetric metric connection in a Kenmotsu manifolds. In [20], the semi-symmetric non-metric connection in a Kenmotsu manifold was studied by M. M. Tripathi and N. Nakkar. In [1], Amit Prakash and Dhruwa Narain studied quarter symmetric non-metric connection in Lorentzian para-Sasakian manifolds. Also in [21], M. M.Tripathi proved the existence of a new connection and showed that in particular cases, this connection reduces to semi-symmetric connections; even some of them are not introduced so far. On the other hand, there is a class of almost paracontact-metric manifolds, namely Lorentzian para-Sasakian manifolds. In 1989, K. Matsumoto [8] introduced the notion of Lorentzian para-Sasakian manifold. Then I.Mihai and R. Rosca [11] introduced the same notion independently and they obtained several results on this

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manifold. Lorentzian para-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [9], U. C. De et al.,[4], A.A. Shaikh and S. Biswas [18], M. M. Tripathi and U. C. De [22].

A linear connection $\tilde{\nabla}$ in an n-dimensional differentiable manifold is said to be a quarter-symmetric connection [6] if its torsion tensor T is of the form

$$T(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1}$$

where η is a 1-form and ϕ is a tensor field of type $(1,1)$. In particular, if we replace ϕX by X and ϕY by Y, then the quarter-symmetric connection reduces to the semi-symmetric connection [5]. Thus, the notion of quarter-symmetric connection generalizes the idea of semi-symmetric connection. And if quarter-symmetric linear connection $\tilde{\nabla}$ satisfies the condition $(\tilde{\nabla}_X g)(Y,Z) = 0$ for all $X, Y, Z \in \mathcal{X}(M)$, where $\mathcal{X}(M)$ is the Lie algebra of vector fields on the manifold M, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection.

In this paper we study the geometry of LP-Sasakian manifolds with respect to quarter-symmetric metric connection. However these manifolds have been studied by many geometers like K. Matsumoto [8], K. Matsumoto and I. Mihai [9], I. Mihai and R. Rosca [11], I. Mihai, A.A. Shaikh and U.C. De [10], Venkatesha and C.S. Bagewadi [23] and they obtained several results on this manifold.

2. Preliminaries

An n-dimensional differentiable manifold M is called an LP-Sasakian manifold ([8], [11]) if it admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \tag{2}$$
$$\phi^2 X = X + \eta(X)\xi, \tag{3}$$
$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{4}$$
$$g(X,\xi) = \eta(X), \tag{5}$$
$$\nabla_X \xi = \phi X, \tag{6}$$
$$(\nabla_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{7}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

It can be easily seen that in a LP-Sasakian manifold, the following relations hold:

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad rank\phi = n-1. \tag{8}$$

Again, if we put

$$\Phi(X,Y) = g(X,\phi Y), \tag{9}$$

for any vector fields X and Y, then the tensor field $\Phi(X,Y)$ is a symmetric (0, 2) tensor field [8]. Also, since the 1-form η is closed in an LP-Sasakian manifold, we have ([8], [10])

$$(\nabla_X \eta)(Y) = \Phi(X,Y), \quad \Phi(X,\xi) = 0, \tag{10}$$

for any vector fields X and Y.
Also in an LP-Sasakian manifold, the following relations hold ([9], [10]):

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{11}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{12}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{13}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{14}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{15}$$

for any vector fields X, Y and Z, where R is the Riemannian curvature tensor and S is the Ricci tensor of M.

Definition 1. *An LP-Sasakian manifold M is said to be symmetric if*

$$(\nabla_W R)(X,Y)Z = 0, \tag{16}$$

for all vector fields X, Y, Z and W.

Definition 2. *An LP-Sasakian manifold M is said to be ϕ-symmetric if*

$$\phi^2(\nabla_W R)(X,Y)Z = 0, \tag{17}$$

for all vector fields X, Y, Z and W.

Definition 3. *An LP-Sasakian manifold M is said to be concircular symmetric if*

$$(\nabla_W \bar{C})(X,Y)Z = 0, \tag{18}$$

for all vector fields X, Y, Z and W, where $\bar{C}$ is the concircular curvature tensor and is given by [24]

$$\bar{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{19}$$

for any vector fields X, Y and Z, where R and r are the Riemannian curvature tensor and scalar curvature respectively.

Definition 4. *An LP-Sasakian manifold M is said to be concircular ϕ-symmetric if*

$$\phi^2(\nabla_W \bar{C})(X,Y)Z = 0, \tag{20}$$

for all vector fields X, Y, Z and W.

3. Expression of $\tilde{R}(X,Y)Z$ in terms of $R(X,Y)Z$

In this section we express $\tilde{R}(X,Y)Z$ the curvature tensor w.r.t quarter-symmetric metric connection in terms of $R(X,Y)Z$ the curvature tensor w.r.t Riemannian connection.

Let $\tilde{\nabla}$ be a linear connection and ∇ be a Riemannian connection of an almost contact metric manifold M such that

$$\tilde{\nabla}_X Y = \nabla_X Y + U(X,Y), \tag{21}$$

where U is a tensor of type $(1,1)$. For $\tilde{\nabla}$ to be a quarter-symmetric metric connection in M, we have [6]

$$U(X,Y) = \frac{1}{2}[T(X,Y) + T^{'}(X,Y) + T^{'}(Y,X)] \tag{22}$$

and

$$g(T^{'}(X,Y),Z) = g(T(Z,X),Y). \tag{23}$$

From (1) and (23), we get

$$T^{'}(X,Y) = \eta(X)\phi Y - g(X,\phi Y)\xi. \tag{24}$$

Using (1) and (24) in (22), we obtain

$$U(X,Y) = \eta(Y)\phi X - g(X,\phi Y)\xi.$$

Thus the quarter-symmetric metric connection $\tilde{\nabla}$ in an LP-Sasakian manifold is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(X,\phi Y)\xi. \tag{25}$$

Hence (25) is the relation between Riemannian connection and the quarter-symmetric metric connection on an LP-Sasakian manifold.

A relation between the curvature tensor of M with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ and the Riemannian connection ∇ is given by

$$\begin{aligned} \tilde{R}(X,Y)Z &= R(X,Y)Z + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &+ [\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\xi \\ &+ [\eta(Y)X - \eta(X)Y]\eta(Z), \end{aligned} \tag{26}$$

where $\tilde{R}$ and R are the Riemannian curvatures of the connections $\tilde{\nabla}$ and ∇, respectively. From (26) it follows that

$$\tilde{S}(Y,Z) = S(Y,Z) + (n-1)\,\eta(Y)\eta(Z), \tag{27}$$

where $\tilde{S}$ and S are the Ricci tensors of the connections $\tilde{\nabla}$ and ∇, respectively.
Contracting (27), we get

$$\tilde{r} = r - (n-1), \tag{28}$$

where $\tilde{r}$ and r are the scalar curvatures of the connections $\tilde{\nabla}$ and ∇, respectively.

4. Symmetry of LP-Sasakian manifold with respect to quarter-symmetric metric connection

Analogous to the definition of symmetric LP-Sasakian manifold with respect to Riemannian connection, we define a symmetric LP-Sasakian manifold with quarter-symmetric metric connection by

$$(\tilde{\nabla}_W \tilde{R})(X, Y)Z = 0, \tag{29}$$

for all vector fields X, Y, Z and W.
Using (25), we have

$$\begin{aligned}(\tilde{\nabla}_W \tilde{R})(X, Y)Z &= (\nabla_W \tilde{R})(X, Y)Z + \eta(\tilde{R}(X, Y)Z)\phi W - g(W, \phi\tilde{R}(X, Y)Z)\xi \\ &- \eta(X)\tilde{R}(\phi W, Y)Z - \eta(Y)\tilde{R}(X, \phi W)Z - \eta(Z)\tilde{R}(X, Y)\phi W \\ &+ g(W, \phi X)\tilde{R}(\xi, Y)Z + g(W, \phi Y)\tilde{R}(X, \xi)Z + g(W, \phi Z)\tilde{R}(X, Y)\xi.\end{aligned} \tag{30}$$

Now differentiating (26) with respect to W and using (6), (7) and (10), we obtain

$$\begin{aligned}&(\nabla_W \tilde{R})(X, Y)Z \\ =\ & (\nabla_W R)(X, Y)Z - [\eta(Y)g(W, Z) + \eta(Z)g(Y, W) + 2\eta(Y)\eta(Z)\eta(W)]\phi X \\ +\ & [\eta(X)g(W, Z) + \eta(Z)g(X, W) + 2\eta(X)\eta(Z)\eta(W)]\phi Y \\ +\ & g(X, \phi Z)[g(W, Y)\xi + \eta(Y)W + 2\eta(W)\eta(Y)\xi] \\ -\ & g(Y, \phi Z)[g(W, X)\xi + \eta(X)W + 2\eta(W)\eta(X)\xi] \\ +\ & [g(W, \phi X)g(Y, Z) - g(W, \phi Y)g(X, Z)]\xi \\ +\ & [\eta(X)g(Y, Z) - \eta(Y)g(X, Z)]\phi W \\ +\ & [g(W, \phi Y)X - g(W, \phi X)Y]\eta(Z) \\ +\ & g(W, \phi Z)[\eta(Y)X - \eta(X)Y].\end{aligned} \tag{31}$$

Using (2), (8) and (31) in (30), we obtain

$$(\tilde{\nabla}_W \tilde{R})(X, Y)Z = (\nabla_W R)(X, Y)Z. \tag{32}$$

Therefore, we can state the following:

Theorem 1. *An LP-Sasakian manifold is symmetric with quarter-symmetric metric connection $\tilde{\nabla}$ if and only if it is so with respect to Riemannian connection ∇.*

Corollary 1. *An LP-Sasakian manifold is ϕ-symmetric with respect to quarter-symmetric metric connection $\tilde{\nabla}$ if and only if it is so with respect to Riemannian connection ∇.*

5. Concircular symmetry of LP-Sasakian manifold with respect to quarter-symmetric metric connection

An LP-Sasakian manifold M is said to be a concircular symmetric with respect to quarter-symmetric metric connection if

$$(\tilde{\nabla}_W\tilde{\tilde{C}})(X,Y)Z = 0, \tag{33}$$

for all vector fields X, Y, Z and W, where $\tilde{\tilde{C}}$ is the concircular curvature tensor with respect to quarter-symmetric metric connection given by

$$\tilde{\tilde{C}}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{\tilde{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{34}$$

where $\tilde{R}$ is the Riemannian curvature tensor and $\tilde{r}$ is the scalar curvature with quarter-symmetric metric connection $\tilde{\nabla}$.
Using (25), we can write

$$\begin{aligned}(\tilde{\nabla}_W\tilde{\tilde{C}})(X,Y)Z &= (\nabla_W\tilde{\tilde{C}})(X,Y)Z + \eta(\tilde{\tilde{C}}(X,Y)Z)\phi W - g(W,\phi\tilde{\tilde{C}}(X,Y)Z)\xi \\ &- \eta(X)\tilde{\tilde{C}}(\phi W,Y)Z - \eta(Y)\tilde{\tilde{C}}(X,\phi W)Z - \eta(Z)\tilde{\tilde{C}}(X,Y)\phi W \\ &+ g(W,\phi X)\tilde{\tilde{C}}(\xi,Y)Z + g(W,\phi Y)\tilde{\tilde{C}}(X,\xi)Z + g(W,\phi Z)\tilde{\tilde{C}}(X,Y)\xi.\end{aligned} \tag{35}$$

Now differentiating (34) with respect to W, we obtain

$$(\nabla_W\tilde{\tilde{C}})(X,Y)Z = (\nabla_W\tilde{R})(X,Y)Z - \frac{\nabla_W\tilde{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{36}$$

By making use of (28) and (31) in (36), we get

$$\begin{aligned}&(\nabla_W\tilde{\tilde{C}})(X,Y)Z) \\ =\ & (\nabla_W R)(X,Y)Z - [\eta(Y)g(W,Z) + \eta(Z)g(Y,W) + 2\eta(Y)\eta(Z)\eta(W)]\phi X \\ +\ & [\eta(X)g(W,Z) + \eta(Z)g(X,W) + 2\eta(X)\eta(Z)\eta(W)]\phi Y \\ +\ & g(X,\phi Z)[g(W,Y)\xi + \eta(Y)W + 2\eta(W)\eta(Y)\xi] \\ -\ & g(Y,\phi Z)[g(W,X)\xi + \eta(X)W + 2\eta(W)\eta(X)\xi] \\ +\ & [g(W,\phi X)g(Y,Z) - g(W,\phi Y)g(X,Z)]\xi \\ +\ & [\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\phi W + [g(W,\phi Y)X - g(W,\phi X)Y]\eta(Z) \\ +\ & g(W,\phi Z)[\eta(Y)X - \eta(X)Y] - \frac{\nabla_W r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{37}$$

Taking account of (19), we rewrite (37) as

$$\begin{aligned}&(\nabla_W\tilde{\tilde{C}})(X,Y)Z \\ =\ & (\nabla_W\bar{C})(X,Y)Z - [\eta(Y)g(W,Z) + \eta(Z)g(Y,W) + 2\eta(Y)\eta(Z)\eta(W)]\phi X\end{aligned}$$

$$
\begin{aligned}
&+ \quad [\eta(X)g(W,Z) + \eta(Z)g(X,W) + 2\eta(X)\eta(Z)\eta(W)]\phi Y \\
&+ \quad g(X,\phi Z)[g(W,Y)\xi + \eta(Y)W + 2\eta(W)\eta(Y)\xi] \\
&- \quad g(Y,\phi Z)[g(W,X)\xi + \eta(X)W + 2\eta(W)\eta(X)\xi] \\
&+ \quad [g(W,\phi X)g(Y,Z) - g(W,\phi Y)g(X,Z)]\xi \\
&+ \quad [\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\phi W \\
&+ \quad [g(W,\phi Y)X - g(W,\phi X)Y]\eta(Z) \\
&+ \quad g(W,\phi Z)[\eta(Y)X - \eta(X)Y].
\end{aligned} \tag{38}
$$

Using (2), (8) and (38) in (35), we get

$$
(\tilde{\nabla}_W \tilde{\bar{C}})(X,Y)Z = (\nabla_W \bar{C})(X,Y)Z. \tag{39}
$$

Hence we can state the following:

Theorem 2. *An LP-Sasakian manifold is concircular symmetric with respect to* $\tilde{\nabla}$ *if and only if it is so with respect to Riemannian connection* ∇.

Corollary 2. *An LP-Sasakian manifold is concircular* ϕ*-symmetric with respect to* $\tilde{\nabla}$ *if and only if it is so with respect to Riemannian connection* ∇.

Now taking (2), (8) and (37) in (35), we get

$$
(\tilde{\nabla}_W \tilde{\bar{C}})(X,Y)Z \quad = \quad (\nabla_W R)(X,Y)Z - \frac{\nabla_W r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{40}
$$

If scalar curvature r is constant then (40) reduces to

$$
(\tilde{\nabla}_W \tilde{\bar{C}})(X,Y)Z \quad = \quad (\nabla_W R)(X,Y)Z. \tag{41}
$$

Hence we can state the following:

Theorem 3. *An LP-Sasakian manifold is concircular symmetric with respect to quarter-symmetric metric connection* $\tilde{\nabla}$ *if and only if it is symmetric with respect to Riemannian connection* ∇, *provided* r *is constant.*

Corollary 3. *An LP-Sasakian manifold is concircular* ϕ*-symmetric with respect to quarter-symmetric metric connection* $\tilde{\nabla}$ *if and only if it is symmetric with respect to Riemannian connection* ∇, *provided* r *is constant.*

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