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On the Noetherness of the Riemann Problem in Generalized Weighted Hardy Classes

T.I. Najafov*, N.P. Nasibova

Abstract. The Riemann boundary value problem of the theory of analytic functions in generalized weighted Hardy classes is considered. In the case where the coefficient of this problem is a piecewise continuous, the Noetherness of this problem is studied. The general solution for homogeneous and non-homogeneous problem in these classes is constructed.

Key Words and Phrases: Riemann boundary value problem, generalized Hardy classes, weighted space, Noetherness

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1. Introduction

When solving many equations of mixed type and elliptic equations by Fourier method (see e.g., [1-4]), there appear trigonometric systems with a linear or piecewise-linear phase of the following form

$$\left\{e^{i(nt+\alpha(t)signn)}\right\}_{n\in Z}, \tag{1}$$

$$\{\sin(nt+\beta(t))\}_{n\in N},\ \{\cos(nt+\beta(t))\}_{n\in Z_+}, \tag{2}$$

where N is a set of all positive integers, $Z_+ = \{0\}\bigcup N$, $Z = (-Z_+)\bigcup N$. Justification of this method requires the study frame properties (completeness, minimality, basicity, an atomic decomposition and etc.) of these systems in different functional spaces. These problems with respect to the systems of the form (1), (2) have been well studied in Lebesgue and Sobolev spaces [see 5-16].

Recently, in connection with the application there arose a great interest in studying various problems in Lebesgue and Sobolev spaces with a variable summability exponent. More details with respect to the related issues can be found in [17]. It should be noted that one of the methods (in most cases the only possible) for studying the basis properties of systems (1), (2) is a method of the boundary value problems of the theory of analytic functions. This method dates back to A.V.Bitzadze [18]. To usage of this method in the study

*Corresponding author.

of basis properties of systems (1), (2) in generalized weighted Lebesgue spaces requires the study of Noetherness of corresponding Riemann problem in generalized weighted Hardy classes. It should be noted that similar questions were earlier considered in [19-22].

In the present paper, the homogeneous and non-homogeneous Riemann boundary value problems in the generalized weighted Hardy classes $H^{\pm}_{p(\cdot),\rho}$ are considered. Under certain conditions on the coefficients of the problem and the weight function, the general solution of these problems is constructed. Note that in the case of $p(x) \equiv const$, these problems have been well studied. "Weightless" case has been treated in [23]. In the case of variable $p(x)$ these problems have been studied by Kokilashvili and Samko [24] in a different setting.

2. Needful information

We will use the usual notations. C will denote the field of complex numbers; $(\bar{\cdot})$ will be the complex conjugate; $\omega \equiv \{z \in C : |z| < 1\}$.

Let $p : [-\pi, \pi] \to [1, +\infty)$ be some Lebesgue-measurable function. By $\mathscr{L}_0$ we denote the class of all functions measurable on $[-\pi, \pi]$ with respect to Lebesgue measure. Denote

$$I_p(f) \stackrel{def}{\equiv} \int_{-\pi}^{\pi} |f(t)|^{p(t)}\, dt.$$

Let

$$\mathscr{L} \equiv \{f \in \mathscr{L}_0 : I_p(f) < +\infty\}.$$

With respect to the usual linear operations of addition and multiplication by a number $\mathscr{L}$ is a linear space as $p^+ = \sup\limits_{[-\pi,\pi]} vrai\, p(t) < +\infty$. With respect to the norm

$$\|f\|_{p(\cdot)} \stackrel{def}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

$\mathscr{L}$ is a Banach space, and we denote it by $L_{p(\cdot)}$. Let

$$WL \stackrel{def}{\equiv} \{p : p(-\pi) = p(\pi); \exists C > 0, \quad \forall t_1, t_2 \in [-\pi, \pi] : |t_1 - t_2| \le \tfrac{1}{2} \Rightarrow \\ \Rightarrow |p(t_1) - p(t_2)| \le \tfrac{C}{-\ln|t_1 - t_2|}\Big\}.$$

Throughout this paper, $q(t)$ will denote the conjugate of a function $p(t)$: $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Denote $p^- = \inf\limits_{[-\pi,\pi]} vrai\, p(t)$. The following generalized Hölder inequality is true

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|\, dt \le c\left(p^-; p^+\right) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)},$$

where $c(p^-; p^+) = 1 + \frac{1}{p^-} - \frac{1}{p^+}$. Directly from the definition we get the property, which will be used in the sequel.

Property A. *If* $|f(t)| \le |g(t)|$ *a.e. on* $(-\pi, \pi)$, *then* $\|f\|_{p(\cdot)} \le \|g\|_{p(\cdot)}$.

We will need the following easy-to-prove statement.

Statement 1. *Let $p \in WL,\ p(t) > 0, \forall t \in [-\pi, \pi]$; $\{\alpha_i\}_0^m \subset R$. The weighted function*

$$\rho(t) = |t|^{\alpha_0} \prod_{i=1}^{m} |t - \tau_i|^{\alpha_i},$$

belongs to the space $L_{p(\cdot)}$, if

$$\alpha_i > -\frac{1}{p(\tau_i)}, \forall i = \overline{0, m};$$

where $-\pi = \tau_1 < \tau_2 < ... < \tau_m = \pi$, $\tau_0 = 0$.

The following facts play an important role in obtaining our main results.

Property B [17]. *If $p(t) : 1 < p^- \leq p^+ < +\infty$, then the class $C_0^\infty(-\pi, \pi)$ (class of finite and indefinitely differentiable functions) is everywhere dense in $L_{p(\cdot)}$.*

By S we denote the singular integral

$$Sf = \frac{1}{2\pi i} \int_\Gamma \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \Gamma,$$

where $\Gamma \subset C$ is some piecewise Hölder curve on C. Define weight class $L_{p(\cdot),\rho(\cdot)}$:

$$L_{p(\cdot),\rho(\cdot)} \overset{def}{\equiv} \left\{ f : \ \rho f \in L_{p(\cdot)} \right\},$$

furnished with the norm $\|f\|_{p(\cdot),\rho(\cdot)} \overset{def}{\equiv} \|\rho f\|_{p(\cdot)}$. The validity of the following statement is established in [22].

Statement 2. *[22]. Let $p \in WL$, $1 < p^-$. Then, singular operator S is acting boundedly from $L_{p(\cdot),\rho(\cdot)}$ to $L_{p(\cdot),\rho(\cdot)}$ if and only if*

$$-\frac{1}{p(\tau_k)} < \alpha_k < \frac{1}{q(\tau_k)}, \quad k = \overline{0, m}. \tag{3}$$

Define the generalized weighted Hardy classes.

By $H_{p_0}^+$ we denote the usual Hardy class, where $p_0 \in [1, +\infty)$ is some number. Define $H_{p(\cdot),\rho}^{\pm} \equiv \left\{ f \in H_1^+ : f^+ \in L_{p(\cdot),\rho}(\partial\omega) \right\}$, f^+ are non-tangential boundary values of f on $\partial\omega$. The following theorem is proved in [25].

Theorem 1. *Let $p \in WL, p^- > 1$, and let the inequalities (3) be satisfied. If $F \in H_{p(\cdot),\rho}^+$, then $\exists f \in L_{p(\cdot),\rho}$:*

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_z(t) f(t) dt, \tag{4}$$

where $K_z(t) \equiv \frac{1}{1 - ze^{-it}}$ is the Poisson kernel. Vice versa, if $f \in L_{p(\cdot),\rho}$, then function F, defined by (4), belongs to the class $H_{p(\cdot),\rho}^+$.

The weighted Hardy class ${}_mH^-_{p(\cdot),\rho}$ of functions analytic in $C\backslash\bar{\omega}$ $(\bar{\omega}=\omega\bigcup\partial\omega)$ with the orders $m_0\le m$ at infinity is defined similarly to the classical one. Let $f\left(z\right)$ be an analytic function in $C\backslash\bar{\omega}$ of finite order $m_0\le m$ at infinity, i.e.

$$f(z)=f_1(z)+f_2(z),$$

where $f_1(z)$ is a polynomial of degree $m_0\le m$ $(f_1(z)\equiv 0,\ m_0<0)$, $f_2(z)$ is a regular part of Laurent series expansion of $f(z)$ in the neighborhood of an infinitely remote point. If the function $\varphi(z)\equiv\overline{f_2\left(\frac{1}{\bar{z}}\right)}$ belongs to the class $H^+_{p(\cdot),\rho}$, then we will say that the function $f\left(z\right)$ belongs to the class ${}_mH^-_{p(\cdot),\rho}$.

The validity of the following theorem is proved just like in the classical case.

Theorem 2. *Let $p\in WL, p^->1$, and let the inequalities* (3) *be satisfied. If $f\in H^+_{p(\cdot),\rho}$, then*

$$\left\|f(re^{it})-f^+(e^{it})\right\|_{p(\cdot),\rho}\to 0,\ r\to 1-0,$$

where f^+ are non-tangential boundary values of f on $\partial\omega$.

We also have

Theorem 3. *Let $p\in WL, p^->1$, and let the inequalities* (3) *be satisfied. If $f\in {}_mH^-_{p(\cdot),\rho}$, then*

$$\left\|f(re^{it})-f^-(e^{it})\right\|_{p(\cdot),\rho}\to 0,\ r\to 1+0,$$

where $f^-$are non-tangential boundary values of f on $\partial\omega$ from the outside of ω.

Let us show the validity of an analogue of the classical Smirnov theorem.

Assume that $p\in WL, p^->1$, and let the inequality (1) be fulfilled. Let $u\in H_1^+$ and $u^+\in L_{p(\cdot),\rho}$, where u^+ is a non-tangential boundary value of u on $\partial\omega$. Then it is known that $\exists\, f\in L_1(\partial\omega)$:

$$u(z)=\frac{1}{2\pi i}\int_{\partial\omega}\frac{f(\tau)}{\tau-z}d\tau.$$

Consequently, $u(re^{i\theta})\to f(e^{i\theta})$, a.e. on $(-\pi,\pi)$ as $r\to 1-0$. Hence it directly follows that $f\in L_{p(\cdot),\rho}$. Then by Theorem 1 we obtain $u\in H^+_{p(\cdot),\rho}$. Thus, the following theorem is true.

Theorem 4. *Let $p\in WL, p^->1$, and let the inequalities* (3) *be satisfied. If $u\in H_1^+$ and $u^+\in L_{p(\cdot),\rho}$, then $u\in H^+_{p(\cdot),\rho}$.*

3. The general solution of the homogeneous problem

Consider the following homogeneous Riemann problem in $H^{+}_{p(\cdot),\rho} \times {}_{m_0}H^{-}_{p(\cdot),\rho}$ classes

$$F^{+}(\tau) - G(\tau)F^{-}(\tau) = 0,\ \tau \in \partial\omega. \tag{5}$$

By the solution of problem (5) we mean a pair of analytic functions

$$\left(F^{+}(z); F^{-}(z)\right) \in H^{+}_{p(\cdot),\rho} \times {}_{m_0}H^{-}_{p(\cdot),\rho},$$

boundary values of which satisfy the relation (5) almost everywhere. Introduce the following functions $X_i\left(z\right), i = 1,2$, which are analytic inside (with the "+" sign) and outside (with the "-" sign) the unit circle, respectively:

$$X_1\left(z\right) \equiv \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\left|G\left(e^{it}\right)\right| \frac{e^{it}+z}{e^{it}-z}dt\right\},$$

$$X_2\left(z\right) \equiv \exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi} \theta\left(t\right) \frac{e^{it}+z}{e^{it}-z}dt\right\},$$

where $\theta\left(t\right) \equiv \arg G\left(e^{it}\right)$. Define

$$Z_i\left(z\right) \equiv \begin{cases} X_i\left(z\right), & |z| < 1, \\ \left[X_i\left(z\right)\right]^{-1}, & |z| > 1, \quad i = 1,2. \end{cases}$$

Sokhotski-Plemelj formulas yield

$$\left|G\left(e^{it}\right)\right| = \frac{Z_1^{+}\left(e^{it}\right)}{Z_1^{-}\left(e^{it}\right)},\ e^{i\theta(t)} = \frac{Z_2^{+}\left(e^{it}\right)}{Z_2^{-}\left(e^{it}\right)}.$$

Assume

$$Z^{\pm}\left(z\right) \equiv Z_1^{\pm}\left(z\right) Z_2^{\pm}\left(z\right).$$

We have

$$Z^{+}\left(\tau\right) - G\left(\tau\right) Z^{-}\left(\tau\right) = 0, \tau \in \partial\omega. \tag{6}$$

Introduce the piecewise analytic function

$$Z\left(z\right) \equiv \begin{cases} Z^{+}\left(z\right), & |z| < 1, \\ Z^{-}\left(z\right), & |z| > 1. \end{cases}$$

Following the classics, we call function $Z\left(z\right)$ the canonical solution of the problem (5). Substituting the expression (6) for $G\left(\tau\right)$ in (5), we obtain

$$\frac{F^{+}\left(\tau\right)}{Z^{+}\left(\tau\right)} = \frac{F^{-}\left(\tau\right)}{Z^{-}\left(\tau\right)}, \tau \in \partial\omega.$$

Let $\Phi^{\pm}(z) \equiv \frac{F^{\pm}(z)}{Z^{\pm}(z)}$, and define the piecewise analytic function

$$\Phi(z) \equiv \begin{cases} \Phi^{+}(z), & |z| < 1, \\ \Phi^{-}(z), & |z| > 1. \end{cases}$$

It is not difficult to see that the function $Z(z)$ has neither poles nor zeros for $z \notin \partial\omega$. Therefore, functions $\Phi(z)$ and $F(z)$ have the same order at infinity. The results of [23] imply directly that the function $\Phi(z)$ belongs to the Hardy class $H_{\delta}^{\pm}$ for sufficiently small values of $\delta > 0$. Let us show that $\Phi(z) \in H_{1}^{\pm}$. To do so, it suffices to prove that $\Phi^{\pm}(\tau) \in L_1(\partial\omega)$, because the rest will immediately follow from the Smirnov theorem [23].

We will suppose that the coefficient $G(\tau)$ satisfies the following conditions:

i) $G^{\pm 1} \in L_{\infty}(\partial\omega)$;

ii) $\theta(t) \equiv \arg G(e^{it})$ *is a piecewise Hölder function on* $[-\pi, \pi]$.

Let $\{s_k\}_1^r \, : \, -\pi < s_1 < ...s_r < \pi-$ be the points of discontinuity of the function $\theta(t)$ and

$$\{h_k\}_1^r \, : \, h_k = \theta(s_k + 0) - \theta(s_k - 0), k = \overline{1, r};$$

be the corresponding jumps of $\theta(\cdot)$ at these points. Denote

$$\theta(t) \equiv \theta_0(t) + \theta_1(t), \; t \in [-\pi, \pi],$$

where $\theta_0(\cdot)$ is the continuous part , and $\theta_1(\cdot)$ is the jump function defined by the expression

$$\theta(-\pi) = 0, \; \theta(t) = \sum_{k:0<s_k<t} h_k, \, t \in [-\pi, \pi].$$

Denote

$$h_0 = \theta(-\pi) - \theta(\pi), h_0^{(0)} = \theta_0(\pi) - \theta_0(-\pi).$$

Let

$$u_0(t) \equiv \left\{ \sin\left|\frac{t-\pi}{2}\right| \right\}^{-\frac{h_0^{(0)}}{2\pi}} \exp\left\{ -\frac{1}{4\pi} \int_{-\pi}^{\pi} \theta_0(\tau)\, ctg\frac{t-\tau}{2} d\tau \right\}.$$

Assume

$$u(t) = \prod_{k=0}^{r} \left\{ \sin\left|\frac{t-s_k}{2}\right| \right\}^{\frac{h_k}{2\pi}}, \text{ where } s_0 = \pi.$$

As is known (see [23]), the boundary values $\left|Z_2^{-}(\tau)\right|$ are expressed by the formula

$$\left|Z_2^{-}(e^{it})\right| = u_0(t) [u(t)]^{-1} \left\{ \sin\left|\frac{t-\pi}{2}\right| \right\}^{-\frac{h_0}{2\pi}},$$

i.e.

$$\left|Z_2^{-}(e^{it})\right| = u_0(t) \prod_{k=0}^{r} \left| \sin\frac{t-s_k}{2} \right|^{-\frac{h_k}{2\pi}}.$$

It follows directly from the Sokhotski-Plemelj formula that

$$\sup_{(-\pi,\pi)} vrai \left\{ \left|Z_1^-\left(e^{it}\right)\right|^{\pm 1} \right\} < +\infty.$$

Thus, the following representation is true for $\left|Z^-\left(e^{it}\right)\right|^{-1}$:

$$\left|Z^-\left(e^{it}\right)\right|^{-1} = \left|Z_1^-\left(e^{it}\right)\right|^{-1} \left|u_0\left(t\right)\right|^{-1} \prod_{k=0}^{r} \left|\sin\frac{t-s_k}{2}\right|^{\frac{h_k}{2\pi}}. \tag{7}$$

By the definition of solution, we have $F^-(z) \in H^-_{p(\cdot),\rho}$. Consequently, $F^-(\tau) \in L_{p(\cdot),\rho}(\partial\omega)$. Therefore, if $\left|Z^-(\tau)\right|^{-1} \in L_{q(\cdot),\rho^{-1}}(\partial\omega)$, then we obtain directly from the Hölder inequality that $\Phi^-(\tau) \in L_1(\partial\omega)$.

We will need the following easy-to-prove lemma that follows directly from definition of the weighted space $L_{p(\cdot),\rho}$.

Lemma 1. *Let $p \in C\left[-\pi,\pi\right]$ and $p(t) > 0,\ \forall t \in \left[-\pi,\pi\right]$. Then the function $\xi(t) = |t-c|^{\alpha}$ belongs to $L_{p(\cdot),\rho}$, if $\alpha > -\frac{1}{p(c)}$, as $c \neq \tau_k,\ \forall\, k=\overline{1,m}$, and $\alpha + \alpha_{k_0} > -\frac{1}{p(c)}$, as $c = \tau_{k_0}$.*

Represent the product $\left|Z^-\rho\right|^{-1}$ in the following form

$$\left|Z^-\rho\right|^{-1} = \left|Z_1^-\right|^{-1} \left|u_0\right|^{-1} \prod_{k=0}^{l} \left|t-t_k\right|^{\beta_k},$$

where

$$\{t_k\}_{k=0}^{l} \equiv \{\tau_k\}_{k=1}^{m} \bigcup \{s_k\}_{k=0}^{r},$$

and β_k is defined by the relation

$$\beta_k = -\sum_{i=1}^{m} \alpha_i \chi_{\{t_k\}}\left(\tau_i\right) + \frac{1}{2\pi}\sum_{i=0}^{r} h_i \chi_{\{t_k\}}\left(s_i\right),\ \ k=\overline{0,l}. \tag{8}$$

Taking into account Lemma 1, we obtain that if the inequalities

$$\beta_k > -\frac{1}{q(t_k)}\ ,\ \ k=\overline{0,r}\ , \tag{9}$$

are true, then the product $\left|Z^-\rho\right|^{-1}$ belongs to the space $L_{q(\cdot)}$, i.e. $\left|Z^-\right|^{-1} \in L_{q(\cdot),\rho^{-1}}$. So, if the inequalities (9) are true, then the function $\Phi(z)$ belongs to classes $H_1^{\pm}$. Consequently, according to the results of [23], $\Phi(z)$ is a polynomial $P_{m_0}(z)$ of order $m_0 \le m$. Thus, $F^-(z) = P_{m_0}(z)Z^-(z)$. Let's find out under which conditions the function $F^-(z)$ belongs to the space $H^-_{p(\cdot),\rho}$. We have

$$\left|Z^-\rho\right| = \left|Z_1\right|\left|u_0\right| \prod_{k=0}^{l} \left|t-t_k\right|^{-\beta_k}.$$

Consequently, if the inequalities

$$\beta_k < \frac{1}{p(t_k)}\ ,\ k = \overline{0, r}\ ,$$

are true, then it is clear that $F^{-}(\tau) \in L_{p(\cdot),\rho}$, and hence $F^{-} \in {}_{m_0}H^{-}_{p(\cdot),\rho}$. So, if the inequalities

$$-\frac{1}{q(t_k)} < \beta_k < \frac{1}{p\,(t_k)},\ k = \overline{0, r}, \tag{10}$$

are true, then the general solution of the homogeneous problem

$$F_0^{+}(\tau) = G(\tau)\, F_0^{-}(\tau)\,, \quad \tau \in \partial\omega,$$

in classes $H^{+}_{p(\cdot),\,\rho} \times_{m_0} H^{-}_{p(\cdot),\,\rho}$ can be represented as $F_0(z) = P_{m_0}(z)\, Z(z)$, where $P_{m_0}(z)$ is an arbitrary polynomial of order $k \leq m_0$. Thus, the following theorem is true.

Theorem 5. *Let* $\{\beta_k\}_1^r$ *be defined by* (8) *and the inequality* (10) *be satisfied. If the inequality*

$$-\frac{1}{p\,(\tau_k)} < \alpha_k < \frac{1}{q\,(\tau_k)}, \quad k = \overline{1, m},$$

is fulfilled, then the general solution of the homogeneous Riemann problem (5) *in classes* $H^{+}_{p(\cdot),\,\rho} \times_{m_0} H^{-}_{p(\cdot),\,\rho}$ *can be represented as*

$$F(z) = P_{m_0}(z)\, Z(z)\,,$$

where $Z(\cdot)$ *is the canonical solution of homogeneous problem,* $P_{m_0}(\cdot)$ *is a polynomial of order* $k \leq m_0$.

4. Non-homogeneous Riemann problem in generalized weighted Hardy classes

Consider the following non-homogeneous Riemann boundary value problem in classes $H^{+}_{p(\cdot),\rho} \times {}_{m_0}H^{-}_{p(\cdot),\rho}$:

$$F^{+}(\tau) - G(\tau)\, F^{-}(\tau) = f(\tau)\,, \tau \in \partial\omega\,, \tag{11}$$

where $f \in L_{p(\cdot),\rho}$ is some given function, and the weight $\rho(\cdot)$ is defined by the expression

$$\rho(t) = \prod_{i=1}^{m} |t - \tau_i|^{\alpha_i}\,,\ t \in [-\pi, \pi],$$

where $-\pi < \tau_1 < ... < \tau_m < \pi$ is some number. It is absolutely clear that the general solution of the problem (11) in classes $H^{+}_{p(\cdot),\rho} \times_{m_0} H^{-}_{p(\cdot),\rho}$ can be represented in the following form

$$F(z) = F_0(z) + F_1(z)\ ,$$

where $F_0(z)$ is a general solution of the homogeneous problem

$$F_0^+(\tau) - G(\tau) F_0^-(\tau) = 0,\ \tau \in \partial\omega\,, \tag{12}$$

in classes $H^+_{p(\cdot),\rho} \times {}_{m_0}H^-_{p(\cdot),\rho}$ and $F_1(z)$ is some particular solution of the problem (11) in the same classes. Let $Z(z)$ be the canonical solution of the homogeneous problem (12), i.e.

$$Z(z) = Z_1(z) Z_2(z),$$

where the functions $Z_k(z)$, $k = 1, 2$, are defined by the expressions

$$Z_k(z) \equiv \begin{cases} X_k(z), & |z| < 1, \\ [X_k(z)]^{-1}, & |z| > 1,\ k = 1, 2; \end{cases}$$

and the functions $X_k(z)$ are defined by the following integrals with Schwartz kernel

$$X_1(z) \equiv \exp\left\{\tfrac{1}{4\pi}\int_{-\pi}^{\pi} \ln\left|G\left(e^{it}\right)\right| \tfrac{e^{it}+z}{e^{it}-z} dt\right\},$$

$$X_2(z) \equiv \exp\left\{\tfrac{i}{4\pi}\int_{-\pi}^{\pi} \theta(t) \tfrac{e^{it}+z}{e^{it}-z} dt\right\}.$$

Consider the following integral with Cauchy kernel $K_z : K_z(t) = \frac{1}{e^{it}-z}$:

$$F_1(z) = \frac{Z(z)}{2\pi}\int_{-\pi}^{\pi} K_z(t)\left[Z^+\left(e^{it}\right)\right]^{-1} f(t)\, dt. \tag{13}$$

Applying the Sokhotski-Plemelj formula to the expression (14), we obtain

$$F_1^{\pm}(\tau) = Z^{\pm}(\tau)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left[Z^+\left(e^{it}\right)\right]^{-1} f(t)}{e^{it}-z} dt\right]^{\pm} =$$

$$= Z^{\pm}(\tau)\left[\pm\frac{1}{2}\left[Z^+(\tau)\right]^{-1} f(\arg\tau) - \left[Z^+(\tau)\right]^{-1}(Sf)(\tau)\right],$$

where S is the singular operator with Cauchy kernel

$$(Sf)(\tau) = \frac{Z^+(\tau)}{2\pi}\int_{-\pi}^{\pi}\frac{\left[Z^+\left(e^{it}\right)\right]^{-1} f(t)}{e^{it}-\tau} dt,\ \tau \in \partial\omega.$$

Hence it directly follows that

$$\frac{F_1^+(\tau)}{Z^+(\tau)} - \frac{F_1^-(\tau)}{Z^-(\tau)} = \left[Z^+(\tau)\right]^{-1} f(\arg\tau)\ ,\ \tau \in \partial\omega\,. \tag{14}$$

Taking into account that $Z(z)$ satisfies the homogeneous boundary condition (12), from (14) we obtain $\frac{Z^+(\tau)}{Z^-(\tau)} = G(\tau)$, a.e. $\tau \in \partial\omega \Rightarrow F_1^+(\tau) - G(\tau) F_1^-(\tau) = f(\arg\tau)$, a.e. $\tau \in \partial\omega$. Thus, the boundary values $F_1^{\pm}(\tau)$ satisfy the relation (11) a.e. on $\partial\omega$. We have

$$F_1^+(\tau) = \frac{1}{2} f(\arg\tau) - Z^+(\tau)(Sf)(\tau),\ \tau \in \partial\omega\,.$$

By definition, from $f \in L_{p(\cdot),\rho}$ it follows that $f\rho \in L_{p(\cdot)}$. Restate Statement 2 [22] in an equivalent form. Consider the singular operator S_ρ :

$$(S_\rho g)(\tau) = \frac{\rho(t)}{2\pi} \int_{-\pi}^{\pi} \frac{g(t)}{\rho(t)\left(e^{it}-\tau\right)} dt \, , \, \tau \in \partial\omega.$$

So, the following statement is valid.

Statement 3. *Let* $(p \in WL) \wedge (p^- > 1)$*. Then, singular operator* S_ρ *is acting boundedly in* $L_{p(\cdot)}$ *if and only if*

$$-\frac{1}{p(\tau_k)} < \alpha_k < \frac{1}{q(\tau_k)}, \ k = \overline{1,m}\,.$$

Following the previous section we have

$$\left|Z^-\left(e^{it}\right)\right| = \left|Z_1^-\left(e^{it}\right)\right| \left|u_0(t)\right| \prod_{k=0}^{r} \left|\sin\frac{t-s_k}{2}\right|^{-\frac{h_k}{2\pi}},$$

where

$$u_0(t) = \left[\sin\left|\frac{t-\pi}{2}\right|\right]^{-\frac{h_0^{(0)}}{2\pi}} \exp\left\{-\frac{1}{4\pi}\int_{-\pi}^{\pi} \theta_0(\tau)\, ctg\frac{t-\tau}{2} d\tau\right\}.$$

It is clear that

$$\left|Z^+\left(e^{it}\right)\right| \sim \left|Z^-\left(e^{it}\right)\right| \, , \, t \in [-\pi, \pi] \, . \tag{15}$$

Assume

$$\tilde{\rho}(t) = Z^+\left(e^{it}\right)\rho(t)\, , t \in [-\pi, \pi]\, .$$

Then for $F_1^+(\tau)$ we obtain

$$F_1^+(\tau) = \frac{1}{2} f(\arg\tau) - \rho^{-1}(\arg\tau)\left(S_{\tilde{\rho}}\tilde{f}\right)(\tau)\, , \tag{16}$$

where

$$\tilde{f}(t) = f(t)\rho(t) \, , \ t \in [-\pi, \pi]\, .$$

It is clear that $\tilde{f} \in L_{p(\cdot)}$. We have

$$(Sf)(\tau) = \frac{Z^+(\tau)}{2\pi}\int_{-\pi}^{\pi} \frac{f(t)}{Z^+\left(e^{it}\right)} \frac{dt}{e^{it}-\tau} =$$

$$= \rho^{-1}(\arg\tau)\frac{\tilde{\rho}(\arg\tau)}{2\pi}\int_{-\pi}^{\pi}\frac{\tilde{f}(t)}{\tilde{\rho}(t)}\frac{dt}{e^{it}-\tau} = \rho^{-1}(\arg\tau)\left(S_{\tilde{\rho}}\tilde{f}\right)(\tau)\, ,$$

i.e.

$$(Sf)(\tau) = \rho^{-1}(\arg\tau)\left(S_{\tilde{\rho}}\tilde{f}\right)(\tau)\, . \tag{17}$$

Assume that the inequalities

$$-\frac{1}{q(t_k)} < \beta_k < \frac{1}{p(t_k)}, \; k = \overline{0,l}\,, \tag{18}$$

are fulfilled. From the results of the previous section and from the relation (16) it follows directly that

$$|\tilde{\rho}(t)| = \left|Z^+\left(e^{it}\right)\right| \rho(t) \sim \left|Z^-\left(e^{it}\right)\right| \rho(t) \sim \left|Z_1^-\left(e^{it}\right)\right| |u_0(t)| \prod_{k=0}^{l} |t-t_k|^{-\beta_k}\,,\; t \in [-\pi,\pi]\;.$$

Applying Statement 2 [22] to the operator $S_{\tilde{\rho}}$ we obtain that if the inequality (18) holds, then the singular operator $S_{\tilde{\rho}}$ acts boundedly in $L_{p(\cdot)}$. Then from expression (17) it follows that the operator S acts boundedly in $L_{p(\cdot),\rho}$. As a result, from (16) we immediately obtain that F_1^+ belongs to $L_{p(\cdot),\rho}$. We have

$$\int_{-\pi}^{\pi} \left|F_1^+\left(e^{it}\right)\right| dt \le c\left(p^-;p^+\right) \left\|F_1^+\rho\right\|_{L_{p(\cdot)}} \left\|\rho^{-1}\right\|_{L_{q(\cdot)}}\;. \tag{19}$$

Thus, if the inequalities

$$\alpha_k < \frac{1}{q(\tau_k)}\,,\;\; k = \overline{1,m}\;, \tag{20}$$

hold, then, as it follows from (19), the function F_1^+ belongs to L_1. We have

$$F_1(z) = Z(z)\, I(z)\;,$$

where

$$I(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(t)}{Z^+\left(e^{it}\right)} \frac{dt}{e^{it}-z}\;.$$

Applying Hölder's inequality, we get

$$\int_{-\pi}^{\pi} |f(t)| \left|Z^+\left(e^{it}\right)\right|^{-1} dt \le c\left(p^-;p^+\right) \|f\rho\|_{L_{p(\cdot)}} \left\|\left(Z^+\rho\right)^{-1}\right\|_{L_{q(\cdot)}}\;.$$

According to the results of [23], we have

$$\left\|Z_1^-\left(e^{it}\right)\right\|_{L_\infty}^{\pm 1} < +\infty\;,$$

where $\|\cdot\|_{L_\infty}$ is a sup-norm in $L_\infty(-\pi,\pi)$. Consequently

$$\left|Z^+\left(e^{it}\right)\rho(t)\right|^{-1} \sim |u_0(t)|^{-1} \prod_{k=0}^{l} |t-t_k|^{\beta_k}\,,\; x,t \in [-\pi,\pi]\,. \tag{21}$$

Let $\varepsilon > 0$ be some positive number. Assume $q_\varepsilon(t) = \frac{q(t)+\varepsilon}{q(t)}$,and let

$$P_\varepsilon^{-1}(t) = 1 - q_\varepsilon^{-1}(t)\;\;,\, t \in [-\pi,\pi]\,.$$

Applying (generalized) Hölder's inequality to (21), we get

$$\int_{-\pi}^{\pi} \left|Z^{+}\left(e^{it}\right)\rho\left(t\right)\right|^{-q(t)} dt \leq c_1 \int_{-\pi}^{\pi} \left|u_0\left(t\right)\right|^{-q(t)} \left|\rho_0\left(t\right)\right|^{q(t)} dt \leq$$

$$c_2 \left\|\left(u_0\left(t\right)\right)^{-q(t)}\right\|_{L_{p_\varepsilon(\cdot)}} \left\|\left(\rho_0\left(t\right)\right)^{q(t)}\right\|_{L_{q_\varepsilon(\cdot)}},$$

where c_k, $k = 1, 2-$are the absolute constants, and the weight function $\rho_0\left(\cdot\right)$ is defined by the expression

$$\rho_0\left(t\right) \equiv \prod_{k=0}^{l} \left|t - t_k\right|^{\beta_k}.$$

It is clear that

$$\left\|\left(u_0\left(t\right)\right)^{-q(t)}\right\|_{L_{p_\varepsilon(\cdot)}} < +\infty.$$

Now, take $\varepsilon > 0$ such that the inequalities

$$-\frac{1}{q\left(t_k\right) + \varepsilon} < \beta_k, \ k = \overline{0, l}, \tag{22}$$

hold. Since $\lim\limits_{\varepsilon \to 0} q_\varepsilon\left(t\right) = q\left(t\right)$, then choice of such ε is always possible. Let

$$q_\varepsilon^{-} = \inf_{[-\pi,\pi]} vrai \ \ q_\varepsilon\left(t\right); q_\varepsilon^{+} = \sup_{[-\pi,\pi]} vrai \ \ q_\varepsilon\left(t\right).$$

We have

$$\|f\|_{L_{q_\varepsilon(\cdot)}} \leq \left(\int_{-\pi}^{\pi} \left|f\left(x\right)\right|^{q_\varepsilon(x)} dx\right)^{\frac{1}{q_\varepsilon^{-}}}, \text{ for } \|f\|_{L_{q_\varepsilon(\cdot)}} > 1;$$

and

$$\|f\|_{L_{q_\varepsilon(\cdot)}} \leq \left(\int_{-\pi}^{\pi} \left|f\left(x\right)\right|^{q_\varepsilon(x)} dx\right)^{\frac{1}{q_\varepsilon^{+}}}, \text{ for } \|f\|_{L_{q_\varepsilon(\cdot)}} < 1.$$

Let $f\left(x\right) = \left(\rho_0\left(x\right)\right)^{q(x)}$. Then, it is true

$$\int_{-\pi}^{\pi} \left(\rho_0\left(x\right)\right)^{q_\varepsilon(x) q(x)} dx = \int_{-\pi}^{\pi} \left(\rho_0\left(x\right)\right)^{q(x)+\varepsilon} dx < +\infty.$$

The last inequality follows directly from the inequality (22). Then it is obvious that

$$\left\|\left(\rho_0\left(t\right)\right)^{q(t)}\right\|_{L_{q_\varepsilon(\cdot)}} < +\infty,$$

and as a result

$$\int_{-\pi}^{\pi} \left|Z^{+}\left(e^{it}\right)\rho\left(t\right)\right|^{-q(t)} dt < +\infty,$$

i.e. $(Z^+\rho)^{-1} \in L_{q(\cdot)}$. Thus, we established that

$$\int_{-\pi}^{\pi} \left| \frac{f(t)}{Z^+(e^{it})} \right| dt < +\infty.$$

Then, by Riesz-Fikhtengolts theorem, Cauchy type integral $I(z)$ belongs to the Hardy class $H_1^{\pm}$. According to the results of I.I.Danilyuk [23], it follows from the expression $F_1(z)$ that it belongs to the Hardy class H_μ^+ for some $\mu > 0$. Since $F_1^+ \in L_1$, by Smirnov's theorem we have the inclusion $F_1 \in H_1^+$. Then from the definition it directly follows that $F_1 \in H_{p(\cdot),\rho}^+$.

Similarly we can prove that $F_1 \in H_1^-$ and $F_1^- \in L_{p(\cdot),\rho}$. It is easy to see that $F_1(\infty) = 0$, and as a result, it is clear that $F_1 \in {}_{-1}H_{p(\cdot),\rho}^-$. So, the following statement is true.

Statement 4. *Let the inequalities* (18) *be fulfilled with respect to the quantities* $\{\beta_k\}_0^l$ *and the relations* (20) *hold. Then the Cauchy type integral* (13) *is a particular solution of the Riemann boundary value problem (11) in classes* $H_{p(\cdot),\rho}^+ \times {}_{-1}H_{p(\cdot),\rho}^-$, *where* $f \in L_{p(\cdot),\rho}$ *is an arbitrary function.*

The general solution of the Riemann problem (11) in the classes $H_{p(\cdot),\rho}^+ \times {}_{m_0}H_{p(\cdot),\rho}^-$ depends on the number $m_0 \in Z$. For $m_0 \geq -1$ it has a representation

$$F(z) = Z(z) P_{m_0}(z) + F_1(z),$$

where $P_{m_0}(z)$ is an arbitrary polynomial of degree $k \leq m_0$ (for $m_0 = -1$ we assume $P_{m_0} \equiv 0$), and $F_1(z)$ is a particular solution of the form (13). $Z(z)$ is the canonical solution of homogeneous problem (12).

For $m_0 < -1$ the non-homogeneous problem (11) is solvable in classes $H_{p(\cdot),\rho}^+ \times {}_{m_0}H_{p(\cdot),\rho}^-$ if and only if the relations of orthogonality $[-(m_0+1)]$ are fulfilled

$$\int_{-\pi}^{\pi} \frac{f(t)}{Z^+(e^{it})} e^{ikt} dt, \; k = \overline{0, -m_0 - 2}. \tag{23}$$

These relations follow directly from the expansion of Cauchy type integral

$$\int_{-\pi}^{\pi} K_z(t) \left[Z^+(e^{it})\right]^{-1} f(t)\, dt,$$

in a Taylor series in powers of z in the neighborhood of an infinitely remote point

$$K(z) \equiv \int_{-\pi}^{\pi} K_z(t) \left[Z^+(e^{it})\right]^{-1} f(t)\, dt = -\frac{1}{z} \int_{-\pi}^{\pi} \frac{f(t)}{Z^+(e^{it})} \frac{dt}{1 - e^{it} z^{-1}} =$$

$$= -\sum_{k=1}^{\infty} \int_{-\pi}^{\pi} \frac{f(t)}{Z^+(e^{it})} e^{i(k-1)t} dt z^{-k}.$$

Since $|Z^-(\infty)|^{\pm 1}< +\infty$, the order of the function $F_1(z)$ at the infinitely remote point coincides with the order of the Cauchy integral $K(z)$. It is clear that in this case we need to put $P_{m_0}(z) \equiv 0$. If the orthogonality relations (23) hold, then in this case the problem (11) is uniquely solvable in the classes $H^+_{p(\cdot),\rho} \times {}_{m_0}H^-_{p(\cdot),\rho}$. As a result, we have the following main

Theorem 6. *Let $p(\cdot) \in WL$, $p^- > 1$, and the weighted function $\rho(\cdot)$ be defined by*

$$\rho(t) = \prod_{i=1}^{m} |t - \tau_i|^{\alpha_i},$$

where $-\pi < \tau_1 < ... < \tau_m < \pi$ are some points. Let the coefficients $G\left(e^{it}\right) = \left|G\left(e^{it}\right)\right| e^{i\theta(t)}$ satisfy the conditions i), ii) and $h_k = \theta(s_k + 0) - \theta(s_k - 0)$, $k = \overline{1,r}$ be the jumps of function $\theta(t)$ at the points of discontinuity

$$\{s_k\}_1^r \subset (-\pi, \pi)\ ;\ h_0 = \theta(-\pi) - \theta(\pi)\ .$$

Assume

$$\{t_k\}_{k=0}^l = \{\tau_k\}_{k=1}^m \bigcup \{s_k\}_{k=0}^r,$$

and define

$$\beta_k = \frac{1}{2\pi} \sum_{i=0}^{r} h_i \chi_{\{t_k\}}(s_i) - \sum_{i=1}^{m} \alpha_i \chi_{\{t_k\}}(\tau_i)\ ,\ k = \overline{0,l}\,.$$

Let the inequalities

$$-\frac{1}{q(t_k)} < \beta_k < \frac{1}{p(t_k)},\ k = \overline{0,l}\,,$$

$$\frac{1}{q(\tau_k)} < \alpha_k\ ,\ k = \overline{1,m},$$

hold. Then:

α) for $m_0 \geq -1$ the non-homogeneous Riemann problem (11) has a general solution of the form

$$F(z) = Z(z) P_{m_0}(z) + F_1(z)\,,$$

where $Z(\cdot)$ is a canonical solution of the homogeneous problem (12), $P_{m_0}(z)$ is an arbitrary polynomial of order $k \leq m_0$ $(P_{-1}(z) \equiv 0)$, and $F_1(\cdot)$ is a particular solution of the form

$$F_1(z) = \frac{Z(z)}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)}{Z^+(e^{it})} K_z(t)\,dt, \tag{24}$$

where, $K_z(\cdot)$ is a Cauchy kernel, and $f \in L_{p(\cdot),\rho}$ is an arbitrary function;

β) for $m_0 < -1$ non-homogeneous problem (11) is solvable in the class $H^+_{p(\cdot),\rho} \times {}_{m_0}H^-_{p(\cdot),\rho}$ if and only if the orthogonality conditions (23) are true and $F(z) = F_1(z)$ is a solution of this problem.

This theorem has the following

Corollary 1. *Let all the conditions of Theorem 6 be fulfilled. Then the non-homogeneous problem (3) has a unique solution of the form (24) in classes $H^+_{p(\cdot),\rho} \times {}_{-1}H^-_{p(\cdot),\rho}$.*

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Tofig I.Najafov
Nakhchivan State University, University campus, AZ7012, Nakhchivan, Azerbaijan
E-mail: tofiq-necefov@mail.ru

Natavan P. Nasibova
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Az1141, Baku, Azerbaijan
E-mail: natavan2008@gmail.com

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