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Mixed Problem for Countable Hyperbolic System of Linear Equations

T. Firman*, V. Kyrylych

Abstract. In this paper the correct solvability of mixed problem for linear hyperbolic countable system of first order equations with two independent variables is proved.

Key Words and Phrases: Mixed problem, countable hyperbolic system, characteristic, integro-functional equation.

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1. Introduction

A variety of theoretical and application problems require the investigation of the theory of countable systems of partial differential equations.

For example, application of Lyapunov's method to the study of the problems of stability solutions for countable systems of ordinary differential equations leads to countable systems of first-order partial differential equations [1]. Using asymptotic method or Fourier method to solve nonlinear problem also leads to corresponding problems for countable systems of partial differential equations [2, 3].

The study of various problems of the theory of countable systems dates back to the works by Tikhonov, Persidskiy, Zhautykov and is systematically extended in the papers of Teplinsky and Samoylenko (see, for example, [4]).

2. Statement of problem

In the rectangle $\Pi = \{(x,t) : 0 < x < l, 0 < t < T\}$ we consider a countable hyperbolic system of first order linear differential equations

$$\frac{\partial u_i}{\partial t} + \lambda_i(x,t)\frac{\partial u_i}{\partial x} = \sum_{j=1}^{\infty} a_{ij}(x,t)u_j(x,t) + f_i(x,t), \quad i \in \mathbb{N}. \tag{1}$$

*Corresponding author.

Let us define sets of indices $I^+ = \{2k-1|\, k \in \mathbb{N}\}$, $I^- = \mathbb{N}/I^+$. Functions λ_i in (1) are ordered as follows:

$$\lambda_1(x,t) \geq \lambda_3(x,t) \geq ... \geq \lambda_{2k-1}(x,t) \geq ...,$$
$$\lambda_2(x,t) \leq \lambda_4(x,t) \leq ... \leq \lambda_{2k}(x,t) \leq ...,$$

where $\lambda_i > 0$ for $i \in I^+$ and $\lambda_i < 0$ for $i \in I^-$.

Together with system (1), we consider the initial and the boundary conditions

$$u_i(x,0) = g_i(x), \;\; x \in [0,l], \;\;\; i \in \mathbb{N}, \tag{2}$$

$$\begin{aligned} u_i(0,t) &= \sum_{j\in I^-} \alpha_{ij}(t)u_j(0,t) + h_i(t), \;\; t \in [0,T], \;\; i \in I^+, \\ u_i(l,t) &= \sum_{j\in I^+} \beta_{ij}(t)u_j(l,t) + r_i(t), \;\; t \in [0,T], \;\; i \in I^-. \end{aligned} \tag{3}$$

We consider problem (1)–(3) in the space C^∞, that consists of a countable set of continuous functions. Besides, these functions are bounded by some constant. In space C^∞ define the norm by the formula

$$||u|| = \sup_{i\in\mathbb{N}} \max_{(x,t)\in\overline{\Pi}} \{|u_i(x,t)|\},$$

where $u(x,t) = (u_1(x,t), u_2(x,t), ...)$.

We introduce the notation $u = (u_1, u_2, ...)$, $\lambda = (\lambda_1, \lambda_2, ...)$, $g = (g_1, g_2, ...)$.

We denote $F = \sup\limits_{i\in\mathbb{N}} \max\limits_{(x,t)\in\overline{\Pi}} \{|f_i(x,t)|\}$, $G = \sup\limits_{i\in\mathbb{N}} \max\limits_{x\in[0,l]} \{|g_i(x)|\}$, $H = \sup\limits_{i\in I^+} \max\limits_{t\in[0,T]} \{|h_i(t)|\}$ and $R = \sup\limits_{i\in I^-} \max\limits_{t\in[0,T]} \{|r_i(t)|\}$.

Suppose the function $\lambda : \mathbb{R}^2 \to \mathfrak{M}$ satisfies the Lipschitz condition with respect to x in $\overline{\Pi}$, if $\lambda_i \in Lip_x(\overline{\Pi})$ for all $i \in \mathbb{N}$.

3. The reduction of problem (1)–(3) to a system of integro-functional equations

We denote by $\varphi_i(\tau; x, t)$ the solution of the equation

$$\frac{d\xi}{d\tau} = \lambda_i(\xi, \tau), \qquad i \in \mathbb{N}, \tag{4}$$

satisfying the initial condition

$$\xi|_{\tau=t} = x. \tag{5}$$

Let $L_i(x,t)$ be an integral curve defined by the equation $\xi = \varphi_i(\tau; x, t)$, which starts from the point $(x,t) \in \overline{\Pi}$, and $t_i(x,t)$ is an ordinate of the point of intersection of the i-th characteristic with the line $x = 0$ or $x = l$ in the direction of decreasing t.

Integrating (1) along the corresponding characteristics, we obtain the following system of integro-functional equations:

$$u_i(x,t) = \omega_i[u](x,t) + \int\limits_{\max\{t_i(x,t),0\}}^{t} \Big(\sum_{j=1}^{\infty} a_{ij}u_j + f_i\Big)[(\varphi_i(\tau;x,t),\tau)]d\tau, \tag{6}$$

where

$$\omega_i[u](x,t) = \begin{cases} u_i(0,t_i(x,t)), & \varphi_i(t_i(x,t);x,t) = 0, \\ g_i(\varphi_i(0;x,t)), & \\ u_i(l,t_i(x,t)), & \varphi_i(t_i(x,t);x,t) = l. \end{cases} \tag{7}$$

Definition 1. *The continuous function $u : \overline{\Pi} \to \mathfrak{M}$ is a generalized solution of problem (1)-(3) if it satisfies the system of integro-functional equations (6).*

We consider the rectangle $[0,l]\times[0,T]$, where T is chosen so, that $L_1(0,0)$ and $L_2(l,0)$ disjoint [5], namely, all characteristics that come from the lower corner points will not intersect in $\overline{\Pi}$. So, the rectangle $\overline{\Pi}$ is divided into an infinite number of subdomains $\overline{\Pi} = \bigcup\limits_{i=0}^{\infty} \overline{\Pi}_i$ (Figure 1).

Figure 1: Breaking the rectangle Π

Firstly, we consider subdomain Π_0. Let (x,t) be an arbitrary point in Π_0. From this point we omit all characteristics $L_i(x,t)$ in the direction of decreasing t. They all cross the lower base $\overline{\Pi}$ (Figure 2). In the domain Π_0 we obtain the Cauchy problem equivalent to the countable system of integro-functional equations

$$u_i(x,t) = g_i(\varphi_i(0;x,t)) + \int\limits_{0}^{t} \Big(\sum_{j=1}^{\infty} a_{ij}u_j + f_i\Big)[(\varphi_{2k}(\tau;x,t),\tau)]d\tau. \tag{8}$$

The sufficient conditions for the existence of a generalized solution of the Cauchy problem are given in [7].

Now consider one of domains Π_{2s-1}, $s\in\mathbb{N}$. In this subdomain, all characteristics with odd numbers up to the number $2s-1$ omitted from arbitrary point (x,t) will fall to

Figure 2: Characteristics in domain Π_0

the left side of the rectangle $\overline{\Pi}$. The remaining characteristics with odd numbers and all characteristics with even numbers will fall to the lower base of the rectangle $\overline{\Pi}$ (Figure 3).

Figure 3: Characteristics in domain Π_{2s-1}

Therefore, in this domain we obtain the following system of integro-functional equations:

$$u_{2k}(x,t) = g_{2k}(\varphi_{2k}(0;x,t)) + \int_0^t \Big(\sum_{j=1}^{\infty} a_{2k,j}u_j + f_{2k}\Big)[(\varphi_{2k}(\tau;x,t),\tau)]d\tau, \quad k \in \mathbb{N}, \tag{9}$$

$$u_{2k-1}(x,t) = u_{2k-1}(0,t_{2k-1}(x,t)) + \int_{t_{2k-1}(x,t)}^t \Big(\sum_{j=1}^{\infty} a_{2k-1,j}u_j + f_{2k-1}\Big)[(\varphi_{2k-1}(\tau;x,t),\tau)]d\tau, \tag{10}$$

$$k = 1,2,...,s,$$

$$u_{2k-1}(x,t) = g_{2k-1}(\varphi_{2k-1}(0;x,t)) + \int_0^t \Big(\sum_{j=1}^{\infty} a_{2k-1,j}u_j + f_{2k-1}\Big)[(\varphi_{2k-1}(\tau;x,t),\tau)]d\tau, \tag{11}$$

$$k = s+1, s+2, ...$$

Consider (10) and use the boundary conditions on the left boundary of Π, i.e.

$$u_{2k-1}(0,t_{2k-1}(x,t))=\sum_{j=1}^{\infty}\alpha_{2k-1,2j}(t_{2k-1}(x,t))u_{2j}(0,t_{2k-1}(x,t))+h_{2k-1}(t_{2k-1}(x,t)). \quad (12)$$

Substituting this equality in (10), we obtain

$$\begin{aligned} u_{2k-1}(x,t) &= \sum_{j=1}^{\infty}\alpha_{2k-1,2j}(t_{2k-1}(x,t))u_{2j}(0,t_{2k-1}(x,t))+h_{2k-1}(t_{2k-1}(x,t))+ \qquad (13)\\ &+\int\limits_{t_{2k-1}(x,t)}^{t}\Big(\sum_{j=1}^{\infty}a_{2k-1,j}u_j+f_{2k-1}\Big)[(\varphi_{2k-1}(\tau;x,t),\tau)]d\tau, \qquad k=1,2,...,s. \end{aligned}$$

For $u_{2j}(0,t_{2k-1}(x,t))$, where $k=1,2,...,s$ we have

$$\begin{aligned} u_{2j}(0,t_{2k-1}(x,t)) &= g_{2j}(\varphi_{2j}(0;0,t_{2k-1}(x,t)))+ \qquad (14)\\ &+\int\limits_{0}^{t_{2k-1}(x,t)}\Big(\sum_{p=1}^{\infty}a_{2j,p}u_p+f_{2j}\Big)[(\varphi_{2j}(\tau;0,t_{2k-1}(x,t)),\tau)]d\tau, \qquad j\in\mathbb{N}. \end{aligned}$$

Taking into account (13) and (14), we get

$$\begin{aligned} u_{2k-1}(x,t) &= h_{2k-1}(t_{2k-1}(x,t))+\sum_{j=1}^{\infty}\alpha_{2k-1,2j}(t_{2k-1}(x,t))\Big[g_{2j}(\varphi_{2j}(0;0,t_{2k-1}(x,t)))+\\ &+\int\limits_{0}^{t_{2k-1}(x,t)}\Big(\sum_{p=1}^{\infty}a_{2j,p}u_p+f_{2j}\Big)[(\varphi_{2j}(\tau;0,t_{2k-1}(x,t)),\tau)]\Big]d\tau+ \qquad (15)\\ &+\int\limits_{t_{2k-1}(x,t)}^{t}\Big(\sum_{j=1}^{\infty}a_{2k-1,j}u_j+f_{2k-1}\Big)[(\varphi_{2k-1}(\tau;x,t),\tau)]d\tau, \qquad k=1,2,...,s. \end{aligned}$$

Thus, in domain Π_{2s-1} the problem (1)-(3) is reduced to the system of integro-functional equations (9),(11),(15).

4. The main results

Theorem 1. *Suppose the following conditions hold:*

1) $\lambda\in C^{\infty}(\overline{\Pi})\cap \mathrm{Lip}_x(\overline{\Pi})$;

2) the functions

$$a_{ij}(x,t),\; a_i\equiv\sum_{j=1}^{\infty}|a_{ij}(x,t)|,\; f_i(x,t),$$

$$\alpha_{2i-1,2j}(t),\ \alpha_{2i-1} \equiv \sum_{j=1}^{\infty} |\alpha_{2i-1,2j}(t)|,\ h_{2i-1}(t),$$
$$\beta_{2i,2j-1}(t),\ \beta_{2i} \equiv \sum_{j=1}^{\infty} |\beta_{2i,2j-1}(t)|,\ r_{2i}(t)$$

are continuous for all $i, j \in \mathbb{N}$;

3) the inequalities

$$a_i(x,t) \le a(x,t),\ \alpha_{2i-1}(t) \le \alpha(t),\ \beta_{2i}(t) \le \beta(t),$$

are satisfied, where $a(x,t) \in C(\overline{\Pi}), \alpha(t), \beta(t) \in C[0,T]$;

4) zero-order compatibility conditions

$$g_{2i-1}(0) = \sum_{j=1}^{\infty} \alpha_{2i-1,2j}(0) g_{2j}(0) + h_{2i-1}(0), \qquad i \in \mathbb{N},$$
$$g_{2i}(l) = \sum_{j=1}^{\infty} \beta_{2i,2j-1}(0) g_{2j-1}(l) + r_{2i}(0), \qquad i \in \mathbb{N},$$

are satisfied. Then there exists a unique generalized solution of problem (1)–(3).

Proof. Let $A = \max\limits_{(x,t)\in\overline{\Pi}} \{|a(x,t)|\}$, $\Lambda = \max\limits_{t\in[0,T]} \{|\alpha(t)|\}$, $B = \max\limits_{t\in[0,T]} \{|\beta(t)|\}$. We will prove the existence and uniqueness of the solution of the integro-functional equations (9),(11),(15) using the method of successive approximations.

First, we prove the existence of a solution. Zero approximation is constructed as follows:

$$u_{2k}^{(0)}(x,t) = g_{2k}(\varphi_{2k}(0;x,t)), \qquad k \in \mathbb{N},$$
$$u_{2k-1}^{(0)}(x,t) = h_{2k-1}(t_{2k-1}(x,t)) + \sum_{j=1}^{\infty} \alpha_{2k-1,2j}(t_{2k-1}(x,t)) g_{2j}(\varphi_{2j}(0;0,t_{2k-1}(x,t))), \qquad k = 1,2,...,s, \tag{16}$$
$$u_{2k-1}^{(0)}(x,t) = g_{2k-1}(\varphi_{2k-1}(0;x,t)), \qquad k = s+1, s+2, ...$$

Then we obtain the following successive approximations for $m = 1,2,...$:

$$u_{2k}^{(m)}(x,t) = g_{2k}(\varphi_{2k}(0;x,t)) + \int_0^t \Big(\sum_{j=1}^{\infty} a_{2k,j} u_j^{(m-1)} + f_{2k}\Big)[(\varphi_{2k}(\tau;x,t),\tau)]d\tau,\ k \in \mathbb{N},$$

$$u_{2k-1}^{(m)}(x,t) = h_{2k-1}(t_{2k-1}(x,t)) + \sum_{j=1}^{\infty} \alpha_{2k-1,2j}(t_{2k-1}(x,t)) \Big[g_{2j}(\varphi_{2j}(0;0,t_{2k-1}(x,t))) +$$

$$+\int\limits_{0}^{t_{2k-1}(x,t)} f_{2j}(\varphi_{2j}(\tau;0,t_{2k-1}(x,t)),\tau)d\tau\Big]+\int\limits_{t_{2k-1}(x,t)}^{t} f_{2k-1}(\varphi_{2k-1}(\tau;x,t),\tau)d\tau+$$

$$+\sum_{j=1}^{\infty}\alpha_{2k-1,2j}(t_{2k-1}(x,t))\int\limits_{0}^{t_{2k-1}(x,t)}\sum_{p=1}^{\infty}a_{2j,p}u_p^{(m-1)}[(\varphi_{2j}(\tau;0,t_{2k-1}(x,t)),\tau)]d\tau+ \quad (17)$$

$$+\int\limits_{t_{2k-1}(x,t)}^{t}\sum_{j=1}^{\infty}a_{2k-1,j}u_j^{(m-1)}[(\varphi_{2k-1}(\tau;x,t),\tau)]d\tau, \qquad k=1,2,...,s,$$

$$\begin{aligned} u_{2k-1}^{(m)}(x,t) &= g_{2k-1}(\varphi_{2k-1}(0;x,t))+ \\ &+\int\limits_{0}^{t}\Big(\sum_{j=1}^{\infty}a_{2k-1,j}u_j^{(m-1)}+f_{2k-1}\Big)[(\varphi_{2k-1}(\tau;x,t),\tau)]d\tau,\ k=s+1,s+2,... \end{aligned}$$

As is known from the theory of ordinary differential equations, the function $\varphi_i(\tau;x,t)$ is continuous with respect to the variables (x,t) if $\lambda_i(x,t)\in C(\overline{\Pi})\cap Lip_x(\overline{\Pi})$ [6]. If $g_i(\cdot)$ is a continuous function, then $g_i(\varphi_i(\tau;x,t))$ is a continuous function with respect to the variables (x,t), as a composition of continuous functions. If the functions $u_i(x,t)$ are continuous with respect to the variables (x,t) for all $i\in\mathbb{N}$ and satisfy conditions of Theorem 1, then all the successive approximation (16)–(17) are continuous functions, as a composition of continuous functions [4].

We show that the sequence of approximations is uniformly convergent in Π_{2s-1}. The sequence is uniformly convergent if and only if the following functional series is uniformly convergent:

$$u_i^{(0)}(x,t)+\sum_{m=1}^{\infty}\left(u_i^{(m)}(x,t)-u_i^{(m-1)}(x,t)\right),\quad i\in\mathbb{N}. \quad (18)$$

We prove the convergence of the series (18), using Weierstrass convergence theorem. For $i=2k$, $k\in\mathbb{N}$ and $i=2k-1$, $k=s+1,s+2,...$ first successive approximation will be

$$u_i^{(1)}(x,t)=g_i(\varphi_i(0;x,t))+\int\limits_{0}^{t}\Big[\sum_{j=1}^{\infty}a_{i,2j}(\varphi_i(\tau;x,t),\tau)\cdot g_{2j}(\varphi_{2j}(0;\varphi_i(\tau,x,t),\tau))+$$

$$+\sum_{j=1}^{s}a_{i,2j-1}(\varphi_i(\tau;x,t),\tau)\cdot\Big(h_{2j-1}(t_{2j-1}(\varphi_i(\tau;x,t),\tau))+$$

$$+\sum_{p=1}^{\infty}\alpha_{2j-1,2p}(t_{2j-1}(\varphi_i(\tau;x,t),\tau))\cdot g_{2p}(\varphi_{2p}(0;0,t_{2j-1}(\varphi_i(\tau;x,t),\tau)))\Big)+$$

$$+\sum_{j=s+1}^{\infty} a_{i,2j-1}(\varphi_i(\tau;x,t),\tau)\cdot g_{2j-1}(\varphi_{2j-1}(0;\varphi_i(\tau;x,t),\tau))+f_i(\varphi_i(\tau;x,t),\tau)\Big]d\tau.$$

For $i=2k-1,$ where $k=1,2,...,s,$ we have

$$u_i^{(1)}(x,t)=h_i(t_i(x,t))+\sum_{j=1}^{\infty}\alpha_{i,2j}(t_i(x,t))\Big[g_{2j}(\varphi_{2j}(0;0,t_i(x,t)))+$$
$$+\int_0^{t_i(x,t)} f_{2j}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)d\tau\Big]+\int_{t_i(x,t)}^{t}\Big[\sum_{j=1}^{\infty}a_{i,2j}(\varphi_i(\tau;x,t),\tau)\cdot g_{2j}(\varphi_{2j}(0;\varphi_i(\tau;x,t),\tau))+$$
$$+\sum_{j=1}^{s}a_{i,2j-1}(\varphi_i(\tau;x,t),\tau)\cdot\Big(h_{2j-1}(t_{2j-1}(\varphi_i(\tau;x,t),\tau))+$$
$$+\sum_{n=1}^{\infty}\alpha_{2j-1,2n}(t_{2j-1}(\varphi_i(\tau;x,t),\tau))\cdot g_{2n}(\varphi_{2n}(0;0,t_{2j-1}(\varphi_i(\tau;x,t),\tau)))\Big)+$$
$$+\sum_{j=s+1}^{\infty}a_{i,2j-1}(\varphi_i(\tau;x,t),\tau)\cdot g_{2j-1}(\varphi_{2j-1}(0;\varphi_i(\tau;x,t),\tau))+f_i(\varphi_i(\tau;x,t),\tau)\Big]d\tau+$$
$$+\sum_{j=1}^{\infty}\alpha_{i,2j}(t_i(x,t))\cdot\int_0^{t_i(x,t)}\Big[\sum_{p=1}^{\infty}a_{2j,2p}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)\cdot g_{2p}(\varphi_{2p}(0;\varphi_{2p}(\tau,0,t_i(x,t)),\tau))+$$
$$+\sum_{p=1}^{s}a_{2j,2p-1}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)\cdot\Big(h_{2p-1}(t_{2p-1}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau))+$$
$$+\sum_{n=1}^{\infty}\alpha_{2p-1,2n}(t_{2p-1}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau))\cdot g_{2n}(\varphi_{2n}(0;0,t_i(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)))\Big)+$$
$$+\sum_{p=s+1}^{\infty}a_{2j,2p-1}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)\cdot g_{2p-1}(\varphi_{2p-1}(0;\varphi_{2j}(\tau;0,t_i(x,t)),\tau))\Big]d\tau.$$

We estimate the difference $|u_i^{(1)}(x,t)-u_i^{(0)}(x,t)|$ for all $i\in\mathbb{N}$ and $(x,t)\in\Pi_{2s-1}$. For $i=2k,\ k\in\mathbb{N}$ and $i=2k-1,\ k=s+1,s+2,...$ we obtain

$$|u_i^{(1)}(x,t)-u_i^{(0)}(x,t)|=\Big|\int_0^{t}\Big[\sum_{j=1}^{\infty}a_{i,2j}(\varphi_i(\tau;x,t),\tau)\cdot g_{2j}(\varphi_{2j}(0;\varphi_i(\tau,x,t),\tau))+$$
$$+\sum_{j=1}^{s}a_{i,2j-1}(\varphi_i(\tau;x,t),\tau)\cdot\Big(h_{2j-1}(t_{2j-1}(\varphi_i(\tau;x,t),\tau))+$$

$$+\sum_{p=1}^{\infty}\alpha_{2j-1,2p}(t_{2j-1}(\varphi_i(\tau;x,t),\tau))\cdot g_{2p}(\varphi_{2p}(0;0,t_{2j-1}(\varphi_i(\tau;x,t),\tau)))\Big)+$$

$$+\sum_{j=s+1}^{\infty}a_{i,2j-1}(\varphi_i(\tau;x,t),\tau)\cdot g_{2j-1}(\varphi_{2j-1}(0;\varphi_i(\tau;x,t),\tau))+f_i(\varphi_i(\tau;x,t),\tau)\Big]d\tau\Bigg|\leq$$

$$\leq\int_0^t\sum_{j=1}^{\infty}\Big|a_{i,j}(\varphi_i(\tau;x,t),\tau)\cdot g_j(\varphi_j(0;\varphi_i(\tau,x,t),\tau))\Big|d\tau+$$

$$+\int_0^t\sum_{j=1}^{\infty}\Big|a_{i,2j-1}(\varphi_i(\tau;x,t),\tau)\cdot\Big(h_{2j-1}(t_{2j-1}(\varphi_i(\tau;x,t),\tau))+$$

$$+\sum_{p=1}^{\infty}\alpha_{2j-1,2p}(t_{2j-1}(\varphi_i(\tau;x,t),\tau))\cdot g_{2p}(\varphi_{2p}(0;0,t_{2j-1}(\varphi_i(\tau;x,t),\tau)))\Big)\Big|d\tau+$$

$$+\int_0^t|f_i(\varphi_i(\tau;x,t),\tau)|d\tau\leq\int_0^t\Big(AG+A(H+\Lambda G)+F\Big)d\tau=Mt\leq(\Lambda+1)Mt,$$

where $M=AG+A(H+\Lambda G)+F$.

For $i=2k-1$, where $k=1,2,...,s$, we get

$$|u_i^{(1)}(x,t)-u_i^{(0)}(x,t)|=\Bigg|\sum_{j=1}^{\infty}\alpha_{i,2j}(t_i(x,t))\int_0^{t_i(x,t)}f_{2j}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)d\tau+$$

$$+\int_{t_i(x,t)}^{t}\Big[\sum_{j=1}^{\infty}a_{i,2j}(\varphi_i(\tau;x,t),\tau)\cdot g_{2j}(\varphi_{2j}(0;\varphi_i(\tau,x,t),\tau))+$$

$$+\sum_{j=1}^{s}a_{i,2j-1}(\varphi_i(\tau;x,t),\tau)\cdot\Big(h_{2j-1}(t_{2j-1}(\varphi_i(\tau;x,t),\tau))+$$

$$+\sum_{n=1}^{\infty}\alpha_{2j-1,2n}(t_{2j-1}(\varphi_i(\tau;x,t),\tau))\cdot g_{2n}(\varphi_{2n}(0;0,t_{2j-1}(\varphi_i(\tau;x,t),\tau)))\Big)+$$

$$+\sum_{j=s+1}^{\infty}a_{i,2j-1}(\varphi_i(\tau;x,t),\tau)\cdot g_{2j-1}(\varphi_{2j-1}(0;\varphi_i(\tau;x,t),\tau))+f_i(\varphi_i(\tau;x,t),\tau)\Big]d\tau+$$

$$+\sum_{j=1}^{\infty}\alpha_{i,2j}(t_i(x,t))\cdot\int_0^{t_i(x,t)}\Big[\sum_{p=1}^{\infty}a_{2j,2p}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)\cdot g_{2p}(\varphi_{2p}(0;\varphi_{2p}(\tau,0,t_i(x,t)),\tau))+$$

$$+\sum_{p=1}^{s}a_{2j,2p-1}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)\cdot\Big(h_{2p-1}(t_{2p-1}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau))+$$

$$+\sum_{n=1}^{\infty}\alpha_{2p-1,2n}(t_{2p-1}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau))\cdot g_{2n}(\varphi_{2n}(0;0,t_i(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)))\Big)+$$

$$+\sum_{p=s+1}^{\infty}a_{2j,2p-1}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)\cdot g_{2p-1}(\varphi_{2p-1}(0;\varphi_{2j}(\tau;0,t_i(x,t)),\tau))\Big]d\tau\Bigg|\leq$$

$$\leq\int\limits_{t_i(x,t)}^{t}\Bigg|\sum_{j=1}^{\infty}a_{i,j}(\varphi_i(\tau;x,t),\tau)\cdot g_j(\varphi_j(0;\varphi_i(\tau,x,t),\tau))+f_i(\varphi_i(\tau;x,t),\tau)+$$

$$+\sum_{j=1}^{\infty}a_{i,2j-1}(\varphi_i(\tau;x,t),\tau)\cdot\Big(h_{2j-1}(t_{2j-1}(\varphi_i(\tau;x,t),\tau))+$$

$$+\sum_{n=1}^{\infty}\alpha_{2j-1,2n}(t_{2j-1}(\varphi_i(\tau;x,t),\tau))\cdot g_{2n}(\varphi_{2n}(0;0,t_{2j-1}(\varphi_i(\tau;x,t),\tau)))\Big)\Bigg|d\tau+$$

$$+\sum_{j=1}^{\infty}\Big|\alpha_{i,2j}(t_i(x,t))\Big|\cdot\Bigg|\int\limits_{0}^{t_i(x,t)}\Big(f_{2j}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)+$$

$$+\sum_{p=1}^{\infty}a_{2j,p}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)\cdot g_p(\varphi_p(0;\varphi_{2j}(\tau;0,t_i(x,t)),\tau))+$$

$$+\sum_{p=1}^{\infty}a_{2j,2p-1}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)\cdot\Big(h_{2p-1}(t_{2p-1}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau))+$$

$$+\sum_{n=1}^{\infty}\alpha_{2p-1,2n}(t_{2p-1}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau))\cdot g_{2n}(\varphi_{2n}(0;0,t_i(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)))\Big)\Big)d\tau\Bigg|\leq$$

$$\leq\int\limits_{t_i(x,t)}^{t}\Big(AG+F+A(H+\Lambda G)\Big)d\tau+\Lambda\int\limits_{0}^{t_i(x,t)}\Big(F+AG+A(H+\Lambda G)\Big)d\tau\leq$$

$$\leq(\Lambda+1)\int\limits_{0}^{t}Md\tau=(\Lambda+1)Mt.$$

Hence, for all $(x,t)\in\Pi_{2s-1}$ and arbitrary $i\in\mathbb{N}$, we get the following estimate:

$$|u_i^{(1)}(x,t)-u_i^{(0)}(x,t)|\leq(\Lambda+1)Mt.$$

Let $m=2$. For $i=2k$, $k\in\mathbb{N}$ and $i=2k-1$, $k=s+1,s+2,...$ we obtain

$$|u_i^{(2)}(x,t)-u_i^{(1)}(x,t)|=\Bigg|\int\limits_{0}^{t}\Big(\sum_{j=1}^{\infty}a_{i,j}(\varphi_{2k}(\tau;x,t),\tau)\cdot\big(u_j^{(1)}(\varphi_i(\tau;x,t),\tau)-$$

$$-u_j^{(0)}(\varphi_i(\tau;x,t),\tau)\Big)d\tau\Bigg| \le (\Lambda+1)MA\int\limits_0^t \tau d\tau = (\Lambda+1)MA\frac{t^2}{2!} \le MA(\Lambda+1)^2\frac{t^2}{2!}.$$

For $i=2k-1$, where $k=1,2,...,s$, we get

$$|u_i^{(2)}(x,t)-u_i^{(1)}(x,t)| = \Bigg|\sum_{j=1}^{\infty}\alpha_{i,2j}(t_i(x,t))\int\limits_0^{t_i(x,t)}\Big(\sum_{p=1}^{\infty}a_{2j,p}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)\times$$

$$\big(u_p^{(1)}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)-u_p^{(0)}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)\big)\Big)d\tau+$$

$$+\int\limits_{t_i(x,t)}^{t}\Big(\sum_{j=1}^{\infty}a_{i,j}(\varphi_i(\tau;x,t),\tau)\cdot\big(u_j^{(1)}(\varphi_i(\tau;x,t),\tau)-u_j^{(0)}(\varphi_i(\tau;x,t),\tau)\big)\Big)d\tau\Bigg| \le$$

$$\le \Lambda\int\limits_0^{t_i(x,t)}A(\Lambda+1)M\tau d\tau+\int\limits_{t_i(x,t)}^{t}A(\Lambda+1)M\tau d\tau \le MA(\Lambda+1)^2\int\limits_0^t\tau d\tau = MA(\Lambda+1)^2\frac{t^2}{2!}.$$

Hence, for all $(x,t)\in\Pi_{2s-1}$ and arbitrary $i,m\in\mathbb{N}$ we get the estimate

$$|u_i^{(m)}(x,t)-u_i^{(m-1)}(x,t)| \le M(\Lambda+1)^m A^{m-1}\frac{t^m}{m!}.$$

Suppose that this inequality holds for some $m=n$. Let's prove it for $m=n+1$. For $i=2k$, $k\in\mathbb{N}$ and $i=2k-1$, $k=s+1,s+2,...$, we obtain

$$|u_i^{(n+1)}(x,t)-u_i^{(n)}(x,t)| = \Bigg|\int\limits_0^t\Big(\sum_{j=1}^{\infty}a_{i,j}(\varphi_i(\tau;x,t),\tau)\cdot\big(u_j^{(n)}(\varphi_i(\tau;x,t),\tau)-$$

$$-u_j^{(n-1)}(\varphi_i(\tau;x,t),\tau)\big)\Big)d\tau\Bigg| \le AM(\Lambda+1)^nA^{n-1}\int\limits_0^t\frac{\tau^n}{n!}d\tau = M(\Lambda+1)^nA^n\frac{t^{n+1}}{(n+1)!} \le$$

$$\le M(\Lambda+1)^{n+1}A^n\frac{t^{n+1}}{(n+1)!}.$$

For $i=2k-1$, where $k=1,2,...,s$ we have

$$|u_i^{(n+1)}(x,t)-u_i^{(n)}(x,t)| = \Bigg|\sum_{j=1}^{\infty}\alpha_{i,2j}(t_i(x,t))\int\limits_0^{t_i(x,t)}\Big(\sum_{p=1}^{\infty}a_{2j,p}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)\times$$

$$\big(u_p^{(n)}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)-u_p^{(n-1)}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)\big)\Big)d\tau+$$

$$+\int\limits_{t_i(x,t)}^{t}\Big(\sum_{j=1}^{\infty}a_{i,j}(\varphi_i(\tau;x,t),\tau)\cdot\big(u_j^{(n)}(\varphi_i(\tau;x,t),\tau)-u_j^{(n-1)}(\varphi_i(\tau;x,t),\tau)\big)\Big)d\tau\Bigg|\le$$

$$\le\Lambda\int\limits_0^{t_i(x,t)}AM(\Lambda+1)^nA^{n-1}\frac{\tau^n}{n!}d\tau+\int\limits_{t_i(x,t)}^{t}AM(\Lambda+1)^nA^{n-1}\frac{\tau^n}{n!}d\tau\le$$

$$\le M(\Lambda+1)^{n+1}A^n\int\limits_0^t\frac{\tau^n}{n!}d\tau=M(\Lambda+1)^{n+1}A^n\frac{t^{n+1}}{(n+1)!}.$$

Hence, for all $(x,t)\in\Pi_{2s-1}$ and arbitrary $i,m\in\mathbb{N}$ we get the following estimate:

$$|u_i^{(m)}(x,t)-u_i^{(m-1)}(x,t)|\le M(\Lambda+1)^mA^{m-1}\frac{t^m}{m!}\le M(\Lambda+1)^mA^{m-1}\frac{T^m}{m!}.$$

Hence, for norm $||u||=\sup\limits_i\max\limits_{(x,t)\in\overline{\Pi}}\{|u_i(x,t)|\}$ we get

$$||u^{(m)}(x,t)-u^{(m-1)}(x,t)||\le M(\Lambda+1)^mA^{m-1}\frac{T^m}{m!}.$$

Therefore, series (18) is uniformly convergent on Π_{2s-1}, which means that the sequence of approximations is uniformly convergent on Π_{2s-1}, namely,

$$u_i^{(m)}(x,t)\rightrightarrows u_i(x,t),\ \ \forall i\in\mathbb{N}.$$

In order to show that the limit functions $u_i(x,t)$ provide a continuous solution of the system (9),(11),(15), we have to pass to the limit in (17) as $m\to\infty$. Since all the functions included in the system are continuous, such transition is possible.

Now we prove the uniqueness of the solution. Suppose there are two different solutions $\widehat{u}$ and $\widetilde{u}$. Consider the difference $u(x,t)=\widehat{u}(x,t)-\widetilde{u}(x,t)$. Denote by $U(t)$ the function $U(t)=\sup\limits_{i,x,\tau\le t}\{|u_i(x,\tau)|\}$. Since $\widehat{u}$ and $\widetilde{u}$ are the solutions of system (9),(11),(15), for $u(x,t)$ we get:

1) for $i=2k$, $k\in\mathbb{N}$ and $i=2k-1$, $k=s+1,s+2,...$

$$u_i(x,t)=\int\limits_0^t\Big(\sum_{j=1}^{\infty}a_{i,j}(\varphi_i(\tau;x,t),\tau)u_j(\varphi_i(\tau;x,t),\tau)\Big)d\tau;\tag{19}$$

2) for $i=2k-1$, where $k=1,2,...,s$

$$u_i(x,t)=\sum_{j=1}^{\infty}\alpha_{i,2j}(t_i(x,t))\int\limits_0^{t_i(x,t)}\sum_{p=1}^{\infty}a_{2j,p}(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)u_p(\varphi_{2j}(\tau;0,t_i(x,t)),\tau)d\tau+\tag{20}$$

$$+\int\limits_{t_i(x,t)}^{t}\sum_{j=1}^{\infty}a_{i,j}(\varphi_i(\tau;x,t),\tau)u_j(\varphi_i(\tau;x,t),\tau)d\tau.$$

Taking into account the inequalities (19),(20), we obtain an estimate for the function $U(t)$

$$U(t)\leq\int\limits_0^t(\Lambda+1)AU(\tau)d\tau.$$

If we apply Gronwall-Bellman lemma to the function $U(t)$, we obtain the inequality

$$U(t)\leq 0,\qquad \forall t\in[0,T],$$

which means that $U(t)\equiv 0$. Thus, the solution is unique.

Similarly, we can prove the existence and uniqueness of the solution in arbitrary domain Π_{2s}, where $s\in\mathbb{N}$.

Thus, we have proved the existence and uniqueness of the continuous solution in all subdomains of the rectangle $\overline{\Pi}$. Thus, solution in the rectangle $\overline{\Pi}$ can have discontinuities only on characteristics.

Consider one of characteristics $L_{2s-1}(0,0)$, $s\in\mathbb{N}$. We obtain the Cauchy problem for the function $u_{2s-1}(x,t)$ in the domain Π_{2s-3}. In the domain Π_{2s-1} we use boundary conditions for the function $u_{2s-1}(x,t)$. Therefore, function $u_{2s-1}(x,t)$ has no gaps if and only if

$$\lim_{\substack{(x,t)\to L_{2s-1}(0,0)\\(x,t)\in\Pi_{2s-1}}}u_{2s-1}(x,t)=\lim_{\substack{(x,t)\to L_{2s-1}(0,0)\\(x,t)\in\Pi_{2s-3}}}u_{2s-1}(x,t).$$

It is easy to see that $\lim\limits_{(x,t)\to L_{2s-1}(0,0)}t_{2s-1}(x,t)=0$, and $\lim\limits_{(x,t)\to L_{2s-1}(0,0)}\varphi_{2s-1}(0;x,t)=0$.

Therefore, from (11),(15) we obtain

$$\lim_{\substack{(x,t)\to L_{2s-1}(0,0)\\(x,t)\in\Pi_{2s-3}}}u_{2s-1}(x,t)=g_{2s-1}(0)+\int\limits_0^t\left(\sum_{j=1}^{\infty}a_{2s-1,j}u_j+f_{2s-1}\right)[(\varphi_{2s-1}(\tau;x,t),\tau)]d\tau$$

$$\lim_{\substack{(x,t)\to L_{2s-1}(0,0)\\(x,t)\in\Pi_{2s-1}}}u_{2s-1}(x,t)=\sum_{j=1}^{\infty}\alpha_{2s-1,2j}(0)g_{2j}(0)+h_{2s-1}(0)+$$

$$+\int\limits_0^t\left(\sum_{j=1}^{\infty}a_{2s-1,j}u_j+f_{2s-1}\right)[(\varphi_{2s-1}(\tau;x,t),\tau)]d\tau.$$

Thus, we obtain compatibility conditions at the point $(0,0)$:

$$g_{2s-1}(0) = \sum_{j=1}^{\infty} \alpha_{2s-1,2j}(0) g_{2j}(0) + h_{2s-1}(0), \qquad s \in \mathbb{N}.$$

Similar compatibility conditions are true at the point $(l, 0)$:

$$g_{2s}(l) = \sum_{j=1}^{\infty} \beta_{2s,2j-1}(0) g_{2j-1}(l) + r_{2s}(0), \qquad s \in \mathbb{N}.$$

If the compatibility conditions hold, then there exists a unique generalized solution of the problem (1)–(3).◄

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Taras I. Firman
Ivan Franko National University of L'viv, Ua79000, L'viv, Ukraine
E-mail: tarasfirman91@ukr.net

Volodymyr M. Kyrylych
Ivan Franko National University of L'viv, Ua79000, L'viv, Ukraine
E-mail: vkyrylych@ukr.net

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