Azerbaijan Journal of Mathematics
V. 5, No 1, 2015, January
ISSN 2218-6816

Invariant and Anti-Invariant Submanifolds of a Conformal Kenmotsu Manifold

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Abstract. In this paper, we introduce conformal Kenmotsu manifolds which are not Kenmotsu. Some results on such manifolds and their associated submanifolds are provided.

Key Words and Phrases: Kenmotsu manifold, Conformal Kenmotsu manifold, Invariant submanifold, Anti-invriant submanifold.

2010 Mathematics Subject Classifications: 53C25, 53C40.

1. Introduction

Let (M^{2n}, J, g) be a Hermitian manifold of complex dimension n, where J denotes its complex structure and g is its Hermitian metric. Then (M^{2n}, J, g) is a locally conformal Käehler manifold if there is an open cover $\{U_i\}_{i\in I}$ of M^{2n} and a family $\{f_i\}_{i\in I}$ of C^∞ functions $f_i : U_i \longrightarrow \mathbb{R}$ such that each local metric $g_i = exp(-f_i)g_{|U_i}$ is Käehlerian. Here $g_{|U_i} = \imath_i^* g$ where $\imath_i : U_i \longrightarrow M^{2n}$ is the inclusion. Also (M^{2n}, J, g) is globally conformal Käehler if there is a C^∞ function $f : M^{2n} \longrightarrow \mathbb{R}$ such that the metric $exp(f)g$ is Käehlerian [5]. The first study on locally conformal Käehler manifolds was done by Libermann in 1955 [8]. Visman [10], put down some geometrical conditions for locally conformal Käehler manifold and in 1982 Tricerri mentioned different examples of locally conformal Käehler manifold [9]. In 2001, Banaru [2] succeeded to classify the sixteen classes of almost Hermitian manifolds by using the two tensors of Kirichenko, which are called Kirichenko tensors. Abood studied the properties of these tensors [1]. The locally conformal Käehler manifold is one of the sixteen classes of almost Hermitian manifolds. In 1972, K. Kenmotsu introduced a class of contact metric manifolds, called Kenmotsu manifolds, which are not Sasakian [7]. In this paper we get the idea of constructing conformal Käehler manifolds and introduce conformal Kenmotsu manifolds which are not Kenmotsu. There are a few differences between the geometry of invariant (anti-invariant) submanifolds of a Kenmotsu manifold and a conformal Kenmotsu manifold. For example, we show that any invariant submanifold $\acute{M}$ of a Kenmotsu manifold M is minimal, but

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if M is a conformal Kenmotsu manifold, then $\acute{M}$ is minimal if and only if the Lee vector field of M is tangent to $\acute{M}$. Moreover, it is proved in [6] that if $\acute{M}$ is an anti-invariant submanifold of a Kenmotsu manifold, then $\acute{M}$ has the flat normal curvature tensor if and only if $\acute{M}$ is flat, but in this paper we show that if $\acute{M}$ is an anti-invariant submanifold of a conformal Kenmotsu manifold with flat normal curvature tensor then $\acute{M}$ is not flat. The present paper is organized as follows:
In Section 2 we recall some definitions on almost contact metric manifolds. We introduce conformal Kenmotsu manifolds in Section 3 and establish a relation between the curvature tensor on a conformal Kenmotsu manifold and a Kenmotsu manifold. According to this relation, we give the Gauss and Ricci equations between a conformal Kenmotsu manifold and its submanifolds. In Sections 4 and 5, we study invariant submanifolds and anti-invariant submanifolds of a conformal Kenmotsu manifold and prove some theorems about the mean curvature vector field, the connection of the normal bundle and the curvature tensor of the submanifolds.

2. Preliminaries

A 2n+1-dimensional differentiable manifold M is said to be an almost contact metric manifold if it admits an almost contact metric structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ where $\tilde{\varphi}$ is a tensor field of type (1,1), $\tilde{\xi}$ is a vector field, $\tilde{\eta}$ is a 1-form and $\tilde{g}$ is the Riemannian metric on M satisfying

$$\tilde{\varphi}^2 = -Id + \tilde{\eta} \otimes \tilde{\xi}, \qquad \tilde{\eta}(\tilde{\xi}) = 1, \qquad \tilde{\varphi}\tilde{\xi} = 0, \qquad \tilde{\eta} o \tilde{\varphi} = 0,$$

$$\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = \tilde{g}(X, Y) - \tilde{\eta}(X)\tilde{\eta}(Y), \qquad \tilde{\eta}(X) = \tilde{g}(X, \tilde{\xi}),$$

for all vector fields X, Y on M.
An almost contact metric manifold $(M^{2n+1}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is said to be a Kenmotsu manifold [7] if the relation

$$(\tilde{\nabla}_X \tilde{\varphi})Y = -\tilde{g}(X, \tilde{\varphi}Y)\tilde{\xi} - \tilde{\eta}(Y)\tilde{\varphi}X, \tag{1}$$

holds on M, where $\tilde{\nabla}$ denotes the Riemannian connection of $\tilde{g}$. From the above equation, for a Kenmotsu manifold we also have

$$\tilde{\nabla}_X \tilde{\xi} = X - \tilde{\eta}(X)\tilde{\xi}. \tag{2}$$

Assume $\acute{M}$ is a submanifold of a Kenmotsu manifold M. Let $\acute{g}$ and $\acute{\nabla}$ be the induced metric and Riemannian connections of $\acute{M}$, respectively. Then the Gauss and Weingarten formulas of $\acute{M}$ are given, respectively, by

$$\tilde{\nabla}_X Y = \acute{\nabla}_X Y + h(X, Y), \qquad \tilde{\nabla}_X N = -A_N X + \acute{\nabla}^{\perp}_X N,$$

for all vector fields X, Y on $\acute{M}$, where $\acute{\nabla}^{\perp}$ is the normal connection and A is the shape operator of $\acute{M}$ with respect to the unit normal vector field N. Let $\acute{R}$ is the curvature tensor of $\acute{M}$, the Gauss and Ricci equations of $\acute{M}$ are given, respectively, by

$$\begin{aligned}\tilde{g}(\tilde{R}(X,Y)Z,W) &= \acute{g}(\acute{R}(X,Y)Z,W) - \tilde{g}(h(X,W),h(Y,Z)) \\ &+ \tilde{g}(h(X,W),h(Y,Z)), \\ \tilde{g}(\tilde{R}(X,Y)N_1,N_2) &= \tilde{g}(\acute{R}^{\perp}(X,Y)N_1,N_2) - \acute{g}([A_1,A_2]X,Y),\end{aligned}$$

for all $X,Y,Z,W \in T\acute{M}$ and $N_1,N_2 \in T\acute{M}^{\perp}$.
A submanifold $\acute{M}^m$ of a Kenmotsu manifold $(M^{2n+1},\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ is called an invariant submanifold if $\tilde{\varphi}T_p\acute{M} \subseteq T_p\acute{M}$ for any $p \in \acute{M}$.

Theorem 1. *Let $\acute{M}^m$ be an invariant submanifold of a Kenmotsu manifold M tangent to $\tilde{\xi}$. Then $\acute{M}$ is minimal.*

Proof. From the Gauss formula and (1) we have

$$h(X,\tilde{\varphi}Y) = \tilde{\varphi}h(X,Y) - (\acute{\nabla}_X\tilde{\varphi})Y - \tilde{g}(X,\tilde{\varphi}Y)\tilde{\xi} - \tilde{\eta}(Y)\tilde{\varphi}X,$$

for all X,Y on $\acute{M}$. Since $\acute{M}$ is invariant, taking the tangent and normal parts, we have

$$h(X,\tilde{\varphi}Y) = \tilde{\varphi}h(X,Y). \tag{3}$$

Let $\{E_\alpha,\tilde{\varphi}E_\alpha,\xi,\alpha:1,\cdots,\frac{m-1}{2}\}$ be an orthonormal basis on $\acute{M}$ and suppose H is the mean curvature vector field of $\acute{M}$. From the Gauss formula and (2) it follows that $h(\xi,\xi)=0$. Hence, using (3) we have

$$H = \frac{1}{m}\{\Sigma_{\alpha=1}^{\frac{m-1}{2}}(h(E_\alpha,E_\alpha) + h(\tilde{\varphi}E_\alpha,\tilde{\varphi}E_\alpha)) + h(\xi,\xi)\} = 0. \blacktriangleleft$$

3. Conformal Kenmotsu Manifolds

A smooth manifold M^{2n+1} with almost contact metric structure (φ,η,ξ,g) is called a conformal Kenmotsu manifold if there is a positive smooth function $f: M^{2n+1} \to \mathbb{R}$ so that

$$\tilde{g} = exp(f)g, \qquad \tilde{\xi} = exp(-f)^{\frac{1}{2}}\xi, \qquad \tilde{\eta} = exp(f)^{\frac{1}{2}}\eta, \qquad \tilde{\varphi} = \varphi,$$

is a Kenmotsu structure on M.
Let M is a conformal Kenmotsu manifold. Suppose $\tilde{\nabla}$ and ∇ denote the Riemannian connections M with respect to metrics $\tilde{g}$ and g, respectively. Using the Koszul formula, we obtain the following relation between the connections $\tilde{\nabla}$ and ∇

$$\tilde{\nabla}_XY = \nabla_XY + \frac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X,Y)\omega^\sharp\}, \tag{4}$$

such that $\omega(X) = X(f)$ and $\omega^\sharp = gradf$ is vector field metrically equivalent to 1-form ω, that is, $g(\omega^\sharp, X) = \omega(X)$, that is called the Lee vector field of a conformal Kenmotsu manifold M. Let $\tilde{R}$ and R denote the curvature tensor on $(M^{2n+1}, \tilde{\varphi}, \tilde{\eta}, \tilde{\xi}, \tilde{g})$ and $(M^{2n+1}, \varphi, \eta, \xi, g)$, respectively. Then the relation between $\tilde{R}$ and R is given by

$$\begin{aligned} \exp(-f)\tilde{g}(\tilde{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) \\ &+ \frac{1}{2}\{B(X,Z)g(Y,W) - B(Y,Z)g(X,W) \\ &+ B(Y,W)g(X,Z) - B(X,W)g(Y,Z)\} \\ &+ \frac{1}{4}\|\omega^\sharp\|^2\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\}, \end{aligned} \tag{5}$$

for all vector fields X, Y, Z, W on M, where $B := \nabla\omega - \frac{1}{2}\omega \otimes \omega$. Furthermore, by the relations (1), (2) and (4) we get

$$\begin{aligned} (\nabla_X\varphi)Y &= (\exp(f))^{\frac{1}{2}}\{-g(X,\varphi Y)\xi - \eta(Y)\varphi X\} \\ &- \frac{1}{2}\{\omega(\varphi Y)X - \omega(Y)\varphi X + g(X,Y)\varphi\omega^\sharp - g(X,\varphi Y)\omega^\sharp\}, \end{aligned} \tag{6}$$

$$\nabla_X\xi = (\exp(f))^{\frac{1}{2}}\{X - \eta(X)\xi\} - \frac{1}{2}\{\omega(\xi)X - \eta(X)\omega^\sharp\}, \tag{7}$$

for all vector fields X, Y on M. Assume $\acute{M}$ is a submanifold of a conformal Kenmotsu manifold M. Let $\acute{\nabla}$ and $\acute{R}$ are the Riemannian connection and curvature tensor on $\acute{M}$, respectively, and $\acute{g}$ is an induced metric on $\acute{M}$. Also let N is an unit vector field normal to $\acute{M}$. We put

$$\begin{aligned} PX &= tan(\varphi X), \qquad FX = nor(\varphi X), \\ tN &= tan(\varphi N), \qquad fN = nor(\varphi N), \end{aligned}$$

for any $X \in T\acute{M}$ and $N \in T\acute{M}^\perp$. Then by the Gauss formula and (6) we obtain the following relations

$$\begin{aligned} (\acute{\nabla}_X P)Y &= A_{FY}X + th(X,Y) + (exp(f))^{\frac{1}{2}}\{-g(X,\varphi Y)\xi - \eta(Y)PX\} \\ &- \frac{1}{2}\{\omega(\varphi Y)X - \omega(Y)PX + g(X,Y)(\varphi\omega^\sharp)^\top - g(X,\varphi Y)(\omega^\sharp)^\top\}, \end{aligned} \tag{8}$$

$$\begin{aligned} (\acute{\nabla}_X F)Y &= fh(X,Y) - h(X,PY) - (exp(f))^{\frac{1}{2}}\eta(Y)FX \\ &+ \frac{1}{2}\{\omega(Y)FX - g(X,Y)(\varphi\omega^\sharp)^\perp + g(X,\varphi Y)(\omega^\sharp)^\perp\}, \end{aligned} \tag{9}$$

$$\begin{aligned} (\acute{\nabla}_X t)N &= A_{fN}X - PA_NX - (exp(f))^{\frac{1}{2}}g(X,\varphi N)\xi \\ &- \frac{1}{2}\{\omega(\varphi N)X - \omega(N)PX + g(X,\varphi N)(\omega^\sharp)^\top\}, \end{aligned} \tag{10}$$

$$\begin{aligned} (\acute{\nabla}_X f)N &= -h(X,tN) - FA_NX \\ &+ \frac{1}{2}\{\omega(N)FX + g(X,\varphi N)(\omega^\sharp)^\perp\}, \end{aligned} \tag{11}$$

for all $X, Y \in T\acute{M}$ and $N \in T\acute{M}^{\perp}$ such that ξ is tangent to $\acute{M}$.
From (5), the Gauss and Ricci equations of $\acute{M}^m \subseteq (M^{2n+1}, \varphi, \xi, \eta, g)$ are given, respectively, by

$$\begin{aligned}\exp(-f)\tilde{g}(\tilde{R}(X,Y)Z,W) &= \acute{g}(\acute{R}(X,Y)Z,W) - g(h(X,W),h(Y,Z)) \\ &- g(h(Y,W),h(X,Z)) + \frac{1}{2}(B \wedge g + g \wedge B)(X,Y,Z,W)\} \\ &+ \frac{1}{4}\|\omega^\sharp\|^2\{g \wedge g(X,Y,Z,W)\}, \end{aligned} \tag{12}$$

$$g(R(X,Y)N_1,N_2) = g(\acute{R}^{\perp}(X,Y)N_1,N_2) - \acute{g}([A_1,A_2]X,Y), \tag{13}$$

for all $X, Y, Z, W \in T\acute{M}$ and $N_1, N_2 \in T\acute{M}^{\perp}$, where the wedge product of tensor fields A and B on $\acute{M}$ is given by

$$(A \wedge B)(X,Y,Z,W) = A(X,Z)B(Y,W) - A(Y,Z)B(X,W),$$

for all $X, Y, Z, W \in T\acute{M}$.

4. Invariant Submanifolds

A submanifold $\acute{M}^m$ of a conformal Kenmotsu manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is called an invariant submanifold if $\varphi T_p\acute{M} \subseteq T_p\acute{M}$ for any $p \in \acute{M}$ then $\varphi X = PX$ for any $X \in T\acute{M}$.

Lemma 1. *Let $\acute{M}^m$ be an invariant submanifold of a conformal Kenmotsu manifold M tangent to ξ. Then*

$$h(X,\varphi Y) = \varphi h(X,Y) + \frac{1}{2}\{g(X,\varphi Y)(\omega^{\sharp\perp}) - g(X,Y)(\varphi\omega^\sharp)^\perp\}, \tag{14}$$

$$A_{\varphi N}X = \varphi A_N X + \frac{1}{2}\{\omega(\varphi N)X - \omega(N)\varphi X\}, \tag{15}$$

$$A_N\varphi X + \varphi A_N X = \omega(N)\varphi X, \tag{16}$$

for all $X, Y \in T\acute{M}$ and $N \in T\acute{M}^{\perp}$.

Proof. Since $\acute{M}$ is invariant, we get $\varphi N = fN$. Then (14) and (15) follow immediately from (9) and (10), respectively. Since h is self-adjoint, from (14) we have

$$h(\varphi X,Y) = \varphi h(X,Y) + \frac{1}{2}\{g(Y,\varphi X)\omega^{\sharp\perp} - g(X,Y)(\varphi\omega^\sharp)^\perp\}, \tag{17}$$

for all $X, Y \in T\acute{M}$. By using (17), we obtain

$$\begin{aligned}\acute{g}(A_N\varphi X,Y) &= g(h(\varphi X,Y),N) = g(\varphi h(X,Y),N) \\ &- \frac{1}{2}\{g(X,Y)g((\varphi\omega^\sharp)^\perp,N) - g(\varphi X,Y)g((\omega^\sharp)^\perp,N)\}\end{aligned}$$

$$= -g(h(X,Y),\varphi N) + \frac{1}{2}\{g(X,Y)\omega(\varphi N) + g(\varphi X,Y)\omega(N)\},$$

hence we get

$$A_N\varphi X = -A_{\varphi N}X + \frac{1}{2}\{\omega(\varphi N)X + \omega(N)\varphi X\}, \tag{18}$$

for all $X \in T\acute{M}$ and $N \in T\acute{M}^{\perp}$. Now (16) follows from (15) and (18).◂

Theorem 2. *Let $\acute{M}^m$ be an invariant submanifold of a conformal Kenmotsu manifold M tangent to ξ. Then $\acute{M}$ is minimal if and only if the Lee vector field $\omega^\sharp$ of M is tangent to $\acute{M}$.*

Proof. From the Gauss formula and (4) we have

$$h(X,Y) = \tilde{\nabla}_X Y - \acute{\nabla}_X Y - \frac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X,Y)\omega^\sharp\}, \tag{19}$$

for all X, Y on $\acute{M}$. Replacing Y by φY in (19), we get

$$\begin{aligned} h(X,\varphi Y) &= \varphi h(X,Y) - (\acute{\nabla}_X\varphi)Y - (\exp(f))^{\frac{1}{2}}\{g(X,\varphi Y)\xi + \eta(Y)PX\} \\ &- \frac{1}{2}\{\omega(\varphi Y)X - \omega(Y)\varphi X + g(X,Y)\varphi\omega^\sharp - g(X,\varphi Y)\omega^\sharp\}, \end{aligned}$$

taking the tangent and normal parts, we have

$$h(X,\varphi Y) = \varphi h(X,Y) - \frac{1}{2}\{g(X,Y)(\varphi\omega^\sharp)^\perp - g(X,\varphi Y)(\omega^\sharp)^\perp\},$$

hence

$$h(\varphi X,\varphi Y) + h(X,Y) = \{g(X,Y) - \frac{1}{2}\eta(X)\eta(Y)\}(\omega^\sharp)^\perp - g(\varphi X,Y)(\varphi\omega^\sharp)^\perp. \tag{20}$$

From the Gauss formula and (7), it follows that

$$h(\xi,\xi) = \frac{1}{2}(\omega^\sharp)^\perp. \tag{21}$$

Now, let $\{E_\alpha, \varphi E_\alpha, \xi, \alpha : 1, \cdots, \frac{m-1}{2}\}$ be an orthonormal basis on $\acute{M}$ and suppose H is the mean curvature vector field of $\acute{M}$. Then, using (20) and (21) we have

$$H = \frac{1}{m}\{\Sigma_{\alpha=1}^{\frac{m-1}{2}}(h(E_\alpha,E_\alpha) + h(\varphi E_\alpha,\varphi E_\alpha)) + h(\xi,\xi)\} = \frac{1}{2}(\omega^\sharp)^\perp.$$

This completes the proof of the theorem.◂

Lemma 2. *Let $\acute{M}^m$ be an invariant submanifold of a conformal Kenmotsu manifold M tangent to ξ. Then*

$$[A_N, A_{\varphi N}] = -2\varphi T_N^2, \tag{22}$$

for any $N \in T\acute{M}^\perp$, where

$$T_N = A_N - \frac{1}{2}\omega(N)I, \tag{23}$$

and I denotes the identity transformation.

Proof. Lemma 2 results from (18).◀

Let Riz is the holomorphic bisectional curvature of a conformal Kenmotsu manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$, and $\sigma, \acute{\sigma}$ are two holomorphic 2-planes tangent to M at a point $p \in M$. Let $u \in \sigma$ and $v \in \acute{\sigma}$ such that $\|u\| = 1, \|v\| = 1$. Then, by definition we have

$$Riz(\sigma, \acute{\sigma}) = g(R(v, \varphi_p v)u, \varphi_p u). \tag{24}$$

Theorem 3. *Let $\acute{M}^m$ be an invariant submanifold of a conformal Kenmotsu manifold M tangent to ξ. If M has the negative holomorphic bisectional curvature, then the normal bundle $T\acute{M}^\perp$ admits no parallel connection.*

Proof. We prove this theorem by contradiction. Let the normal bundle $T\acute{M}^\perp$ admits parallel connection. Then $\acute{R}^\perp(X, Y)N_1 = 0$ for all $X, Y \in T\acute{M}$ and $N_1 \in T\acute{M}^\perp$. We put $N_2 = \varphi N_1$ in the Ricci equation, hence we obtain

$$g(R(X, Y)N_1, \varphi N_1) = -\acute{g}([A_{N_1}, A_{\varphi N_1}]X, Y). \tag{25}$$

Substituting (22) into (25) we get $g(R(X, Y)N_1, \varphi N_1) = 2\acute{g}(\varphi T_{N_1}^2 X, Y)$. Taking $X = \varphi Y$ in this equation, suppose $\sigma, \acute{\sigma}$ are holomorphic 2-planes spanned by $\{u, \varphi_x u\}$ and $\{v, \varphi_x v\}$, respectively, where $u = (\frac{Y}{\|Y\|})_p$ and $v = (\frac{N_1}{\|N_1\|})_p$. From (12) it follows that T_{N_1} is self-adjoint. Hence from (24) we obtain

$$0 \geq \|Y\|^2 \|N_1\|^2 Riz(\sigma, \acute{\sigma}) = 2\|T_{N_1}\|_p^2,$$

that is a contradiction.◀

For further use, we set $\acute{B} = \iota^* B, \acute{\omega} = \iota^* \omega$ and $\acute{B} = \acute{\nabla}\acute{\omega} - \frac{1}{2}\acute{\omega} \otimes \acute{\omega}$, hence

$$B(X, Y) = \acute{B}(X, Y) - \omega(h(X, Y)),$$

for all $X, Y \in T\acute{M}$. Moreover

$$B(X, Y) = g(B(X, .)^\sharp, Y),$$

for all $X, Y \in TM$.

5. Anti-Invariant Submanifolds

A submanifold $\acute{M}^m$ of a conformal Kenmotsu manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is called an anti-invariant submanifold if $\varphi T_p\acute{M} \subseteq T_p\acute{M}^\perp$ for any $p \in \acute{M}$. Then we have $PX = 0$ and $fN = 0$ for any $X \in T\acute{M}$ and $N \in T\acute{M}^\perp$.

Lemma 3. *Let $\acute{M}^m$ be an anti-invariant submanifold of a conformal Kenmotsu manifold M tangent to ξ. Then*

$$
\begin{aligned}
A_{\varphi Y}X &= -\varphi h(X,Y) + \frac{1}{2}\{\omega(\varphi Y)X + g(X,Y)(\varphi\omega^\sharp)^\top\}, && (26)\\
\acute{g}([A_{\varphi Z}, A_{\varphi W}]X, Y) &= g(h(X,W), h(Y,Z)) - g(h(X,Z), h(Y,W))\\
&- \frac{1}{2}\{g(Y,Z)\omega(h(X,W)) + g(X,W)\omega(h(Y,Z))\\
&- g(Y,W)\omega(h(X,Z)) - g(X,Z)\omega(h(Y,W))\\
&+ \omega(\varphi Z)\Phi(Y, h(X,W)) + \omega(\varphi W)\Phi(X, h(Y,Z))\\
&- \omega(\varphi W)\Phi(Y, h(X,Z)) - \omega(\varphi Z)\Phi(X, h(Y,W))\}\\
&- \frac{1}{4}\{\omega(\varphi W)\omega(\varphi X)g(Y,Z) + \omega(\varphi Z)\omega(\varphi Y)g(X,W)\\
&- \omega(\varphi Z)\omega(\varphi X)g(Y,W) - \omega(\varphi W)\omega(\varphi Y)g(X,Z)\\
&+ \|\omega^{\sharp\perp}\|^2(g(X,Z)g(Y,W) - g(X,W)g(Y,Z))\}, && (27)
\end{aligned}
$$

for all $X, Y, Z, W \in T\acute{M}$, where the fundamental 2-form Φ is defined by $\Phi(X,Y) = g(X, \varphi Y)$ for all vector fields X, Y on M.

Proof. Since $\acute{M}$ is anti-invariant, (26) results from (8). Now, substituting (26) into $\acute{g}([A_{\varphi Z}, A_{\varphi W}]X, Y) = \acute{g}(A_{\varphi W}X, A_{\varphi Z}Y) - \acute{g}(A_{\varphi Z}X, A_{\varphi W}Y)$, we get (27).◀

Theorem 4. *Let $\acute{M}^m$ be an anti-invariant submanifold of a conformal Kenmotsu manifold M tangent to ξ. Then $\acute{M}$ has the flat normal curvature tensor if and only if*

$$
\begin{aligned}
\acute{R}(X,Y)Z &= \eta(R(X,Y)Z)\xi - \frac{1}{2}\{\acute{B}(X,Z)Y + \acute{B}(Y,Z)X\\
&+ g(X,Z)\acute{B}(Y,.)^\sharp - g(Y,Z)\acute{B}(X,.)^\sharp - B(X,Z)\eta(Y)\xi\\
&+ B(Y,Z)\eta(X)\xi + B(X,\xi)g(Y,Z)\xi - B(Y,\xi)g(X,Z)\xi\}\\
&- \frac{1}{4}\|\omega^{\sharp\perp}\|^2\{g(X,Z)Y - g(Y,Z)X\} + (\frac{1}{4}\|\omega^\sharp\|^2 - exp(f))\{g(Y,Z)X\\
&- g(X,Z)Y + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, && (28)
\end{aligned}
$$

for all $X, Y, Z \in T\acute{M}$. ◀

Proof. Since $(\tilde{\nabla}_X\tilde{\varphi})Y = -\tilde{g}(X, \tilde{\varphi}Y)\tilde{\xi} - \tilde{\eta}(Y)\tilde{\varphi}X$, we have

$$
\begin{aligned}
\tilde{R}(X,Y)\tilde{\varphi}Z &= \tilde{\varphi}\tilde{R}(X,Y)Z + \tilde{g}(Y,Z)\tilde{\varphi}X\\
&- \tilde{g}(X,Z)\tilde{\varphi}Y + \tilde{g}(X,\tilde{\varphi}Z)Y - \tilde{g}(Y,\tilde{\varphi}Z)X, && (29)
\end{aligned}
$$

for all vector fields X, Y, Z on a Kenmotsu manifold $(M^{2n+1}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ [7]. Substituting (5) into (29), we obtain

$$\begin{aligned} R(X,Y)\varphi Z &= \varphi R(X,Y)Z - \frac{1}{2}\{B(X,\varphi Z)Y - B(Y,\varphi Z)X + B(Y,Z)\varphi X \\ &- B(X,Z)\varphi Y + B(Y,.)^\sharp g(X,\varphi Z) - B(X,.)^\sharp g(Y,\varphi Z) \\ &- \varphi B(Y,.)^\sharp g(X,Z) + \varphi B(X,.)^\sharp g(Y,Z)\} + \frac{1}{4}(\|\omega^\sharp\|^2 - exp(f)) \\ &\{g(X,Z)\varphi Y - g(Y,Z)\varphi X + g(Y,\varphi Z)X - g(X,\varphi Z)Y\}, \end{aligned}$$

for all vector fields X, Y, Z on M. Taking the inner product of the above equation with the vector field φW and using the Ricci and Gauss equations, we get

$$\begin{aligned} g(\acute{R}^\perp(X,Y)\varphi Z,\varphi W) &- \acute{g}([A_{\varphi Z}, A_{\varphi W}]X, Y) \\ &= \acute{g}(\acute{R}(X,Y)Z,W) - \eta(R(X,Y)Z)\eta(W) \\ &- g(h(X,W),h(Y,Z)) + g(h(Y,W),h(X,Z)) \\ &+ \frac{1}{2}\{B(X,Z)g(Y,W) - B(Y,Z)g(X,W) \\ &+ B(Y,W)g(X,Z) - B(X,W)g(Y,Z) \\ &- B(X,Z)\eta(Y)\eta(W) + B(Y,Z)\eta(X)\eta(W) \\ &+ B(X,\xi)g(Y,Z)\eta(W) - B(Y,\xi)g(X,Z)\eta(W)\} \\ &- \frac{1}{4}(\|\omega^\sharp\|^2 - exp(f))\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ &+ g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)\}. \end{aligned} \tag{30}$$

for all vector fields $X, Y, Z, W \in T\acute{M}$. From (26), we have

$$\Phi(Y,h(X,Z)) = \Phi(Z,h(X,Y)) + \frac{1}{2}\{\omega(\varphi Z)g(X,Y) - \omega(\varphi Y)g(X,Z)\}, \tag{31}$$

for all vector fields $X, Y, Z \in T\acute{M}$. Putting (27) in (30) and using (31), we obtain

$$\begin{aligned} -\varphi\acute{R}^\perp(X,Y)\varphi Z &= \acute{R}(X,Y)Z - \eta(R(X,Y)Z)\xi + \frac{1}{2}\{\acute{B}(X,Z)Y \\ &+ \acute{B}(Y,Z)X + g(X,Z)\acute{B}(Y,.)^\sharp - g(Y,Z)\acute{B}(X,.)^\sharp \\ &- B(X,Z)\eta(Y)\xi + B(Y,Z)\eta(X)\xi + B(X,\xi)g(Y,Z)\xi \\ &- B(Y,\xi)g(X,Z)\xi\} + \frac{1}{4}\|\omega^{\sharp\perp}\|^2\{g(X,Z)Y - g(Y,Z)X\} \\ &- (\frac{1}{4}\|\omega^\sharp\|^2 - exp(f))\{g(Y,Z)X - g(X,Z)Y + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi\}, \end{aligned}$$

for all $X, Y, Z \in T\acute{M}$. Thus $\acute{R}^\perp = 0$ if and only if (28) holds.◄

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Received 08 March 2014
Accepted 23 September 2014