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The Two-Phase Problem for One Quasilinear Hyperbolic System

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Abstract. We investigate the two-phase problem for one quasilinear hyperbolic system in one space variable that cannot be reduced to a system in the Riemann invariants. We are looking for a generalized solution of the problem in the class of piecewise Lipschitz continuous functions. This solution admits a jump discontinuity on the line $x = 0$. Applying the method of characteristics and the Banach fixed point theorem, we prove a local existence-uniqueness theorem.

Key Words and Phrases: two-phase problem, hyperbolic system, the Schauder canonical form, piecewise Lipschitz continuity, jump discontinuity, the method of characteristics, the Banach fixed point theorem

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1. Introduction

Partial differential equations are one of the basic areas of applied analysis, and it is difficult to imagine any area of applications where their impact is not felt. In recent decades there has been tremendous emphasis on understanding and modeling nonlinear processes; such processes are often governed by nonlinear PDEs, and the subject has become one of the most active areas in applied mathematics and central in modern-day mathematical research. Nonlinear equations have come to the forefront because, basically, the world is nonlinear.

A single first-order PDE is a hyperbolic equation, it is wave-like, that is, associated with the propagation of signals at finite speed. The fundamental idea associated with hyperbolic equations is the notion of a characteristic. There are several ways of considering the concept of a characteristic: one definition is that it is a curve in spacetime (a hypersurface in higher dimensions) along which information is carried or a signal propagates. But ultimately, a characteristic is a curve along which the PDE can be reduced to a simpler form, for example, to an ordinary differential equation.

Besides, most physical models involve several unknown functions. For example, the complete description of a fluid mechanical system might require knowledge of the density,

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pressure, temperature, and the particle velocity. So to describe this model and many others we would need to formulate a system of PDEs

$$u_t + Au_x = b$$

in unknowns $u = (u_1, \dots, u_n)$, where the matrix A and vector b may depend on x, t, u. In the case of hyperbolic systems, as well as for a single equation, the oldest, and still useful, approach to this subject is the method of characteristics, or Riemann's method, which originated from the works of G. Monge [12], B. Riemann [13, 14], and H. Lewy [11]. It was used later in numerous publications and books (see e.g. [5, 10, 3, 8, 15]). Until now, a variety of initial and initial-boundary value problems for hyperbolic systems have been solved by this method. It is based on the simple fact that, under certain conditions, we can introduce new dependent variables $r = (r_1, \dots, r_n)$, called the Riemann invariants, so that to reduce a system to the form

$$r_t + \Lambda r_x = f,$$

where Λ is a diagonal matrix similar to A. This system can be regarded as a family of ordinary differential equations along some curves in space, which are called characteristics, precisely:

$$\frac{dr_k}{dt} = f_k \qquad \text{on} \qquad \frac{dx}{dt} = \lambda_k, \qquad k \in \{1, \dots, n\},$$

where λ_k denote the diagonal entries of Λ, which are the eigenvalues of A at the same time. Then, in a simple case, the solution can be found explicitly by integrating these equations along appropriate characteristic curves. In more complicated case, we reduce our problem to some operator equation, so functional analysis methods can be used to prove its solvability. Note that the Riemann invariants do not always exist in the nonlinear hyperbolic system having more than two dependent variables. But such a system can always be written in the Schauder canonical form or the characteristic form (see [4, 15, 18, 19])

$$\sum_{i=1}^{n} l_i^k \left(\frac{\partial u_i}{\partial t} + \lambda_k \frac{\partial u_i}{\partial x} - f_i \right) = 0, \qquad k \in \{1, \dots, n\}, \tag{1}$$

where $l^k = (l_1^k, \dots, l_n^k)$ denote the left eigenvectors of A. Thus, finding approaches to investigate general nonlinear processes is a difficult task in modern mathematics, even if the process is governed by a system of three PDEs.

In many researches, it is useful to consider, in addition to (1), the system obtained by differentiating (1), where the derivatives of the solution u are also unknowns. Putting system (1) and differentiated one together gives the so called augmented system, which was introduced by Courant and Lax [5]. Note that the augmented system of any quasilinear hyperbolic system is already reducible to invariants. This approach was applied e.g. in [16, 17] to study the problem that is close to ours. Of course, it is not possible to speak of the equivalence of (1) and the augmented system because a classical solution of the augmented system requires that u be twice continuously differentiable, but, on the other hand, the definition of a classical solution u of (1) requires only continuous differentiability of it.

Since there are many areas where wave propagation is of fundamental importance, the study of hyperbolic PDEs and systems of PDEs is of great interest in mathematics. They have a broad range of applications, such as for fluid mechanics (water waves, aerodynamics, meteorology, traffic flow), acoustics (sound waves in air and liquids), elasticity (stress waves, earthquakes), physics (optics, electromagnetic waves, quantum mechanics), biology (spread of diseases, population dispersal, nerve signal transmission), chemistry (combustion and detonation waves), and other topics of interest in applied mathematics.

Moreover, many industrial processes involve two or more separate phases. Considering phase changing processes governed by PDEs, we obtain a multiphase (e.g. two-phase) problem whose solution may suffer a jump discontinuity on the common boundary according to given conjugation conditions.

This paper is dedicated to the study of two-phase problem for one quasilinear hyperbolic system of first order PDEs in one space variable that cannot be reduced to a system in the Riemann invariants. We study solutions on the triangular region bounded above by the line $t = T$, laterally by the curves $x = s_1(t)$ and $x = s_2(t)$, whereas $x = 0$ is the interface between two phases. We prove a local existence theorem for a generalized solution, which is meant to be a piecewise Lipschitz continuous function admitting a jump discontinuity on the line $x = 0$ such that the corresponding integro-differential system is satisfied. Note that, generalized solutions of quasilinear equations were first investigated by Hopf [7]. Generalized solutions of nonlinear partial differential equations of first order in the class of Lipschitz continuous functions were considered by Kruzhkov [9]. In this paper we use the method of characteristics and the Banach fixed point theorem. This technique is close to that used in [1, 2, 6]. Note that, when studying solvability, we do not use the augmented system. Therefore our result was obtained without smoothness assumptions on given data.

2. Problem formulation

Put

$$\Omega_T^- = \left\{(x,t) \in \mathbb{R}^2 : \ 0 < t < T, \ s_1(t) < x < 0\right\},$$
$$\Omega_T^+ = \left\{(x,t) \in \mathbb{R}^2 : \ 0 < t < T, \ 0 < x < s_2(t)\right\},$$

where s_1, s_2 are given smooth functions such that $s_1(0) = s_2(0) = 0$, $s_1(t) < 0 < s_2(t)$, $0 < t \le T$, $s_1'(0) < 0 < s_2'(0)$.

On each of the domains Ω_T^-, Ω_T^+ we consider a hyperbolic system of quasilinear equations in unknowns $u = (u_1, u_2, u_3)$

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + a_{11}(x,t,u)\dfrac{\partial u_1}{\partial x} + a_{13}(x,t,u)\dfrac{\partial u_3}{\partial x} = b_1(x,t,u), \\ \dfrac{\partial u_2}{\partial t} + a_{22}(x,t,u)\dfrac{\partial u_2}{\partial x} + a_{23}(x,t,u)\dfrac{\partial u_3}{\partial x} = b_2(x,t,u), \\ \dfrac{\partial u_3}{\partial t} = b_3(x,t,u), \end{cases} \tag{2}$$

where given functions a_{ij}, b_j are required to be continuous on each domain, and they can be continuously extended to the closures of these domains. Note that the system (2) cannot be reduced to the Riemann invariants.

We introduce notation

$$u(-0,t_0) \overset{def}{=} \lim_{\substack{(x,t)\to(0,t_0)\\(x,t)\in\Omega_T^-}} u(x,t), \quad u(+0,t_0) \overset{def}{=} \lim_{\substack{(x,t)\to(0,t_0)\\(x,t)\in\Omega_T^+}} u(x,t),$$

and append to system (2) the initial conditions

$$u(-0,0) = v^-, \quad u(+0,0) = v^+. \tag{3}$$

Assuming that the inequalities

$$\begin{gathered} \max\{a_{11}(-0,0,v^-),\ a_{11}(+0,0,v^+)\} < s_1'(0), \\ \min\{a_{22}(-0,0,v^-),\ a_{22}(+0,0,v^+)\} > s_2'(0) \end{gathered} \tag{4}$$

hold, we impose the boundary conditions

$$\begin{aligned} u_1(s_2(t),t) &= K_1^+(t, u_1(s_1(t),t), u_2(s_2(t),t)), \\ u_2(s_1(t),t) &= K_2^-(t, u_1(s_1(t),t), u_2(s_2(t),t)), \\ u_3(s_1(t),t) &= K_3^-(t, u_1(s_1(t),t), u_2(s_2(t),t)), \\ u_3(s_2(t),t) &= K_3^+(t, u_1(s_1(t),t), u_2(s_2(t),t)), \end{aligned} \tag{5}$$

along with the conjugation conditions

$$\begin{aligned} u_1(-0,t) &= K_1^-(t, u_1(+0,t), u_2(-0,t), u_3(-0,t), u_3(+0,t)), \\ u_2(+0,t) &= K_2^+(t, u_1(+0,t), u_2(-0,t), u_3(-0,t), u_3(+0,t)). \end{aligned} \tag{6}$$

3. Equivalent integro-differential system

Now we reduce the problem (2), (3), (5), (6) to an integro-differential system. In (2), we add the third equation multiplied by $\frac{a_{13}}{a_{11}}$ to the first one. Similarly, we add the third equation multiplied by $\frac{a_{23}}{a_{22}}$ to the second. Then we obtain

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + a_{11}(x,t,u)\dfrac{\partial u_1}{\partial x} + \dfrac{a_{13}}{a_{11}}(x,t,u)\left(\dfrac{\partial u_3}{\partial t} + a_{11}(x,t,u)\dfrac{\partial u_3}{\partial x}\right) = f_1(x,t,u), \\ \dfrac{\partial u_2}{\partial t} + a_{22}(x,t,u)\dfrac{\partial u_2}{\partial x} + \dfrac{a_{23}}{a_{22}}(x,t,u)\left(\dfrac{\partial u_3}{\partial t} + a_{22}(x,t,u)\dfrac{\partial u_3}{\partial x}\right) = f_2(x,t,u), \\ \dfrac{\partial u_3}{\partial t} = f_3(x,t,u), \end{cases} \tag{7}$$

where $\dfrac{a_{i3}}{a_{ii}}(x,t,u) \stackrel{\text{def}}{=} \dfrac{a_{i3}(x,t,u)}{a_{ii}(x,t,u)}$, $f_i \stackrel{\text{def}}{=} b_i + \dfrac{a_{i3}}{a_{ii}}b_3$, $i \in \{1,2\}$, $f_3 \stackrel{\text{def}}{=} b_3$.

In the domain Ω_T^- (in Ω_T^+ in the second case), fixing $i \in \{1,2\}$, we consider the Cauchy problem

$$\frac{d\xi}{d\tau} = a_{ii}(\xi,\tau,u(\xi,\tau)), \quad \xi(t) = x, \tag{8}$$

where a_{ii} is assumed to be locally Lipschitz continuous on the domain $\Omega_T^- \times R^3$ (or alternatively,on $\Omega_T^+ \times R^3$), u is a Lipschitz continuous map of Ω_T^- to R^3 (or a map of Ω_T^+ to R^3), and (x,t) is a point in the corresponding domain. In this domain, the problem (8) has a unique solution, which we denote by $\xi = \varphi_i[u](\tau;x,t)$, where τ is an argument and x,t are parameters. This solution can be extended in the direction of decrease of τ to the boundary of the domain. By $\chi_i[u](x,t)$ we denote the infimum value of τ such that the point $(\varphi_i[u](\tau;x,t),\tau)$ belongs to Ω_T^- (or to Ω_T^+ ,respectively).

Using the introduced notation, we rewrite (7) in the characteristic form:

$$\begin{cases} \dfrac{du_1(\varphi_1[u](\tau;x,t),\tau)}{d\tau} + \dfrac{a_{13}}{a_{11}}\Big(\varphi_1[u](\tau;x,t),\tau,u(\varphi_1[u](\tau;x,t),\tau)\Big)\dfrac{du_3(\varphi_1[u](\tau;x,t),\tau)}{d\tau} = \\ \qquad = f_1\Big(\varphi_1[u](\tau;x,t),\tau,u(\varphi_1[u](\tau;x,t),\tau)\Big), \\ \dfrac{du_2(\varphi_2[u](\tau;x,t),\tau)}{d\tau} + \dfrac{a_{23}}{a_{22}}\Big(\varphi_2[u](\tau;x,t),\tau,u(\varphi_2[u](\tau;x,t),\tau)\Big)\dfrac{du_3(\varphi_2[u](\tau;x,t),\tau)}{d\tau} = \\ \qquad = f_2\Big(\varphi_2[u](\tau;x,t),\tau,u(\varphi_2[u](\tau;x,t),\tau)\Big), \\ \dfrac{du_3(x,\tau)}{d\tau} = f_3(x,\tau,u(x,\tau)). \end{cases} \tag{9}$$

Let us integrate each equation in (9) with respect to τ from $\chi_i[u](x,t)$ to t, where the index i corresponds to equation number. By definition, put

$$\chi_3[u](x,t) = \begin{cases} s_1^{-1}(x), & \text{if } (x,t) \in \Omega_T^-, \\ s_2^{-1}(x), & \text{if } (x,t) \in \Omega_T^+. \end{cases}$$

Then we obtain

$$u_1(x,t) = u_1\Big(\varphi_1[u](\chi_1[u](x,t);x,t), \chi_1[u](x,t)\Big) - \\ - \int\limits_{\chi_1[u](x,t)}^{t} \frac{a_{13}}{a_{11}}\Big(\varphi_1[u](\tau;x,t), \tau, u(\varphi_1[u](\tau;x,t),\tau)\Big)\frac{du_3(\varphi_1[u](\tau;x,t),\tau)}{d\tau}d\tau + \\ + \int\limits_{\chi_1[u](x,t)}^{t} f_1\Big(\varphi_1[u](\tau;x,t), \tau, u(\varphi_1[u](\tau;x,t),\tau)\Big)d\tau,$$

$$u_2(x,t) = u_2\Big(\varphi_2[u](\chi_2[u](x,t)), \chi_2[u](x,t)\Big) - \\ - \int\limits_{\chi_2[u](x,t)}^{t} \frac{a_{23}}{a_{22}}\Big(\varphi_2[u](\tau;x,t), \tau, u(\varphi_2[u](\tau;x,t),\tau)\Big)\frac{du_3(\varphi_2[u](\tau;x,t),\tau)}{d\tau}d\tau + \\ + \int\limits_{\chi_2[u](x,t)}^{t} f_2\Big(\varphi_2[u](\tau;x,t), \tau, u(\varphi_2[u](\tau;x,t),\tau)\Big)d\tau,$$

$$u_3(x,t) = u_3(x, \chi_3[u](x,t)) + \int\limits_{\chi_3[u](x,t)}^{t} f_3(x,\tau,u(x,\tau))d\tau.$$

Taking into account the boundary conditions (5) and conjugated conditions (6), we have

$$u_1(x,t) = J_1[u](x,t) - \int\limits_{\chi_1[u](x,t)}^{t} \frac{a_{13}}{a_{11}}\Big(\varphi_1[u](\tau;x,t), \tau, u(\varphi_1[u](\tau;x,t),\tau)\Big)\frac{du_3(\varphi_1[u](\tau;x,t),\tau)}{d\tau}d\tau + \\ + \int\limits_{\chi_1[u](x,t)}^{t} f_1\Big(\varphi_1[u](\tau;x,t), \tau, u(\varphi_1[u](\tau;x,t),\tau)\Big)d\tau, \quad (10)$$

where

$$J_1[u](x,t) = \begin{cases} K_1^-\Big(\chi_1[u](x,t), u_1(+0, \chi_1[u](x,t)), u_2(-0, \chi_1[u](x,t)), \\ u_3(-0, \chi_1[u](x,t)), u_3(+0, \chi_1[u](x,t))\Big), \text{ if } (x,t) \in \Omega_T^-, \\ K_1^+\Big(\chi_1[u](x,t), u_1(s_1(\chi_1[u](x,t)), \chi_1[u]), u_2(s_2(\chi_1[u](x,t)), \chi_1[u](x,t))\Big), \\ \text{if } (x,t) \in \Omega_T^+; \end{cases}$$

$$u_2(x,t) = J_2[u](x,t) - \int\limits_{\chi_2[u](x,t)}^{t} \frac{a_{23}}{a_{22}}\Big(\varphi_2[u](\tau;x,t),\tau,u(\varphi_2[u](\tau;x,t),\tau)\Big)\frac{du_3(\varphi_2[u](\tau;x,t),\tau)}{d\tau}d\tau +$$
$$+ \int\limits_{\chi_2[u](x,t)}^{t} f_2\Big(\varphi_2[u](\tau;x,t),\tau,u(\varphi_2[u](\tau;x,t),\tau)\Big)d\tau, \quad (11)$$

where

$$J_2[u](x,t) = \begin{cases} K_2^-\Big(\chi_2[u](x,t), u_1(s_1(\chi_2[u](x,t)),\chi_2[u]), u_2(s_2(\chi_2[u](x,t)),\chi_2[u](x,t))\Big), \\ \text{if } (x,t)\in\Omega_T^-, \\ K_2^+\Big(\chi_2[u](x,t), u_1(+0,\chi_2[u](x,t)), u_2(-0,\chi_2[u](x,t)), \\ u_3(-0,\chi_2[u](x,t)), u_3(+0,\chi_2[u](x,t))\Big), \text{ if } (x,t)\in\Omega_T^+; \end{cases}$$

$$u_3(x,t) = J_3[u](x,t) + \int\limits_{\chi_3[u](x,t)}^{t} f_3(x,\tau,u(x,\tau))d\tau, \quad (12)$$

where

$$J_3[u](x,t) = \begin{cases} K_3^-\Big(\chi_3[u](x,t), u_1(s_1(\chi_3[u])(x,t),\chi_3[u]), u_2(s_2(\chi_3[u](x,t)),\chi_3[u](x,t))\Big), \\ \text{if } (x,t)\in\Omega_T^-, \\ K_3^+\Big(\chi_3[u](x,t), u_1(s_1(\chi_3[u](x,t)),\chi_3[u]), u_2(s_2(\chi_3[u](x,t)),\chi_3[u](x,t))\Big), \\ \text{if } (x,t)\in\Omega_T^+. \end{cases}$$

Definition 1. *A set of functions $u = (u_1, u_2, u_3)$ is called a generalized solution to the problem* (2), (3), (5), (6), *if these functions are Lipschitz continuous on each of the domains Ω_T^-, Ω_T^+, they can be continuously extended to the closures of these domains, and satisfy integro-differential system* (10)–(12).

3.1. Local solvability of the problem

We state our main result as a theorem.

Theorem 1. *Suppose the following conditions hold:*

1) *a_{11}, a_{13}, a_{22}, a_{23}, b_1, b_2, b_3 are locally Lipschitz continuous on each of the domains $\Omega_T^-\times\mathbb{R}^3$, $\Omega_T^+\times\mathbb{R}^3$, and can be extended by continuity to the closures of these domains;*

2) *K_1^+, K_2^-, K_3^+, K_3^- are Lipschitz continuous in a neighborhood of the point $(0, v_1^-, v_2^+)$; K_1^-, K_2^+ are Lipschitz continuous in a neighborhood of $(0, v_1^+, v_2^-, v_3^-, v_3^+)$; the Lipschitz constants of these functions are assumed to be sufficiently small;*

3) inequalities (4) *are satisfied;*

4) the following compatibility conditions are satisfied:

$$v_1^+ = K_1^+(0, v_1^-, v_2^+), \quad v_2^- = K_2^-(0, v_1^-, v_2^+),$$

$$v_3^- = K_3^-(0, v_1^-, v_2^+), \quad v_3^+ = K_3^+(0, v_1^-, v_2^+),$$

$$v_1^- = K_1^-(0, v_1^+, v_2^-, v_3^-, v_3^+), \quad v_2^+ = K_2^+(0, v_1^+, v_2^-, v_3^-, v_3^+);$$

5) the following constants are sufficiently small:

$$\left|\frac{a_{13}(-0,0,v^-)}{a_{11}(-0,0,v^-)}\right|, \quad \left|\frac{a_{13}(+0,0,v^+)}{a_{11}(+0,0,v^+)}\right|, \quad \left|\frac{a_{23}(-0,0,v^-)}{a_{22}(-0,0,v^-)}\right|, \quad \left|\frac{a_{23}(+0,0,v^+)}{a_{22}(+0,0,v^+)}\right|.$$

Then there exists a unique generalized solution to the problem (2), (3), (5), (6) *on* $\Omega_{T_0}^- \cup \Omega_{T_0}^+$ *with a small enough value of* T_0.

Proof. We introduce a metric space $Q = Q(T_0, U, L)$, where parameters $T_0 \in (0, T]$, $U \in (0, 1]$, $L > 0$ are to be determined, as a set of vector functions $u = (u_1, u_2, u_3)$ such that u_1, u_2, u_3 are Lipschitz continuous on each of the domains $\Omega_{T_0}^-$, $\Omega_{T_0}^+$, can be continuously extended to the closures of these domains, and there hold the initial condition (3) along with the following conditions:

1) for every point $(x,t) \in \Omega_{T_0}^-$ we have $|u_i(x,t) - v_i^-| \le U$, $i \in \{1,2,3\}$ and, similarly, for every $(x,t) \in \Omega_{T_0}^+$ we have $|u_i(x,t) - v_i^+| \le U$, $i \in \{1,2,3\}$;

2) for every pair of points $(x^1,t^1), (x^2,t^2) \in \Omega_{T_0}^-$ (or pair of points $(x^1,t^1), (x^2,t^2) \in \Omega_{T_0}^+$) we have $|u_i(x^1,t^1) - u_i(x^2,t^2)| \le L(|x^1 - x^2| + |t^1 - t^2|)$, $i \in \{1,2,3\}$.

For any $u^1, u^2 \in Q$, we define the distance between these elements as

$$\rho(u^1,u^2) = \max\left\{ \sup_{\substack{i\in\{1,2,3\},\\ (x,t)\in\Omega_{T_0}^-}} |u_i^1(x,t) - u_i^2(x,t)|, \sup_{\substack{i\in\{1,2,3\},\\ (x,t)\in\Omega_{T_0}^+}} |u_i^1(x,t) - u_i^2(x,t)| \right\}.$$

Note that (Q, ρ) is a complete metric space. Further, we introduce an operator $\mathcal{A}$ on Q as follows. For any $u \in Q$ write $\mathcal{A}[u] = (\mathcal{A}_1[u], \mathcal{A}_2[u], \mathcal{A}_3[u])$, where the elements $\mathcal{A}_1[u], \mathcal{A}_2[u], \mathcal{A}_3[u]$ are defined as the right-hand sides of the equalities (10)–(12).

Thus, finding a generalized solution of the problem (2), (3), (5), (6) is reduced to finding a fixed point of the operator $\mathcal{A}$ on Q. Applying the Banach fixed point theorem, we will establish the existence and uniqueness of solution. So our aim is to find a set of parameters T_0, U, L such that the operator $\mathcal{A}$ maps the space Q into itself and this operator is a contraction mapping.

Introduce notation. Choose constants A, F such that

$$|a_{ii}(x,t,u)| \le A,\ i \in \{1,2\}, \quad |f_i(x,t,u)| \le F,\ i \in \{1,2,3\}$$

on $D \overset{def}{=} \Omega_T^- \times \{u \in \mathbb{R}^3 : |u_i - v_i^-| \le 1\} \cup \Omega_T^+ \times \{u \in \mathbb{R}^3 : |u_i - v_i^+| \le 1\}$. Let a_0 be the Lipschitz constant of the functions $(x,t,u) \mapsto a_{ii}(x,t,u)$, $i \in \{1,2\}$, $\tilde{a}_0$ be the Lipschitz constant of $(x,t,u) \mapsto \frac{a_{i3}}{a_{ii}}(x,t,u)$, $i \in \{1,2\}$, f_0 be the Lipschitz constant of $(x,t,u) \mapsto f_i(x,t,u)$, $i \in \{1,2,3\}$ on D, and s_0 be the Lipschitz constant of $t \mapsto s_i(t)$, $i \in \{1,2\}$ on $[0,T]$.

Choose constants $\tilde{A}$, B such that

$$\left|\frac{a_{i3}}{a_{ii}}(x,t,u)\right| \le \tilde{A}, \quad \left|\frac{1}{a_{ii}(x,t,u)}\right| \le B,\ i \in \{1,2\}$$

on $\Omega_{T_0}^- \times \{u \in \mathbb{R}^3 : |u_i - v_i^-| \le U\} \cup \Omega_{T_0}^+ \times \{u \in \mathbb{R}^3 : |u_i - v_i^+| \le U\}$. Suppose k_0 is the Lipschitz constant of the functions $(t,u_1,u_2) \mapsto K_1^+(t,u_1,u_2)$, $(t,u_1,u_2) \mapsto K_2^-(t,u_1,u_2)$, $(t,u_1,u_2) \mapsto K_3^+(t,u_1,u_2)$, $(t,u_1,u_2) \mapsto K_3^-(t,u_1,u_2)$ on $[0,T_0] \times \{(u_1,u_2) \in \mathbb{R}^2 : |u_1 - v_1^-| \le U, |u_2 - v_2^+| \le U\}$, and also the Lipschitz constant of the functions $(t,u_1,u_2,u_3,\hat{u}_3) \mapsto K_1^-(t,u_1,u_2,u_3,\hat{u}_3)$, $(t,u_1,u_2,u_3,\hat{u}_3) \mapsto K_2^+(t,u_1,u_2,u_3,\hat{u}_3)$ on $[0,T_0] \times \{(u_1,u_2,u_3,\hat{u}_3) \in \mathbb{R}^4 : |u_1 - v_1^+| \le U,\ |u_2 - v_2^-| \le U,\ |u_3 - v_3^-| \le U,\ |\hat{u}_3 - v_3^+| \le U\}$.

By decreasing T_0, U if necessary, taking into account the conditions of Theorem 1, we may assume that $a_{ii}(x,t,u) \neq 0$, $i \in \{1,2\}$, and $\tilde{A}$, k_0 are sufficiently small (their smallness will be specified in the sequel).

Let us formulate a few auxiliary estimates as lemmas, which are similar to thus used in [1, 2, 6].

Lemma 1. *Let u, u^1, $u^2 \in Q$, $i \in \{1,2\}$. Then the following inequalities hold:*

1) $|\varphi_i[u](\tau;x_1,t) - \varphi_i[u](\tau;x_2,t)| \le e^{a_0(1+L)T_0}|x_1 - x_2|$ *for all* (x_1,t), $(x_2,t) \in \Omega_{T_0}^-$;

2) $|\varphi_i[u](\tau;x,t_1) - \varphi_i[u](\tau;x,t_2))| \le e^{a_0(1+L)T_0}A|t_1 - t_2|$ *for all* (x,t_1), $(x,t_2) \in \Omega_{T_0}^-$;

3) $|\varphi_i[u^1](\tau;x,t) - \varphi_i[u^2](\tau;x,t)| \le e^{a_0(1+L)T_0}a_0T_0\rho(u^1,u^2)$ *for all* $(x,t) \in \Omega_{T_0}^-$.

The same estimates are true in the domain $\Omega_{T_0}^+$.

Lemma 2. *Let u, u^1, $u^2 \in Q$, $i \in \{1,2\}$. Then the following inequalities hold:*

1) $|\chi_i[u](x_1,t) - \chi_i[u](x_2,t)| \le e^{a_0(1+L)T_0}B|x_1 - x_2|$ *for all* (x_1,t), $(x_2,t) \in \Omega_{T_0}^-$;

2) $|\chi_i[u](x,t_1) - \chi_i[u](x,t_2)| \le e^{a_0(1+L)T_0}AB|t_1 - t_2|$ *for all* (x,t_1), $(x,t_2) \in \Omega_{T_0}^-$;

3) $|\chi_i[u^1](x,t) - \chi_i[u^2](x,t)| \le e^{a_0(1+L)T_0}a_0BT_0\rho(u^1,u^2)$ *for all* $(x,t) \in \Omega_{T_0}^-$.

The same estimates are true in the domain $\Omega_{T_0}^+$.

Lemma 3. *Suppose that* $\tau \mapsto \varphi(\tau)$, $\tau \in [a,b]$ *is a smooth function and* $|\varphi'(\tau)| \le A$. *Moreover, assume that the function* $(x,t) \mapsto u(x,t)$ *is Lipschitz in both arguments with a constant* L *and also the function* $(x,t,u) \mapsto g(x,t,u)$ *is Lipschitz in all arguments with a constant* g_0. *Then, on* $[a,b]$, *the functions* $\tau \mapsto u(\varphi(\tau),\tau)$, $\tau \mapsto g(\varphi(\tau),\tau,u(\varphi(\tau),\tau))$ *are almost everywhere differentiable, their derivatives are integrable, and the following inequalities hold almost everywhere:*

1) $\left|\dfrac{du(\varphi(\tau),\tau)}{d\tau}\right| \le L(A+1)$;

2) $\left|\dfrac{dg\big(\varphi(\tau),\tau,u(\varphi(\tau),\tau)\big)}{d\tau}\right| \le g_0(L+1)(A+1)$.

First let us prove that there exists a set of parameters such that the operator $\mathcal{A}$ maps the space Q into itself. For any $u \in Q$, using an estimate

$$|J_1[u](x,t) - J_1[u](-0,0)| \le k_0(1+4L)T_0, \quad (x,t) \in \Omega^-_{T_0},$$

along with an estimate

$$|J_1[u](x,t) - J_1[u](+0,0)| \le k_0(1+2L(s_0+1))T_0, \quad (x,t) \in \Omega^+_{T_0},$$

we obtain

$$\begin{aligned} |\mathcal{A}_1[u](x,t) - v_1^-| &= |J_1[u](x,t) - J_1[u](-0,0)| + \\ &+ \left| \int\limits_{\chi_1[u](x,t)}^{t} \frac{a_{13}}{a_{11}}\Big(\varphi_1[u](\tau;x,t),\tau,u(\varphi_1[u](\tau;x,t),\tau)\Big) \frac{du_3(\varphi_1[u](\tau;x,t),\tau)}{d\tau} d\tau \right| + \\ &+ \left| \int\limits_{\chi_1[u](x,t)}^{t} f_1\Big(\varphi_1[u](\tau;x,t),\tau,u(\varphi_1[u](\tau;x,t),\tau)\Big) d\tau \right| \le \\ &\le \Big(k_0(1+2Ls_0+4L) + \tilde{A}L(A+1) + F\Big)T_0, \quad (x,t) \in \Omega^-_{T_0}. \end{aligned}$$

Similarly, we have

$$|\mathcal{A}_1[u](x,t) - v_1^+| \le \Big(k_0(1+2Ls_0+4L) + \tilde{A}L(A+1) + F\Big)T_0, \quad (x,t) \in \Omega^+_{T_0}.$$

Thus, whenever the inequality

$$\Big(k_0(1+2Ls_0+4L) + \tilde{A}L(A+1) + F\Big)T_0 \le U \tag{13}$$

holds, we derive that $|\mathcal{A}_1[u](x,t) - v_1^-| \le U$ for every point $(x,t) \in \Omega^-_{T_0}$ and also $|\mathcal{A}_1[u](x,t) - v_1^+| \le U$ for $(x,t) \in \Omega^+_{T_0}$. Reasoning as above, we obtain similar estimates for $\mathcal{A}_2[u]$, $\mathcal{A}_3[u]$.

Further, let us show that $\mathcal{A}_1[u]$ is Lipschitz in x on each of the domains Ω_T^-, Ω_T^+ whenever $u \in Q$. For every pair of points (x_1,t), $(x_2,t) \in \Omega_{T_0}^-$, using Lemma 2, we obtain (here for brevity we write τ_j instead of $\chi_1[u](x_j,t)$ and, for any function f, we write $\Delta_j f(x_j)$ instead of $f(x_1) - f(x_2)$)

$$|\Delta_j J_1[u](x_j,t)| = |\Delta_j K_1^-(\tau_j, u_1(+0,\tau_j), u_2(-0,\tau_j), u_3(-0,\tau_j), u_3(+0,\tau_j))| \le$$
$$\le k_0(1+4L)|\tau_1 - \tau_2| \le k_0(1+4L)e^{a_0(1+L)T_0}B|x_1 - x_2|.$$

Similarly, for every pair of points (x_1,t), $(x_2,t) \in \Omega_{T_0}^+$, we have an estimate

$$|\Delta_j J_1[u](x_j,t)| = |\Delta_j K_1^+(\tau_j, u_1(s_1(\tau_j),\tau_j), u_2(s_2(\tau_j),\tau_j))| \le$$
$$\le k_0(1+2L(s_0+1))e^{a_0(1+L)T_0}B|x_1 - x_2|.$$

For clarity, assume that $\chi_1[u](x_1,t) > \chi_1[u](x_2,t)$. Using Lemmas 1, 2, 3, and integration by parts, for every pair of points (x_1,t), $(x_2,t) \in \Omega_{T_0}^-$ (or (x_1,t), $(x_2,t) \in \Omega_{T_0}^+$), we obtain

$$\left|\Delta_j \int\limits_{\chi_1[u](x_j,t)}^{t} \frac{a_{13}}{a_{11}}\Big(\varphi_1[u](\tau;x_j,t),\tau,u(\varphi_1[u](\tau;x_j,t),\tau)\Big)\frac{du_3(\varphi_1[u](\tau;x_j,t),\tau)}{d\tau}d\tau\right| \le$$
$$\le \left|\int\limits_{\chi_1[u](x_1,t)}^{t} \Delta_j\frac{a_{13}}{a_{11}}\Big(\varphi_1[u](\tau;x_j,t),\tau,u(\varphi_1[u](\tau;x_j,t),\tau)\Big)\frac{du_3(\varphi_1[u](\tau;x_1,t),\tau)}{d\tau}d\tau\right| +$$
$$+ \left|\frac{a_{13}}{a_{11}}\Big(\varphi_1[u](\tau;x_2,t),\tau,u(\varphi_1[u](\tau;x_2,t),\tau)\Big)\Delta_j u_3(\varphi_1[u](\tau;x_j,t),\tau)\Big|_{\chi_1[u](x_1,t)}^{t}\right| +$$
$$+ \left|\int\limits_{\chi_1[u](x_1,t)}^{t} \frac{d\frac{a_{13}}{a_{11}}(\varphi_1[u](\tau;x_2,t),\tau,u(\varphi_1[u](\tau;x_2,t),\tau))}{d\tau}\Delta_j u_3(\varphi_1[u](\tau;x_j,t),\tau)d\tau\right| +$$
$$+ \left|\int\limits_{\chi_1[u](x_2,t)}^{\chi_1[u](x_1,t)} \frac{a_{13}}{a_{11}}\Big(\varphi_1[u](\tau;x_2,t),\tau,u(\varphi_1[u](\tau;x_2,t),\tau)\Big)\frac{du_3(\varphi_1[u](\tau;x_2,t),\tau)}{d\tau}d\tau\right| \le$$
$$\le \Big(\tilde{a}_0(L+1)e^{a_0(1+L)T_0}L(A+1)T_0 + \tilde{A}L + \tilde{A}Le^{a_0(1+L)T_0} +$$
$$+ \tilde{a}_0(L+1)(A+1)Le^{a_0(1+L)T_0}T_0 + \tilde{A}L(A+1)e^{a_0(1+L)T_0}B\Big)|x_1 - x_2|.$$

Similarly, we have

$$\left|\Delta_j \int\limits_{\chi_1[u](x_j,t)}^{t} f_1\Big(\varphi_1[u](\tau;x_j,t),\tau,u(\varphi_1[u](\tau;x_j,t),\tau)\Big)d\tau\right| \le$$

$$\le \Big(f_0(L+1)e^{a_0(1+L)T_0}T_0 + Fe^{a_0(1+L)T_0}B\Big)|x_1-x_2|.$$

Finally, using the inequalities above, we obtain an estimate

$$\begin{aligned}|\Delta_j\mathcal{A}_1[u](x_j,t)| \le \Big(&k_0(1+4L+2Ls_0)e^{a_0(1+L)T_0}B+\\ &+\tilde{a}_0(L+1)e^{a_0(1+L)T_0}L(A+1)T_0+\tilde{A}L+\tilde{A}Le^{a_0(1+L)T_0}+\\ &+\tilde{a}_0(L+1)(A+1)Le^{a_0(1+L)T_0}T_0+\tilde{A}L(A+1)e^{a_0(1+L)T_0}B+\\ &+f_0(L+1)e^{a_0(1+L)T_0}T_0+Fe^{a_0(1+L)T_0}B\Big)|x_1-x_2|.\end{aligned}$$

Assuming that the inequality

$$(L^2+L)T_0 \le 1 \tag{14}$$

holds, we can rewrite the previous estimate as

$$|\Delta_j\mathcal{A}_1[u](x_j,t)| \le (C_1 + L(k_0C_2+\tilde{A}C_3))|x_1-x_2|,$$

where C_1, C_2, C_3 are some positive constants, determined by initial data. Note that $\mathcal{A}_2[u]$ and $\mathcal{A}_3[u]$ satisfy the same estimate.

Applying the previous inequality, we show that $\mathcal{A}_1[u]$ is Lipschitz in t whenever $u \in Q$. By definition, put $x_3 \overset{def}{=} \varphi_i[u](t_1;x,t_2)$. For every pair of points (x,t_1), $(x,t_2) \in \Omega^-_{T_0}$ (or (x,t_1), $(x,t_2) \in \Omega^+_{T_0}$), using Lemma 3, we obtain

$$\begin{aligned}|\Delta_j\mathcal{A}_1[u](x,t_j)| &= |\mathcal{A}_1[u](x,t_1)-\mathcal{A}_1[u](x_3,t_1)| + |\mathcal{A}_1[u](x_3,t_1)-\mathcal{A}_1[u](x,t_2)| \le\\ &\le |\mathcal{A}_1[u](x,t_1)-\mathcal{A}_1[u](x_3,t_1)|+\\ &+\Bigg|\int_{t_1}^{t_2}\frac{a_{13}}{a_{11}}\Big(\varphi_1[u](\tau;x,t_2),\tau,u(\varphi_1[u](\tau;x,t_2),\tau)\Big)\frac{du_3(\varphi_1[u](\tau;x,t_2),\tau)}{d\tau}d\tau+\\ &+\int_{t_1}^{t_2} f_1(\varphi_1[u](\tau;x,t_2),\tau,u(\varphi_1[u](\tau;x,t_2),\tau))d\tau\Bigg| \le\\ &\le (C_1+L(k_0C_2+\tilde{A}C_3))|x-x_3| + (\tilde{A}L(A+1)+F)|t_1-t_2| \le\\ &\le \Big((C_1+L(k_0C_2+\tilde{A}C_3))A+\tilde{A}L(A+1)+F\Big)|t_1-t_2|.\end{aligned}$$

The same estimate holds for $\mathcal{A}_2[u]$ and $\mathcal{A}_3[u]$.

Thus, $\mathcal{A}_i[u]$, $i \in \{1,2,3\}$, are Lipschitz continuous on each of the domains Ω^-_T, Ω^+_T as the inequality

$$|\mathcal{A}_i[u](x_1,t_1)-\mathcal{A}_i[u](x_2,t_2)| \le (C_4+L(k_0C_5+\tilde{A}C_6))(|x_1-x_2|+|t_1-t_2|)$$

holds for all (x_1,t_1), $(x_2,t_2) \in \Omega^-_{T_0}$ (or (x_1,t_1), $(x_2,t_2) \in \Omega^+_{T_0}$), where constants C_4, C_5, C_6 are determined by initial data. Consequently, for $\mathcal{A}_i[u]$ to satisfy the Lipschitz condition with a constant L, it is sufficient to require that

$$C_4 + L(k_0 C_5 + \tilde{A} C_6) \leq L.$$

The last condition can be rewritten as

$$L \geq \frac{C_4}{1-(k_0 C_5 + \tilde{A} C_6)}, \tag{15}$$

provided that k_0 and $\tilde{A}$ are small enough to satisfy the following inequality:

$$k_0 C_5 + \tilde{A} C_6 < 1. \tag{16}$$

Now let us prove that the operator $\mathcal{A}$ is a contraction on Q, i.e., there is some non-negative real number $0 \leq \kappa < 1$ such that for all u^1, $u^2 \in Q$, $\rho(\mathcal{A}[u^1], \mathcal{A}[u^2]) \leq \kappa\rho(u^1,u^2)$. Assuming that u^1, $u^2 \in Q$, using Lemma 2 for all $(x,t) \in \Omega^-_{T_0}$, we obtain (for short, we write here τ_j instead of $\chi_1[u^j](x,t)$ and, for any functional F, we write $\Delta_j F[u^j]$ instead of $F[u^1] - F[u^2]$)

$$\begin{aligned} |\Delta_j J_1[u^j](x,t)| = |\Delta_j K_1^-(\tau_j, u_1^j(+0,\tau_j), u_2^j(-0,\tau_j), u_3^j(-0,\tau_j), u_3^j(+0,\tau_j))| \leq \\ \leq k_0\Big(4 + (1+4L)e^{a_0(1+L)T_0} a_0 B T_0\Big)\rho(u^1,u^2). \end{aligned}$$

Similarly, for all $(x,t) \in \Omega^+_{T_0}$, we have

$$\begin{aligned} |\Delta_j J_1[u^j](x,t)| = |\Delta_j K_1^+(\tau_j, u_1^j(s_1(\tau_j),\tau_j), u_2^j(s_2(\tau_j),\tau_j))| \leq \\ \leq k_0\Big(2 + (1+2L(s_0+1))e^{a_0(1+L)T_0} a_0 B T_0\Big)\rho(u^1,u^2). \end{aligned}$$

For clarity, assume that $\chi_i[u^1](x,t) > \chi_i[u^2](x,t)$. Using Lemmas 1, 2, 3, and integration by parts, for all $(x,t) \in \Omega^-_{T_0} \cup \Omega^+_{T_0}$, we obtain

$$\begin{aligned} &\left|\Delta_j \int\limits_{\chi_1[u^j](x,t)}^{t} \frac{a_{13}}{a_{11}}\Big(\varphi_1[u^j](\tau;x,t),\tau,u^j(\varphi_1[u^j](\tau;x,t),\tau)\Big)\frac{du_3^j(\varphi_1[u^j](\tau;x,t),\tau)}{d\tau}d\tau\right| \leq \\ &\leq \left|\int\limits_{\chi_1[u^1](x,t)}^{t} \Delta_j\frac{a_{13}}{a_{11}}\Big(\varphi_1[u^j](\tau;x,t),\tau,u^j(\varphi_1[u^j](\tau;x,t),\tau)\Big)\frac{du_3^1(\varphi_1[u^1](\tau;x,t),\tau)}{d\tau}d\tau\right| + \\ &+ \left|\frac{a_{13}}{a_{11}}\Big(\varphi_1[u^2](\tau;x,t),\tau,u^2(\varphi_1[u^2](\tau;x,t),\tau)\Big)\Delta_j u_3^j(\varphi_1[u^j](\tau;x,t),\tau)\Big|_{\chi_1[u^1](x,t)}^{t}\right| + \end{aligned}$$

$$+\left|\int\limits_{\chi_1[u^1](x,t)}^{t}\frac{d\frac{a_{13}}{a_{11}}(\varphi_1[u^2](\tau;x,t),\tau,u^2(\varphi_1[u^2](\tau;x,t),\tau))}{d\tau}\Delta_j u_3^j(\varphi_1[u^j](\tau;x,t),\tau)d\tau\right|+$$

$$+\left|\int\limits_{\chi_1[u^2](x,t)}^{\chi_1[u^1](x,t)}\frac{a_{13}}{a_{11}}\Big(\varphi_1[u^2](\tau;x,t),\tau,u^2(\varphi_1[u^2](\tau;x,t),\tau)\Big)\frac{du_3^2(\varphi_1[u^2](\tau;x,t),\tau)}{d\tau}d\tau\right|\le$$

$$\le\Big(\tilde a_0\Big(1+(1+L)e^{a_0(1+L)T_0}a_0T_0\Big)L(A+1)T_0+\tilde A+\tilde A\Big(1+Le^{a_0(1+L)T_0}a_0T_0\Big)+$$

$$+\Big(1+Le^{a_0(1+L)T_0}a_0T_0\Big)a_0(L+1)(A+1)T_0+\tilde A L(A+1)e^{a_0(1+L)T_0}a_0BT_0\Big)\rho(u^1,u^2).$$

Similarly, we have

$$\left|\Delta_j\int\limits_{\chi_1[u^j](x,t)}^{t}f_1\Big(\varphi_1[u^j](\tau;x,t),\tau,u^j(\varphi_1[u^j](\tau;x,t),\tau)\Big)d\tau\right|\le$$

$$\le\Big(f_0\Big(1+(1+L)e^{a_0(1+L)T_0}a_0T_0\Big)T_0+Fe^{a_0(1+L)T_0}a_0BT_0\Big)\rho(u^1,u^2).$$

Finally, using the inequalities above, we obtain an estimate

$$|\Delta_j\mathcal{A}_1[u^j](x,t)|\le\Big(4k_0+k_0(1+4L(s_0+1))e^{a_0(1+L)T_0}a_0BT_0+$$

$$+\tilde a_0\Big(1+(1+L)e^{a_0(1+L)T_0}a_0T_0\Big)L(A+1)T_0+\tilde A+\tilde A\Big(1+Le^{a_0(1+L)T_0}a_0T_0\Big)+$$

$$+\Big(1+Le^{a_0(1+L)T_0}a_0T_0\Big)a_0(L+1)(A+1)T_0+\tilde A L(A+1)e^{a_0(1+L)T_0}a_0BT_0+$$

$$+f_0\Big(1+(1+L)e^{a_0(1+L)T_0}a_0T_0\Big)T_0+Fe^{a_0(1+L)T_0}a_0BT_0\Big)\rho(u^1,u^2).$$

Applying assumption (14), we can rewrite the previous estimate as

$$|\Delta_j\mathcal{A}_1[u^j](x,t)|\le(4k_0+2\tilde A+C_7T_0)\rho(u^1,u^2),$$

where constants C_7, C_8 are determined by initial data. Reasoning as above, we obtain the same inequalities for $\mathcal{A}_2[u]$, $\mathcal{A}_3[u]$. Whence we derive an estimate

$$\rho(\mathcal{A}[u^1],\mathcal{A}[u^2])\le(4k_0+2\tilde A+C_7T_0)\rho(u^1,u^2).$$

Therefore, $\mathcal{A}$ is a contraction mapping if

$$C_7T_0<1-4k_0-2\tilde A, \tag{17}$$

provided that k_0 and $\tilde A$ are small enough to satisfy the following inequality:

$$4k_0+2\tilde A<1. \tag{18}$$

Now, suppose k_0 and $\tilde{A}$ are sufficiently small to satisfy inequalities (16), (18), L is large according to (15), and T_0 is small enough to satisfy the inequalities (13), (14), (17). Then the operator $\mathcal{A}$ maps the space Q into itself and, in addition, this operator is a contraction mapping. In this case, by the Banach fixed-point theorem, the operator $\mathcal{A}$ admits a unique fixed point in Q. This fixed point is a generalized solution to the problem (2), (3), (5), (6).◄

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