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Generalized Potentials on Commutative Hypergroups

M. G. Hajibayov

Abstract. By the Hardy-Littlewood-Sobolev theorem, the classical Riesz potential is bounded on Lebesgue spaces. E. Nakai and H. Sumitomo [19] extended that theorem to the Orlicz spaces. We introduce generalized potential operators on commutative hypergroups and under some assumptions on the kernel we show the boundedness of these operators from Lebesgue space into certain Orlicz space. Our result is an analogue of Theorem 1.3 in [19].

Key Words and Phrases: Riesz potential, hypergroup, Lebesgue space, Orlicz spase, Hardy-Littlewood maximal function
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1. Introduction

For $0 < \alpha < n$, the operator

$$R_\alpha f(x) = \int_{R^n} |x - y|^{\alpha - n} f(y) dy$$

is called a classical Riesz potential (fractional integral).

By the classical Hardy-Littlewood-Sobolev theorem, if $1 < p < \infty$ and $\alpha p < n$, then $R_\alpha f$ is a bounded operator from $L^p(R^n)$ into $L^q(R^n)$, where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ (see [13], [22]).

The Hardy-Littlewood-Sobolev theorem is an important result in the potential theory. There are a lot of generalizations and analogues of that theorem. The boundedness of the Riesz potentials on spaces of homogeneous type was studied in [5] and [15]. The Hardy-Littlewood-Sobolev theorem was proved for the Riesz potentials associated to non-doubling measures in [16]. In [4] and [10], generalized potential-type integral operators were considered and (p, q) properties of these operators were proved. In [18], [19], [20], [11] the Hardy-Littlewood-Sobolev theorem was extended to Orlicz spaces for generalized fractional integrals. The analogues of the Hardy-Littlewood- Sobolev theorem for Riesz potentials on different hypergroups were given in [3], [6] [7], [8], [9], [24] and on commutative hypergroups in [12].

In this paper, we define generalized fractional integrals on commutative hypergroups and prove the analogue of Theorem 1.3 in [19] for the generalized fractional integrals on

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commutative hypergroups. The obtained result is an extension of the Hardy-Littlewood-Sobolev theorem given in [7], [8], [9], [24], for Riesz potentials on different hypergroups.

Let K be a set. A function $\rho : K \times K \to [0, \infty)$ is called quasi-metric if:

1. $\rho(x, y) = 0 \Leftrightarrow x = y$;
2. $\rho(x, y) = \rho(y, x)$;
3. there exists a constant $c \geq 1$ such that for every $x, y, z \in K$

$$\rho(x, y) \leq c(\rho(x, z) + \rho(z, y)).$$

Let all balls $B(x, r) = \{y \in K : \rho(x, y) < r\}$ be λ-measurable and assume that the measure λ fulfils the doubling condition

$$0 < \lambda B(x, 2r) \leq D\lambda B(x, r) < \infty. \tag{1}$$

A space (K, ρ, λ) which satisfies all conditions mentioned above is called a space of homogeneous type (see [2]).

In the theory of locally compact groups there arise certain spaces which, though not groups, have some of the structure of groups. Often, the structure can be expressed in terms of an abstract convolution of measures on the space.

A hypergroup $(K, *)$ consists of a locally compact Hausdorff space K together with a bilinear, associative, weakly continuous convolution on the Banach space of all bounded regular Borel measures on K with the following properties:

1. For all $x, y \in K$, the convolution of the point measures $\delta_x * \delta_y$ is a probability measure with compact support.
2. The mapping: $(x, y) \mapsto supp(\delta_x * \delta_y)$ of $K \times K$ into $\mathcal{C}(K)$ is continuous, where $\mathcal{C}(K)$ is the space of compact subsets of K endowed with the Michael topology, that is the topology generated by the subbasis of all

$$U_{V,W} = \{L \in \mathcal{C}(K) : L \cap V \neq \varnothing, L \subset W\},$$

where V, W are open subsets of K.

3. There exists an identity $e \in K$ such that $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all $x \in K$.
4. There exists a topological involution $\sim$ from K onto K such that $(x^\sim)^\sim = x$, for $x \in K$, with

$$(\delta_x * \delta_y)^\sim = \delta_{y^\sim} * \delta_{x^\sim}$$

and $e \in supp(\delta_x * \delta_y)$ if and only if $x = y^\sim$ for $x, y \in K$, where for any Borel set B, $\mu^\sim(B) = \mu(\{x^\sim : x \in B\})$ (see [14], [21], [1], [17]).

If $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in K$, then the hypergroup K is called commutative. It is known that every commutative hypergroup K possesses a Haar measure which will be denoted by λ (see [21]). That is, for every Borel measurable function f on K,

$$\int\limits_K f(\delta_x * \delta_y) d\lambda(y) = \int\limits_K f(y) d\lambda(y) \ \ (x \in K).$$

Define the generalized translation operators T^x, $x \in K$, by

$$T^x f(y) = \int\limits_K f d(\delta_x * \delta_y)$$

for all $y \in K$. If K is a commutative hypergroup, then $T^x f(y) = T^y f(x)$ and the convolution of two functions is defined by

$$(f * g)(x) = \int\limits_K T^x f(y) g(y^{\sim}) d\lambda(y).$$

Let $p > 0$. By $L^p(K, \lambda)$ we denote a class of all λ-measurable functions $f : K \to (-\infty, +\infty)$ with $\|f\|_{L^p(K,\lambda)} = \left(\int\limits_K |f(x)|^p \, d\lambda(x) \right)^{\frac{1}{p}} < \infty$.
A function $\Phi : [0, \infty] \to [0, \infty]$ is called an N-function if it can be represented as

$$\Phi(r) = \int\limits_0^r \phi(t) \, dt,$$

where $\phi : [0, \infty] \to [0, \infty]$ is a left continuous nondecreasing function such that $\phi(0) = 0$ and $\lim_{t \to \infty} \phi(t) = \infty$.
Let Φ be an N-function. Define the Orlicz space $L^\Phi(K, \lambda)$ to be the set of all locally integrable functions f in K for which

$$\int\limits_K \Phi\left(\frac{|f(x)|}{\eta} \right) d\lambda(x) \ < \infty$$

for some $\eta > 0$. Here $L^\Phi(K, \lambda)$ is equipped with the norm

$$\|f\|_\Phi = \inf\{\eta > 0 : \int\limits_K \Phi\left(\frac{|f(x)|}{\eta} \right) d\lambda(x) \le 1\}.$$

For $\Phi(r) = r^p, 1 < p < \infty$, we have $L^\Phi(K, \lambda) = L^p(K, \lambda)$.
The notation $\chi_A(x)$ denotes the characteristic function of set A.
Define a function $\Lambda_x(y) = T^x \chi_{B(e,r)}(y^{\sim})$.

We will assume that there exist constants $c_1 > 0$, $c_2 > 0$ and $c_3 > 0$ such that for every $x, y \in K$ and $r > 0$

$$supp\Lambda_x(\cdot) \subset B(x, c_1 r) \tag{2}$$

and

$$\lambda B(x,r) T^x \chi_{B(e,r)}(y^{\sim}) \leq c_2 \lambda B(e,r) \leq c_3 r^N. \tag{3}$$

As examples of hypergroups satisfying the conditions (2) and (3), we can mention Laguerre, Dunkl and Bessel hypergroups (see [7], [8], [9]).
A non-negative function $a(r)$ defined on $[0, \infty)$ is called almost increasing (almost decreasing), if there exists a constant $C > 0$ such that

$$a(t_1) \leq C a(t_2)$$

for all $0 < t_1 < t_2 < \infty$ ($0 < t_2 < t_1 < \infty$, respectively).
For an increasing function $a : (0, \infty) \to (0, \infty)$, define

$$I_a f(x) = \int\limits_K T^x \left(\frac{a(\rho(e,y))}{\rho(e,y)^N} \right) f(y^{\sim}) d\lambda(y)$$

on the commutative hypergroup $(K, *)$ equipped with the quasi-metric ρ. If $a(r) = r^{\alpha}, 0 < \alpha < N$, then I_a is the Riesz potential of order α.
Now we formulate a main result of the paper.

Theorem 1. *Let $(K, *)$ be a commutative hypergroup, with the quasi-metric ρ and doubling Haar measure λ satisfying the conditions (2) and (3). Assume that $1 < p < \infty$ and $a = a(r)$ is a non-negative almost increasing function on $[0, \infty)$, $\frac{a(r)}{r^{\lambda}}$ is almost decreasing for some $0 < \lambda < \frac{N}{p}$ and*

$$\int\limits_0^1 \frac{a(t)}{t} dt < \infty.$$

Then the operator I_a is bounded from $L^p(K, \lambda)$ into the Orlicz space $L^{\Phi}(K, \lambda)$, where the N-function is defined by its inverse

$$\Phi^{-1}(r) = \int\limits_0^r A\left(t^{-\frac{1}{N}}\right) t^{-\frac{1}{p'}} dt,$$

with $A(r) = \int\limits_0^r \frac{a(t)}{t} dt$.

If we take $a(r) = r^{\alpha}, 0 < \alpha < N$, then we have Hardy-Littlewood-Sobolev theorem for the Riesz potential

$$I_{\alpha} f(x) = \int\limits_K T^x \rho(e,y)^{\alpha - N} f(y^{\sim}) d\lambda(y)$$

on the commutative hypergroup $(K, *)$.

Corollary 1. *Let* $(K,*)$ *be a commutative hypergroup, with the quasi-metric* ρ *and doubling Haar measure* λ *satisfying the conditions* (2) *and* (3)*. If* $0<\alpha<N$, $1<p<\frac{N}{\alpha}$ *and* $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{N}$, *then* I_α *is a bounded operator from* $L^p(K,\lambda)$ *into* $L^q(K,\lambda)$.

2. Preliminaries

Define Hardy-Littlewood maximal function

$$Mf(x)=\sup_{r>0}\frac{1}{\lambda B(e,r)}\left(|f|*\chi_{B(e,r)}\right)(x)$$

on commutative hypergroup $(K,*)$ equipped with the pseudo-metric ρ.

Lemma 1. *Let* $(K,*)$ *be a commutative hypergroup, with quasi-metric* ρ *and doubling Haar measure* λ*. Assume that there exist constants* $c_1>0$ *and* $c_2>0$ *such that for every* $x,y\in K$ *and* $r>0$

$$supp\Lambda_x(\cdot)\subset B(x,c_1r)$$

and

$$\lambda B(x,r)T^x\chi_{B(e,r)}(y^\sim)\le c_2\lambda B(e,r).$$

Then

1) *The maximal operator* M *satisfies a weak type* $(1,1)$ *inequality, that is, there exists a constant* $C>0$ *such that for every* $f\in L^1(K,\lambda)$ *and* $\alpha>0$

$$\lambda\{x:Mf(x)>\alpha\}\le\frac{C}{\alpha}\int\limits_K|f(x)|d\lambda(x).$$

2) *The maximal operator* M *is of strong type* (p,p)*, for* $1<p\le\infty$*, that is,*

$$\|Mf\|_{L^p(K,\lambda)}\le C_p\|f\|_{L^p(K,\lambda)}, \tag{4}$$

for some constant C_p *and every* $f\in L^p(K,\lambda)$.

Proof. It is clear that there exists nonnegative integer m such that $c_1\le 2^m$ and $\lambda B(x,c_1r)\le D^m\lambda B(x,r)$, where D is a constant in doubling condition (1). Then we have

$$Mf(x)=\sup_{r>0}\frac{1}{\lambda B(e,r)}\int\limits_K T^x|f(y)|\chi_{B(e,r)}(y^\sim)d\lambda(y)$$

$$=\sup_{r>0}\frac{1}{\lambda B(e,r)}\int\limits_K|f(y)|T^x\chi_{B(e,r)}(y^\sim)d\lambda(y)$$

$$\le\sup_{r>0}\frac{1}{\lambda B(e,r)}\int\limits_{B(x,c_1r)}|f(y)|T^x\chi_{B(e,r)}(y^\sim)d\lambda(y)$$

$$= \sup_{r>0} \frac{1}{\lambda B(x,r)} \int_{B(x,c_1 r)} |f(y)| \frac{T^x \chi_{B(e,r)}(y^{\sim}) \lambda B(x,r)}{\lambda B(e,r)} d\lambda(y)$$

$$\leq c_2 \sup_{r>0} \frac{1}{\lambda B(x,r)} \int_{B(x,c_1 r)} |f(y)| d\lambda(y) \leq c_2 D^m M_\rho f(x),$$

where

$$M_\rho f(x) = \sup_{r>0} \frac{1}{\lambda B(x,r)} \int_{B(x,r)} |f(y)| d\lambda(y)$$

is a maximal operator on (K, ρ, λ). It is well known that the maximal operator M_ρ is of weak type $(1,1)$ and is bounded on $L^p(K,\lambda)$ (see [2], [23]). This fact and the inequality $Mf(x) \leq c_2 D^m M_\rho f(x)$ complete the proof.

3. Proof of Theorem 1

We may suppose that $f(x) \geq 0$ and, by the linearity of the operator I_a, it suffices to prove that $\|I_a f\|_\Phi \leq C < \infty$ for $\|f\|_{L^p(K,\lambda)} \leq 1$. In accordance with Hedberg's trick, we split $I_a f\,(x)$ in the standard way:

$$I_a f\,(x) = \int_{B(e,r)} \frac{a\,(\rho\,(e,y))}{\rho\,(e,y)^N} T^x f\,(y^{\sim})\, d\lambda\,(y)$$

$$+ \int_{X \setminus B(e,r)} \frac{a\,(\rho\,(e,y))}{\rho\,(e,y)^N} T^x f\,(y^{\sim})\, d\lambda(y) = \mathcal{A}_r(x) + \mathcal{B}_r(x).$$

Estimate $\mathcal{A}_r(x)$. Since $\frac{a(t)}{t^N}$ is almost decreasing, we have

$$\mathcal{A}_r(x) = \sum_{k=0}^{\infty} \int_{2^{-k-1}r \leq \rho(e,y) < 2^{-k}r} \frac{a\,(\rho\,(e,y))}{\rho\,(e,y)^N} T^x f\,(y^{\sim})\, d\lambda(y)$$

$$\leq C \sum_{k=0}^{\infty} \frac{a\left(2^{-k-1}r\right)}{(2^{-k-1}r)^N} \int_{2^{-k-1}r \leq \rho(e,y) < 2^{-k}r} T^x f\,(y^{\sim})\, d\lambda(y) \leq CMf\,(x) \sum_{k=0}^{\infty} a\left(2^{-k-1}r\right)$$

$$\leq CMf\,(x) \sum_{k=0}^{\infty} \int_{2^{-k-1}r}^{2^{-k}r} \frac{a(t)}{t} dt.$$

Therefore,

$$\mathcal{A}_r(x) \leq CA(r)Mf(x), \quad A(r) = \int_0^r \frac{a(t)}{t}\, dt. \tag{5}$$

Now estimate $\mathcal{B}_r(x)$. By the Hölder inequality and the condition $\|f\|_{L^p(K,\lambda)} \le 1$, we obtain

$$\mathcal{B}_r(x) \le \left(\int\limits_{K\setminus B(e,r)} (T^x f(y^\sim))^p \, d\lambda(y) \right)^{\frac{1}{p}} \left(\int\limits_{K\setminus B(e,r)} \left(\frac{a(\rho(e,y))}{\rho(e,y)^N} \right)^{p'} d\lambda(y) \right)^{\frac{1}{p'}}$$

$$\le \left(\int\limits_{K\setminus B(e,r)} \left(\frac{a(\rho(e,y))}{\rho(e,y)^N} \right)^{p'} d\lambda(y) \right)^{\frac{1}{p'}}$$

$$= \left(\sum_{k=0}^{\infty} \int\limits_{2^k r \le \rho(e,y) < 2^{k+1} r} \left(\frac{a(\rho(e,y))}{\rho(e,y)^N} \right)^{p'} d\lambda(y) \right)^{\frac{1}{p'}}$$

$$\le C \left(\sum_{k=0}^{\infty} \left(\frac{a(2^k r)}{(2^k r)^N} \right)^{p'} \int\limits_{\rho(e,y) < 2^{k+1} r} d\lambda(y) \right)^{\frac{1}{p'}}$$

$$\le C \left(\sum_{k=0}^{\infty} \left(\frac{a(2^k r)}{(2^k r)^N} \right)^{p'} \left(2^{k+1} r \right)^N \right)^{\frac{1}{p'}}$$

$$\le C \left(\sum_{k=0}^{\infty} \left(\frac{a(2^k r)}{(2^k r)^{\frac{N}{p}}} \right)^{p'} \right)^{\frac{1}{p'}}$$

$$\le C \left(\sum_{k=0}^{\infty} \left(a\left(2^k r\right) \right)^{p'} \int\limits_{2^k r}^{2^{k+1} r} \left(\frac{1}{t^{\frac{N}{p}}} \right)^{p'} \frac{1}{t} dt \right)^{\frac{1}{p'}}$$

$$\le C \left(\sum_{k=0}^{\infty} \int\limits_{2^k r}^{2^{k+1} r} \left(\frac{a(t)}{t^{\frac{N}{p}}} \right)^{p'} \frac{1}{t} dt \right)^{\frac{1}{p'}}$$

$$= C \left(\int\limits_{r}^{\infty} \left(\frac{a(t)}{t^{\frac{N}{p}}} \right)^{p'} \frac{1}{t} dt \right)^{\frac{1}{p'}}$$

$$\le C \frac{a(r)}{r^{\beta}} \left(\int\limits_{r}^{\infty} \left(t^{\beta - \frac{N}{p}} \right)^{p'} t^{-1} dt \right)^{\frac{1}{p'}}$$

$$\leq C\frac{a(r)}{r^{\frac{N}{p}}}.$$

Therefore

$$\mathcal{B}_r(x) \leq CA(r)r^{-\frac{N}{p}}. \tag{6}$$

From (5) and (6), we have

$$I_a f(x) \leq C\left(Mf\,(x) + r^{-\frac{N}{p}}\right)A\,(r)\,.$$

Then

$$I_a f(x) \leq C\left[Mf(x)r^{\frac{N}{p}} + 1\right]\Phi^{-1}\left(\frac{1}{r^N}\right) \tag{7}$$

by Theorem 4.9 in [11]. If we choose $r = [Mf(x)]^{-\frac{p}{N}}$, then the inequality (7) turns into

$$I_a f(x) \leq C\Phi^{-1}\left([Mf(x)]^p\right)$$

and consequently,

$$\int\limits_K \Phi\left(\frac{I_a f(x)}{C}\right)d\lambda(x) \leq \int\limits_K [Mf(x)]^p d\lambda(x) \leq 1,$$

where we have used (4) and the fact that $\|f\|_{L^p(K,\lambda)} \leq 1$. Hence

$$\|I_a f\|_\Phi \leq C,$$

which completes the proof.

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Mubariz G. Hajibayov
National Aviation Academy, AZ1104, Baku, Azerbaijan
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Az1141, Baku, Azerbaijan
E-mail: hajibayovm@yahoo.com

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