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On the Properties of the Resolvent of Two-Dimensional Magnetic Schrödinger Operator

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Abstract. This paper studies the basic equation of the theory of perturbations for two-dimensional magnetic Schrödinger operator in $C(\mathbf{R}_2)$ under some conditions on magnetic and electrical potentials. The applicability of Fredholm theory to this equation is established.

Key Words and Phrases: Magnetic Schrödinger operator; magnetic potential; electrical potential; resolvent equation.

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1. Introduction

Magnetic Schrödinger operators arise in many branches of physics, such as Ginzburg-Landau theory of superconductivity, Bose-Einstein condensation theory and state studies in quantum mechanics. Over the past 35 years, Schrödinger operators with magnetic fields evolved into a special mathematical subject (see, e.g., [4-6], [8-16] and [21]).

Consider a magnetic Schrödinger differential expression

$$H_{a,V} = \sum_{k=1}^{2} \left(\frac{1}{i} \frac{\partial}{\partial x_k} + a_k(x) \right)^2 + V(x) \tag{1}$$

on two-dimensional space $\mathbf{R}_2$, where $i = \sqrt{-1}$ is an imaginary unit, $x = (x_1, x_2) \in \mathbf{R}_2$, $a(x) = (a_1(x), a_2(x))$ and $V(x)$ are magnetic and electrical potentials, respectively. These potentials are both real functions. Note that if the magnetic field is perpendicular to the plane $x_1 O x_2$ and holds a three-dimensional charged particle in this plane, then, leaving aside the free movement of the particle along the axis x_3, we obtain the Hamiltonian of the form $H_{a,V}$ on the state space (see [7] or [4]).

Before stating the purpose of this work, let's recall some singularities of two-dimensional Schrödinger operators. It is well-known that the two-dimensional Schrödinger operators (with no magnetic potential) have some singularities that make them very difficult to study. First, the fundamental solution of the free Hamiltonian has a logarithmic singularity. Second, the classical Hardy inequality does not hold. Third, the Cwikel-Lieb-Rosenblum inequality does not hold. And finally, Sobolev's imbedding theorem for a limiting exponent equal to infinity does not hold, i.e. the space $W_2^1(\mathbf{R}_2)$ (first-order Sobolev space) is not continuously imbedded in $L_\infty(\mathbf{R}_2)$. There is an example in [2, p.118] which shows that a function from $W_2^1(\mathbf{R}_2)$, in general, is not necessarily essentially bounded in $\mathbf{R}_2$.

Denote by $C(\mathbf{R}_2)$ a Banach space of all bounded continuous functions on $\mathbf{R}_2$ equipped with the norm

$$\|f\|_{C(\mathbf{R}_2)} = \sup_{x\in\mathbf{R}_2} |f(x)| < +\infty.$$

In this work, we study the basic equation of the theory of perturbations for two-dimensional magnetic Schrödinger operator in the space $C(\mathbf{R}_2)$ generated by the differential expression (1) where the real magnetic and electrical potentials $a(x)$ and $V(x)$ satisfy the following conditions:

1) $\int_{\mathbf{R}_2} |a(x)|^\nu \, dx < +\infty$,

with $\nu > 2$, $|a(x)| = \sqrt{a_1^2(x_1,x_2) + a_2^2(x_1,x_2)}$;

2) $\int_{\mathbf{R}_2} |\Phi(x)|^\mu \, dx < +\infty$,

with $\mu > 1$, $\Phi(x) \equiv \Phi(x_1,x_2) = a^2(x_1,x_2)+V(x_1,x_2)+idiva(x_1,x_2)$, $a^2(x) \equiv a^2(x_1,x_2) = a_1^2(x_1,x_2) + a_2^2(x_1,x_2)$, $diva(x_1,x_2) = \frac{\partial a_1(x_1,x_2)}{\partial x_1} + \frac{\partial a_2(x_1,x_2)}{\partial x_2}$.

Note that the similar problems have been considered in [3] for one-dimensional case and in [17], [18] for three-dimensional case.

2. Properties of the fundamental solution of free Hamiltonian

It is known (see, e.g., [24, p.204]) that the generalized function

$$G_0(x,y,\lambda) = \frac{i}{4} H_0^{(1)}(\lambda\,|x-y|)$$

is a fundamental solution of the operator $-\Delta - \lambda^2$, i.e.

$$(-\Delta - \lambda^2) G_0(x,y,\lambda) = \delta(x-y),$$

where Δ is the two-dimensional Laplace operator, $\delta(x-y)$ is the Dirac δ-function, $\lambda \neq 0$ is a complex number, $H_0^{(1)}(\lambda\,|x-y|)$ is the Hankel function of the first kind. It is known (see, e.g., [19] or [25]) that the analytic function $H_0^{(1)}(z)$ is multi-valued in the domain $\Omega = \{z : 0 < |z| < +\infty\}$, and the points 0 and ∞ are the branch points of this function. There are many ways to "cut" the function $H_0^{(1)}(z)$ into regular branches. It depends on the problem we solve. As we consider the basic equation of the theory of perturbations, it is reasonable to cut the complex plane along the half-axis $[0,+\infty)$ and choose the branch of the Hankel function of the first kind which is expressed through the Poisson integral on the upper side of the cut line (see [19, p.169]), i.e.

$$\forall x > 0,\ \ H_0^{(1)}(x) = \frac{1}{\pi}\sqrt{\frac{2}{x}} e^{i\left(x-\frac{\pi}{4}\right)} \int_0^{+\infty} e^{-t} t^{-\frac{1}{2}} \left(1 + \frac{it}{2x}\right)^{-\frac{1}{2}} dt.$$

Denote by $\mathbf{C}_+ = \{\lambda \in \mathbf{C} : Im\lambda > 0\}$ the upper half plane ($\mathbf{C}$ is the complex plane).

Let H_0 be the operator $-\Delta$ (it is called a free Hamiltonian) on $L_2(\mathbf{R}_2)$ with a domain of definition $D(H_0) = W_2^2(\mathbf{R}_2)$ (second-order Sobolev space). As the spectrum of the self-adjoint operator H_0 coincides with the positive half-axis $[0, +\infty)$, the operator $H_0 - \lambda^2$ is a bijection from $W_2^2(\mathbf{R}_2)$ to $L_2(\mathbf{R}_2)$ with a bounded inverse $R_0(\lambda^2) = \left(H_0 - \lambda^2\right)^{-1}$ for every complex number from $\mathbf{C}_+$. Operator $R_0(\lambda^2)$ is an integral operator (see, e.g., [20, p.73, example 2]) with the kernel

$$G_0(x, y, \lambda) = \frac{i}{4} H_0^{(1)}(\lambda |x - y|),$$

i.e. for every $f(x) \in L_2(R_2)$

$$R_0(\lambda^2) f(x) = \int_{\mathbf{R}_2} G_0(x, y, \lambda) f(y) dy.$$

In the sequel, we will need the following proposition which includes some useful information about the properties of function $G_0(x, y, \lambda)$.

Proposition. *Function $G_0(x, y, \lambda)$ has the following properties:*

1^0. $\frac{\partial}{\partial x_j} G_0(x, y, \lambda) = -\frac{\partial}{\partial y_j} G_0(x, y, \lambda)$, $j = 1, 2$;

2^0. *if $x \neq y$ $(x, y \in \mathbf{R}_2)$, then the following asymptotic formula holds as $\lambda |x - y| \to 0$:*

$$G_0(x, y, \lambda) \sim -\frac{1}{2\pi} \ln(\lambda |x - y|);$$

3^0. *if $x \neq y$ $(x, y \in \mathbf{R}_2)$ and $0 \leq \arg\lambda \leq \pi$, then the following asymptotic formula holds as $\lambda |x - y| \to 0$:*

$$\frac{\partial}{\partial x_j} G_0(x, y, \lambda) \sim -\frac{1}{2\pi} \frac{x_j - y_j}{|x - y|^2}, \quad j = 1, 2;$$

4^0. *if $x \neq y$ $(x, y \in \mathbf{R}_2)$ and $0 \leq \arg\lambda < \pi$, then the following asymptotic formula holds as $|\lambda|\,|x - y| \to +\infty$:*

$$G_0(x, y, \lambda) = \frac{i}{4} \sqrt{\frac{1}{\pi\lambda |x - y|}} e^{i\left(\lambda|x-y| - \frac{\pi}{4}\right)} \left[1 + O\left(\frac{1}{\lambda |x - y|}\right)\right];$$

5^0. *if $x \neq y$ $(x, y \in \mathbf{R}_2)$, $Im\lambda_0 > 0$, $Im\lambda > 0$ and $|\lambda - \lambda_0| < |\lambda_0|$, then the following decomposition holds:*

$$G_0(x, y, \lambda) = G_0(x, y, \lambda_0) J_0\left((\lambda - \lambda_0)|x - y|\right) +$$

$$\frac{i}{2} \sum_{k=1}^{\infty} (-1)^k H_k^{(1)}(\lambda_0 |x - y|) J_k\left((\lambda - \lambda_0)|x - y|\right),$$

where J_k is the Bessel function and $H_k^{(1)}$ is the Hankel function of the first kind;

6^0. *if $x \neq y$ $(x, y \in \mathbf{R}_2)$ and $0 \leq \arg\lambda^2 \leq \pi$, then there is a number $A > 0$ such that*

$$|G_0(x,y,\lambda)| \le \frac{A}{\sqrt{|\lambda|\,|x-y|}};$$

7^0. *if* $Im\lambda > 0$, *then*

$$\int_{\mathbf{R}_2} |G_0(x,0,\lambda)|^2\,dx < +\infty;$$

8^0. *if* $Im\lambda > 0$, *then*

$$\int_{\mathbf{R}_2} |G_0(x,0,\lambda)|\,dx < +\infty.$$

The properties of $G_0(x,y,\lambda)$ listed in Proposition are in fact the properties of the Hankel function.

Property 1^0 follows directly from the equality

$$\frac{\partial}{\partial x_j}|x-y| = -\frac{\partial}{\partial y_j}|x-y|,$$

where $|x-y| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$. Property 2^0 follows from the formula

$$H_0^{(1)}(z) \sim i\frac{2}{\pi}\ln z \quad (z\to 0)$$

(see [1, p.180, formula 9.1.8]). Property 3^0 follows from the equalities

$$\frac{\partial}{\partial x_j}G_0(x,y,\lambda) = \frac{i}{4}\frac{d}{d(\lambda\,|x-y|)}H_0^{(1)}(\lambda\,|x-y|)\,\frac{\lambda(x_j-y_j)}{|x-y|}, \quad j=1,2,$$

$$\frac{d}{dz}H_0^{(1)}(z) = -H_1^{(1)}(z)$$

(see [25, p.89, Section 3.6, formula (7)]) and the asymptotic formula

$$H_1^{(1)}(z) \sim -\frac{2i}{\pi}\frac{1}{z} \quad (z\to 0,\ 0\le \arg z\le \pi)$$

(see [1, p.180, formula 9.1.9]). Property 4^0 is a direct consequence of the asymptotic formula

$$H_0^{(1)}(z) = \sqrt{\frac{2}{\pi z}}e^{i(z-\frac{\pi}{4})}\left[1+O\left(\frac{1}{z}\right)\right] \quad (|z|\to+\infty,\ 0\le\arg z<\pi)$$

(see [19, p.173, formula (13)]). Property 5^0 is established using Neumann-Sonin addition formula (see [25, p.158, Section 5.3, formula (2)])

$$H_0^{(1)}(z+t) = \sum_{m=-\infty}^{+\infty} H_{-m}^{(1)}(t)J_m(z) \quad (|z|<|t|)$$

and Bessel formulas

$$J_{-k}(z) = (-1)^k J_k(z)$$

(see [1, p.180, formula 9.1.5]),

$$H_{-k}^{(1)}(t) = (-1)^k H_k^{(1)}(t)$$

(see [1, p.180, formula 9.1.6]). Property 6^0 was established in [22, p.192]. Last two properties 7^0 and 8^0 are the corollaries of properties 4^0 and 6^0.

3. Investigation of operator function $K(\lambda)$ in the half plane $\mathbf{C}_+$

Denote by $L\left(C(\mathbf{R}_2)\right)$ a Banach space of linear continuous operators from $C(\mathbf{R}_2)$ to $C(\mathbf{R}_2)$.

To justify the application of Fredholm theory to the resolvent equation of two-dimensional magnetic Schrödinger operator, we consider integral operator

$$K(\lambda)f(x) = \int_{\mathbf{R}_2} K(x,y,\lambda)f(y)dy$$

depending on parameter λ with the kernel

$$K(x,y,\lambda) = G_0(x,y,\lambda)\Phi(y) - 2i\frac{\partial G_0(x,y,\lambda)}{\partial x_1}a_1(y) - 2i\frac{\partial G_0(x,y,\lambda)}{\partial x_2}a_2(y).$$

In the sequel, we will need the following

Lemma. *Let the conditions* 1) *and* 2) *hold. Then the following equalities are true:*

$$\lim_{0<\delta\to 0}\left\{\sup_{x\in\mathbf{R}_2}\int_{|x-y\leq\delta|}\frac{|a(y)|}{|x-y|}dy\right\} = 0, \tag{2}$$

$$\lim_{0<\delta\to 0}\left\{\sup_{x\in\mathbf{R}_2}\int_{|x-y\leq\delta|}\ln\frac{1}{|x-y|}\left|\Phi(y)\right|dy\right\} = 0. \tag{3}$$

Proof. Let's apply Hölder's inequality

$$\int_{|x-y|\leq\delta}\frac{|a(y)|}{|x-y|}dy \leq \left\{\int_{|x-y|\leq\delta}|a(y)|^{\nu}\,dy\right\}^{\frac{1}{\nu}}\left\{\int_{|x-y|\leq\delta}\frac{1}{|x-y|^p}dy\right\}^{\frac{1}{p}} \tag{4}$$

with $\frac{1}{\nu}+\frac{1}{p}=1$ to the integral

$$\int_{|x-y|\leq\delta}\frac{|a(y)|}{|x-y|}dy.$$

From $\nu>2$ it follows that $p=\frac{\nu}{\nu-1}<2$. As the integral

$$\int_{|x-y|\leq\delta}\frac{1}{|x-y|^p}dy$$

converges uniformly with respect to $x \in \mathbf{R}_2$ for $p < 2$, the absolute continuity of the Lebesgue integral and the inequality (4) immediately imply (2). Similarly, using Hölder's inequality we obtain

$$\left|\int_{|x-y|\leq\delta}\ln\frac{1}{|x-y|}\,|\Phi(y)|\,dy\right| \leq \left\{\int_{|x-y|\leq\delta}|\Phi(y)|^\mu\,dy\right\}^{\frac{1}{\mu}}\left\{\int_{|x-y|\leq\delta}|\ln|x-y||^p\,dy\right\}^{\frac{1}{p}}, \quad (5)$$

where $\frac{1}{\mu}+\frac{1}{p}=1$. As for every positive number ε it holds that

$$\lim_{\rho\to 0}\rho^\varepsilon\ln\rho = 0,$$

from inequality (5) we get the validity of equality (3). Lemma is proved.◀

The following theorem is true.

Theorem 1. *If the conditions* 1)*and* 2) *hold, then* $K(\lambda) \in L(C(\mathbf{R}_2))$ *for every* $\lambda \in \mathbf{C}_+$.

Proof. According to properties 2^0 and 4^0 (see Proposition in Section 2) and the relation (3), there exist positive constants M_1, M_2, r and δ_0 such that

$$\sup_{x\in\mathbf{R}_2}\left\{\int_{|x-y|\leq\delta_0}|G_0(x,y,\lambda)\Phi(y)|\,dy\right\} \leq M_1, \quad (6)$$

$$|G_0(x,y,\lambda)| \leq M_2 e^{-Im\lambda|x-y|} \quad (7)$$

with $|x-y| > r$.

Represent the integral

$$K_1(x,\lambda) = \int_{\mathbf{R}_2}G_0(x,y,\lambda)\Phi(y)dy$$

in the following form:

$$K_1(x,\lambda) = \int_{|x-y|<\delta_0}G_0(x,y,\lambda)\Phi(y)dy + \int_{\delta_0\leq|x-y|\leq r}G_0(x,y,\lambda)\Phi(y)dy+$$

$$\int_{|x-y|>r}G_0(x,y,\lambda)\Phi(y)dy \equiv K_1^{(1)}(x,\lambda)+K_1^{(2)}(x,\lambda)+K_1^{(3)}(x,\lambda). \quad (8)$$

Using the inequalities (6), (7), the condition 2) and the boundedness of $G_0(x,y,\lambda)$ for $\delta_0 \leq |x-y| \leq r$, we obtain:

$$\sup_{x\in\mathbf{R}_2}\left|K_1^{(i)}(x,\lambda)\right| < +\infty, \quad i=1,2,3. \quad (9)$$

Now represent the integral

$$K_2(x,\lambda)=\int_{\mathbf{R}_2}\left(\frac{\partial G_0(x,y,\lambda)}{\partial x_1}a_1(y)+\frac{\partial G_0(x,y,\lambda)}{\partial x_2}a_2(y)\right)dy\equiv$$

$$\int_{\mathbf{R}_2}(a(y)\cdot\nabla G_0(x,y,\lambda))\,dy$$

in the form

$$K_2(x,\lambda)=\int_{|x-y|<\delta_0}(a(y)\cdot\nabla G_0(x,y,\lambda))\,dy+\int_{\delta_0\le|x-y|\le r}(a(y)\cdot\nabla G_0(x,y,\lambda))\,dy+$$

$$\int_{|x-y|>r}(a(y)\cdot\nabla G_0(x,y,\lambda))\,dy\equiv K_2^{(1)}(x,\lambda)+K_2^{(2)}(x,\lambda)+K_2^{(3)}(x,\lambda), \quad (10)$$

where $\nabla=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}\right)$ is a symbolic Hamilton vector. Using property 3^0 (see Proposition in Section 2), relation (10), condition 1) and inequality

$$|< x-y,a(y) >|\equiv|(x_1-y_1)a_1(y)+(x_2-y_2)a_2(y)|\le|x-y|\,|a(y)|\,,$$

and proceeding in the same way as we did when estimating the integrals $K_1^{(i)}(x,\lambda)$ ($i=1,2,3$), we have

$$\sup_{x\in\mathbf{R}_2}\left|K_2^{(i)}(x,\lambda)\right|<+\infty,\ \ i=1,2,3. \quad (11)$$

Let $u(x)\in C(\mathbf{R}_2)$ and

$$h(x)=\int_{\mathbf{R}_2}K(x,y,\lambda)u(y)dy.$$

From representations (8) and (10), by virtue of inequalities (9) and (11), we obtain

$$\sup_{x\in\mathbf{R}_2}|h(x)|\le\widehat{k}\,\|u\|_{C(\mathbf{R}_2)}\,,$$

where

$$\widehat{k}=\sum_{i=1}^{2}\sum_{j=1}^{3}\sup_{x\in\mathbf{R}_2}\left|K_i^{(j)}(x,\lambda)\right|<+\infty.$$

From relations (2), (3) and inequality (7), due to the continuity of the Hankel function of the first kind $H_0^{(1)}(z)$ in domain $\Omega=\{z\in\mathbf{C}:z\notin[0,+\infty)\}$, we obtain $h(x)\in C(\mathbf{R}_2)$. Theorem is proved.◄

The following theorem is true.

Theorem 2. *If the conditions* 1) *and* 2) *hold, then the operator-valued function* $K(\lambda)$ *is analytic with respect to* λ *in the half plane* $\mathbf{C}_+$ *in the uniform operator topology.*

Proof. Choose the positive number r great enough and the positive number δ_0 small enough to satisfy, in addition to (7), the inequalities

$$\left|\frac{\partial}{\partial x_j} G_0(x,y,\lambda)\right| \le M_2 e^{-Im\lambda|x-y|}, \;\; j=1,2, \tag{12}$$

for $|x-y|>r$ and the asymptotic formulas

$$J_k(z) \sim \frac{z^k}{k!2^k} \;\; (0<\arg z<\pi, \;\; |z|<\delta_0), \;\; k=0,1,2,..., \tag{13}$$

$$H_k^{(1)}(z) \sim -\frac{i}{\pi}\frac{(k-1)!2^k}{z^k} \;\; (0<\arg z<\pi, \;\; |z|<\delta_0), \;\; k=1,2,..., \tag{14}$$

(see [1, p.180, formulas 9.1.7 and 9.1.9]).

Let $\lambda_0 \in \mathbf{C}_+$. Represent the operator $K(\lambda)$ in the form

$$K(\lambda) = K_{1,\lambda_0}(\lambda) + K_{1,\lambda_0,r}(\lambda) + K_{1,r}(\lambda),$$

where

$$K_{1,\lambda_0}(\lambda)u(x) = \int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} K(x,y,\lambda)u(y)dy,$$

$$K_{1,\lambda_0,r}(\lambda)u(x) = \int_{\frac{\delta_0}{|\lambda_0|}\le|x-y|\le r} K(x,y,\lambda)u(y)dy,$$

$$K_{1,r}(\lambda)u(x) = \int_{|x-y|>r} K(x,y,\lambda)u(y)dy$$

for $u(x) \in C(\mathbf{R}_2)$. From inequalities (7) and (12), by virtue of conditions 1) and 2), it follows that $K_{1,\lambda_0,r}(\lambda)$ and $K_{1,r}(\lambda)$ are analytic functions in $\mathbf{C}_+$ in the uniform operator topology. Let's prove that the function $K_{1,\lambda_0}(\lambda)$ is analytic in δ-neighborhood $U_\delta(\lambda_0) = \{\lambda \in \mathbf{C}_+ : |\lambda-\lambda_0|<\delta\}$ of the point λ_0, where $0<\delta<\min\{\delta_0, Im\lambda_0\}$. With this aim, represent the operator $K_{1,\lambda_0}(\lambda)$ in the form

$$K_{1,\lambda_0}(\lambda) = K_{1,\lambda_0}^{(1)}(\lambda) - 2iK_{1,\lambda_0}^{(2)}(\lambda),$$

where

$$K_{1,\lambda_0}^{(1)}(\lambda)u(x) = \int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} G_0(x,y,\lambda)\Phi(y)u(y)dy,$$

$$K_{1,\lambda_0}^{(2)}(\lambda)u(x) = \int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} \left(\nabla G_0(x,y,\lambda)\cdot a(y)\right)u(y)dy$$

for $u(x) \in C(\mathbf{R}_2)$. Using property 5^0 (see Proposition in Section 2), rewrite the integral $K_{1,\lambda_0}^{(1)}(\lambda)u(x)$ as follows:

$$K^{(1)}_{1,\lambda_0}(\lambda)u(x) = \int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} G_0(x,y,\lambda)\Phi(y)u(y)dy =$$

$$\int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} \Big\{ G_0(x,y,\lambda_0)J_0\left((\lambda-\lambda_0)|x-y|\right) + \frac{i}{2}\times$$

$$\sum_{k=1}^{\infty}(-1)^k H^{(1)}_k(\lambda_0|x-y|)J_k\left((\lambda-\lambda_0)|x-y|\right)\Big\}\Phi(y)u(y)dy.$$

From the asymptotic formulas (13) and (14) we have

$$H^{(1)}_k(\lambda_0|x-y|)J_k\left((\lambda-\lambda_0)|x-y|\right) \sim -\frac{i}{\pi k}\left(\frac{\lambda-\lambda_0}{\lambda_0}\right)^k, \quad k=1,2,.... \tag{15}$$

Taking into account the inequalities

$$\left|\frac{1}{k}\left(\frac{\lambda-\lambda_0}{\lambda_0}\right)^k\right| < \left(\frac{|\lambda-\lambda_0|}{\delta}\right)^k, \quad k=1,2,...,$$

$$\left|\int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} \Phi(y)u(y)dy\right| \le \|u(x)\|_{C(\mathbf{R}_2)}\int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} |\Phi(y)|\,dy \le$$

$$\|u(x)\|_{C(\mathbf{R}_2)}\int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} |\Phi(y)|^{\mu}\,dy \le \|u(x)\|_{C(\mathbf{R}_2)}\int_{\mathbf{R}_2} |\Phi(y)|^{\mu}\,dy,$$

where μ is a number appearing in condition 2), and the convergence of numerical series $\sum_{k=1}^{\infty}\left(\frac{|\lambda-\lambda_0|}{\delta}\right)^k$, we conclude from (15) that the series of analytic operator-valued functions $\sum_{k=1}^{\infty}K^{(1)}_{1,\lambda_0,k}(\lambda)$ converges uniformly on $U_\delta(\lambda_0)$, where

$$K^{(1)}_{1,\lambda_0,k}(\lambda)u(x) =$$

$$\int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} H^{(1)}_k(\lambda_0|x-y|)J_k\left((\lambda-\lambda_0)|x-y|\right)\Phi(y)u(y)dy, \quad k=1,2,....$$

The analyticity of $J_0\left((\lambda-\lambda_0)|x-y|\right)$ on $U_\delta(\lambda_0)$ and the asymptotics

$$J_0\left((\lambda-\lambda_0)|x-y|\right) \sim 1 \;\; \left(|(\lambda-\lambda_0)|x-y|| < \delta_0\right)$$

imply by virtue of (3) that the operator

$$K^{(1)}_{1,\lambda_0,0}(\lambda)u(x) = \int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} G_0(x,y,\lambda_0)J_0\left((\lambda-\lambda_0)|x-y|\right)\Phi(y)u(y)dy$$

is analytic on $U_\delta(\lambda_0)$. So it follows from the Weierstrass theorem on uniformly convergent series of analytic functions that the operator-valued function $K^{(1)}_{1,\lambda_0}(\lambda) = \sum_{k=0}^{\infty} K^{(1)}_{1,\lambda_0,k}(\lambda)$ is analytic on the domain $U_\delta(\lambda_0)$. Now let's prove that the operator-valued function $K^{(2)}_{1,\lambda_0}(\lambda)$ is analytic on $U_\delta(\lambda_0)$. Taking into account the equalities

$$\frac{\partial}{\partial x_j} G_0(x,y,\lambda) = \frac{i}{4}\frac{d}{d(\lambda\,|x-y|)} H_0^{(1)}(\lambda\,|x-y|)\,\frac{\lambda(x_j-y_j)}{|x-y|},\ \ j=1,2,$$

$$\frac{d}{dz}H_0^{(1)}(z) = -H_1^{(1)}(z)$$

(see [25, p.89, Section 3.6, formula (7)]), we obtain that for every $u(x) \in C(\mathbf{R}_2)$ it holds that

$$K^{(2)}_{1,\lambda_0}(\lambda)u(x) = -\frac{i\lambda}{4}\int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} H_1^{(1)}(\lambda\,|x-y|)\,\frac{< x-y, a(y) >}{|x-y|}u(y)dy. \tag{16}$$

Using the identities

$$J_{-k}(z) = (-1)^k J_k(z),\ \ k=1,2,...$$

(see [1, p.180, formula 9.1.5]),

$$H^{(1)}_{1-k}(z) = H^{(1)}_{-(k-1)}(z) = e^{(k-1)\pi i}H^{(1)}_{k-1}(z),\ \ k=1,2,...$$

(see [1, p.180, formula 9.1.6]),

$$H^{(1)}_{k+1}(z) - H^{(1)}_{k-1}(z) = -2\frac{d}{dz}H^{(1)}_k(z),\ \ k=1,2,...$$

(see [1, p.182, formula 9.1.27]) and Neumann expansion (see [1, p.184, formula 9.1.75] for $|\lambda-\lambda_0| < |\lambda_0|$), we rewrite $H_1^{(1)}(\lambda\,|x-y|)$ as follows:

$$H_1^{(1)}(\lambda\,|x-y|) = H_1^{(1)}(\lambda_0\,|x-y| + (\lambda-\lambda_0)\,|x-y|) =$$

$$\sum_{k=-\infty}^{+\infty} H^{(1)}_{1-k}(\lambda_0\,|x-y|)J_k\left((\lambda-\lambda_0)\,|x-y|\right) =$$

$$H_1^{(1)}(\lambda_0\,|x-y|)J_0\left((\lambda-\lambda_0)\,|x-y|\right) + \sum_{k=1}^{+\infty} H^{(1)}_{1-k}(\lambda_0\,|x-y|)J_k\left((\lambda-\lambda_0)\,|x-y|\right) +$$

$$\sum_{k=1}^{+\infty} H^{(1)}_{k+1}(\lambda_0\,|x-y|)J_{-k}\left((\lambda-\lambda_0)\,|x-y|\right) = H_1^{(1)}(\lambda_0\,|x-y|)J_0\left((\lambda-\lambda_0)\,|x-y|\right) +$$

$$\sum_{k=1}^{+\infty} e^{(k-1)\pi i} H_{k-1}^{(1)}(\lambda_0 |x-y|) J_k ((\lambda-\lambda_0)|x-y|) +$$

$$\sum_{k=1}^{+\infty} H_{k+1}^{(1)}(\lambda_0 |x-y|)(-1)^k J_k ((\lambda-\lambda_0)|x-y|) =$$

$$H_1^{(1)}(\lambda_0 |x-y|) J_0 ((\lambda-\lambda_0)|x-y|) +$$

$$\sum_{k=1}^{+\infty} (-1)^k \left[H_{k+1}^{(1)}(\lambda_0 |x-y|) - H_{k-1}^{(1)}(\lambda_0 |x-y|) \right] J_k ((\lambda-\lambda_0)|x-y|) =$$

$$H_1^{(1)}(\lambda_0 |x-y|) J_0 ((\lambda-\lambda_0)|x-y|) +$$

$$2 \sum_{k=1}^{+\infty} (-1)^{k+1} \frac{dH_k^{(1)}(\lambda_0 |x-y|)}{d(\lambda_0 |x-y|)} J_k ((\lambda-\lambda_0)|x-y|). \tag{17}$$

Substituting the expansion (17) into the integral (16), we have

$$K_{1,\lambda_0}^{(2)}(\lambda) u(x) =$$

$$-\frac{i\lambda}{4} \int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} H_1^{(1)}(\lambda_0 |x-y|) J_0 ((\lambda-\lambda_0)|x-y|) \frac{< x-y, a(y) >}{|x-y|} u(y) dy +$$

$$\frac{i\lambda}{2} \int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} \left\{ \sum_{k=1}^{+\infty} (-1)^k \frac{dH_k^{(1)}(\lambda_0 |x-y|)}{d(\lambda_0 |x-y|)} J_k ((\lambda-\lambda_0)|x-y|) \right\} \frac{< x-y, a(y) >}{|x-y|} u(y) dy.$$

Taking into account the asymptotic formulas (13) and

$$\frac{d}{dz} H_k^{(1)}(z) \sim \frac{i}{\pi} \frac{k! 2^k}{z^{k+1}} \ \ (0 < \arg z < \pi, \ \ |z| < \delta_0), \ \ k = 1, 2, ...,$$

we obtain

$$\frac{dH_k^{(1)}(\lambda_0 |x-y|)}{d(\lambda_0 |x-y|)} J_k ((\lambda-\lambda_0)|x-y|) \sim \frac{i}{\pi \lambda_0 |x-y|} \left(\frac{\lambda-\lambda_0}{\lambda_0} \right)^k, \ \ k = 1, 2, ... \tag{18}$$

Taking into account the inequalities

$$\left| \frac{i}{\pi \lambda_0 |x-y|} \left(\frac{\lambda-\lambda_0}{\lambda_0} \right)^k \right| < \frac{1}{\pi |\lambda_0| |x-y|} \left(\frac{|\lambda-\lambda_0|}{\delta} \right)^k, \ \ k = 1, 2, ...,$$

$$\frac{|< x-y, a(y) >|}{|x-y|} \frac{1}{\pi |\lambda_0| |x-y|} \leq \frac{1}{\pi |\lambda_0|} \frac{|a(y)|}{|x-y|},$$

$$\left| \int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} \frac{a(y)}{|x-y|} u(y) dy \right| \leq$$

$$\|u(x)\|_{C(\mathbf{R}_2)} \left\{ \int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} |a(y)|^\nu \, dy \right\}^{\frac{1}{\nu}} \left\{ \int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} \frac{1}{|x-y|^p} dy \right\}^{\frac{1}{p}} \leq$$

$$C_{\delta_0,\lambda_0,p} \|u(x)\|_{C(\mathbf{R}_2)} \left\{ \int_{\mathbf{R}_2} |a(y)|^\nu \, dy \right\}^{\frac{1}{\nu}},$$

where $C_{\delta_0,\lambda_0,p}$ does not depend on $u(x)$, $\frac{1}{\nu} + \frac{1}{p} = 1$, ν is a number appearing in condition 1), and the convergence of numerical series $\sum_{k=1}^{\infty} \left(\frac{|\lambda - \lambda_0|}{\delta} \right)^k$, we conclude from (18) that the series of analytic operator-valued functions $\sum_{k=1}^{\infty} K^{(2)}_{1,\lambda_0,k}(\lambda)$ converges uniformly on $U_\delta(\lambda_0)$, where

$$K^{(2)}_{1,\lambda_0,k}(\lambda) u(x) =$$

$$\int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} \frac{dH_k^{(1)}(\lambda_0 |x-y|)}{d(\lambda_0 |x-y|)} J_k \left((\lambda - \lambda_0) |x-y| \right) \frac{< x-y, a(y) >}{|x-y|} u(y) dy, \;\; k = 1, 2, ...$$

The analyticity of $J_0 \left((\lambda - \lambda_0) |x-y| \right)$ on $U_\delta(\lambda_0)$, the asymptotics

$$J_0 \left((\lambda - \lambda_0) |x-y| \right) \sim 1 \;\; (|(\lambda - \lambda_0) |x-y|| < \delta_0),$$

$$H_1^{(1)}(\lambda_0 |x-y|) \sim -\frac{i}{\pi} \frac{2}{\lambda_0 |x-y|} \;\; (|\lambda_0 |x-y|| < \delta_0)$$

and the inequality

$$\frac{|< x-y, a(y) >|}{|x-y|} \leq |a(y)|$$

imply by virtue of (2) that the operator

$$K^{(2)}_{1,\lambda_0,0}(\lambda) u(x) = \int_{|x-y|<\frac{\delta_0}{|\lambda_0|}} H_1^{(1)}(\lambda_0 |x-y|) J_0 \left((\lambda - \lambda_0) |x-y| \right) \frac{< x-y, a(y) >}{|x-y|} u(y) dy$$

is analytic on $U_\delta(\lambda_0)$. So it follows from the Weierstrass theorem on uniformly convergent series of analytic functions that the operator-valued function

$$K^{(2)}_{1,\lambda_0}(\lambda) = -\frac{i\lambda}{4} K^{(2)}_{1,\lambda_0,0}(\lambda) + \frac{i\lambda}{2} \sum_{k=1}^{+\infty} (-1)^k K^{(2)}_{1,\lambda_0,k}(\lambda)$$

is analytic on the domain $U_\delta(\lambda_0)$. As $\lambda_0 \in \mathbf{C}_+$ is arbitrary, the proof is complete.◂

Denote by $\sigma_\infty(C(\mathbf{R}_2))$ the set of completely continuous linear operators from $C(\mathbf{R}_2)$ to $C(\mathbf{R}_2)$.

Theorem 3. *If the conditions* 1) *and* 2) *hold, then* $K(\lambda) \in \sigma_\infty(C(\mathbf{R}_2))$ *for every* $\lambda \in \mathbf{C}_+$.

Proof. Let $\lambda \in \mathbf{C}_+$ and $S_1(0) = \left\{ u(x) \in C(\mathbf{R}_2) : \|u(x)\|_{C(\mathbf{R}_2)} \leq 1 \right\}$ be a unit ball in $C(\mathbf{R}_2)$. Let's prove that the set

$$M^1_\lambda = K(\lambda)\left[S_1(0)\right] = \left\{ h(x) \in C(\mathbf{R}_2) : h(x) = \int_{\mathbf{R}_2} K(x,y,\lambda) u(y) dy,\ u(x) \in S_1(0) \right\}$$

is compact in $C(\mathbf{R}_2)$. To prove its compactness, by the corollary of the Hausdorff theorem (see [23, p.205, Corollary 1]) it suffices to show that for every $\varepsilon > 0$ there exists a compact ε-net for M^1_λ. With given $\varepsilon > 0$, by virtue of (2) and (3) we can find a positive number δ such that for every positive integer n and for every $u(x) \in S_1(0)$ the following estimates are true:

$$\sup_{x \in \mathbf{R}_2} \left| \int_{\{y:|y|>n\} \bigcap \{y:|x-y|\leq\delta\}} G_0(x,y,\lambda)\Phi(y)u(y)dy \right| < \frac{\varepsilon}{4}, \tag{19}$$

$$\sup_{x \in \mathbf{R}_2} \left| \int_{\{y:|y|>n\} \bigcap \{y:|x-y|\leq\delta\}} \left(\nabla G_0(x,y,\lambda) \cdot a(y)\right) u(y)dy \right| < \frac{\varepsilon}{4}. \tag{20}$$

After having chosen δ that way, using conditions 1), 2) and property 4^0 (see Proposition in Section 2), we now choose a positive integer $n_0(\varepsilon)$ great enough to satisfy the inequalities

$$\sup_{x \in \mathbf{R}_2} \left| \int_{\{y:|y|>n_0(\varepsilon)\} \bigcap \{y:|x-y|>\delta\}} G_0(x,y,\lambda)\Phi(y)u(y)dy \right| < \frac{\varepsilon}{4}, \tag{21}$$

$$\sup_{x \in \mathbf{R}_2} \left| \int_{\{y:|y|>n_0(\varepsilon)\} \bigcap \{y:|x-y|>\delta\}} \left(\nabla G_0(x,y,\lambda) \cdot a(y)\right) u(y)dy \right| < \frac{\varepsilon}{4} \tag{22}$$

for $u(x) \in S_1(0)$. From the inequalities (19)-(22) it follows that for every $u(x) \in S_1(0)$

$$\sup_{x \in \mathbf{R}_2} \left| \int_{|y|>n_0(\varepsilon)} K(x,y,\lambda)u(y)dy \right| < \varepsilon. \tag{23}$$

From (23) it follows that the set

$$M^1_{\lambda,\varepsilon} = \left\{ h_\varepsilon(x) = \int_{|y|\leq n_0(\varepsilon)} K(x,y,\lambda)u(y)dy,\ u(x) \in S_1(0) \right\}$$

is an ε-net for M^1_λ. The relations (2) and (3) imply that the functions from $M^1_{\lambda,\varepsilon}$ are uniformly bounded and equicontinuous. Consequently, by Arzela theorem (see [23, p.207, Theorem 1]), $M^1_{\lambda,\varepsilon}$ is compact. Theorem is proved.◄

The obtained results allow using Fredholm theory in the study of spectral properties of two-dimensional magnetic Schrödinger operator. The author plans to consider these issues in one of his next works.

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