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Structural Completeness and Unification Problem of the Logic of Chang Algebra

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Abstract. The variety generated by perfect MV-algebras is investigated in the paper. It is shown that for m-generated algebras from this variety to be finitely presented is equivalent to be projective. The variety generated by perfect algebras has unitary unification type and it is shown that the logic corresponding to this variety is structurally complete.

Key Words and Phrases: MV-algebras, structural completeness, unification.

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1. Introduction

MV-algebras are algebraic counterpart of the infinite valued Łukasiewicz sentential calculus, as Boolean algebras are with respect to the classical propositional logic. There are MV-algebras which are not semisimple, i.e. the intersection of their maximal ideals (the radical of A) is different from $\{0\}$. Non-zero elements from the radical of A are called infinitesimals. It is worth to stress that to the existence of infinitesimals in some MV-algebras is due the remarkable difference of behaviour between Boolean algebras and MV-algebras.

Perfect MV-algebras are those MV-algebras generated by their infinitesimal elements or, equivalently, generated by their radical [5]. They generate the smallest non locally finite subvariety of the variety, **MV**, of all MV-algebras.

Perfect MV-algebras do not form a variety and contain non-simple subdirectly irreducible MV-algebras. Any MV-algebra contains its greatest perfect subalgebra, called its *perfect skeleton*. The variety generated by all perfect MV-algebras is indeed generated by a single MV-chain, actually the MV-algebra C, defined by Chang in [8].

Chang's MV-algebra C [8], which is our main interest, is defined on the set

$$C = \{0, c, ..., nc, ..., 1 - nc, ..., 1 - c, 1\},$$

by the following operations (consider $0 = 0c$): $x \oplus y =$

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- $(m+n))c$ if $x = nc$ and $y = mc$;
- $1-(m-n)c$ if $x = 1-nc$ and $y = mc$ and $0 < n < m$;
- $1-(n-m)c$ if $x = nc$ and $y = 1-mc$ and $0 < m < n$;
- 1 otherwise;

$\neg x = 1 - nc$ if $x = nc$, $\neg x = nc$ if $x = 1 - nc$.

The MV-algebra C is isomorphic to the algebra S_1^ω defined by Komori in [23]. In [23] the author presented the so-called Super-Łukasiewicz Logics, certain equational extensions of Łukasiewicz Propositional Logic. He also provided a characterization of irreducible generators of each of such varieties. Finite equational axiomatization of each subvariety of the variety of all MV-algebras is given in [12] and [32].

The MV-algebra C is the simplest MV-algebra with infinitesimals. That is, any non semisimple MV-algebra contains a copy of C as subalgebra. C is generated by an atom c, which we can interpret as a quasi false truth value. The negation of c is a quasi true value. Now, quasi truth or quasi falsehood are vague concepts. Hence, it is quite intriguing to explore such a logic of *quasi true*. About quasi truth in an MV-algebra, it is reasonable to accept the following propositions:

- there are quasi true values which are not 1;
- 0 is not quasi true;
- if x is quasi true, then x^2 is quasi true (where x^2 denotes the MV algebraic product of x with itself).

In C, to satisfy these axioms, it is enough to say that the quasi true values are the co-infinitesimals.

By way of contrast, note that there is no notion of quasi truth in $[0,1]$ satisfying the previous axioms (there are if we replace the MV product with other suitable t-norms, e.g. the product t-norm or the minimum t-norm).

Algebras from the variety generated by C will be called $MV(C) - algebras$. Also we recall that for an $MV(C)$-algebra A, its *Boolean skeleton*, $B(A)$, that is the greatest Boolean subalgebra of A, is a retract of A, via the radical ideal of A, see [11]. Thus, roughly speaking, every $MV(C)$-algebra can be seen as a Boolean algebra, up to infinitesimals.

Let L_P be the logic corresponding to the variety generated by perfect algebras which coincides with the set of all Łukasiewicz formulas that are valid in all perfect MV-chains, or equivalently that are valid in the MV-algebra C. Actually, L_P is the logic obtained by adding to the axioms of Łukasiewicz sentential calculus the following axiom: $(x \oplus x) \odot (x \oplus x) \leftrightarrow (x \odot x) \oplus (x \odot x)$, see [5]. Notice that the above axiom is used in [9] to define an interesting class of Glivenko MTL-algebras and that the Lindenbaum algebra of L_P is an $MV(C)$-algebra.

The importance of the class $MV(C)$-algebras and of the logic L_P can be percieved looking further at the role that infinitesimals play in MV-algebras and in Łukasiewicz logic. Indeed, the pure first order Łukasiewicz predicate logic is not complete with respect to the canonical set of truth values $[0,1]$, [34]. However, a completeness theorem is obtained if the truth values are allowed to vary through all linearly ordered MV-algebras, [3]. From the incompleteness theorem arises the problem of the algebraic significance of the true but unprovable formulas. In [4] it is remarked that the Lindenbaum algebra of first order Łukasiewicz logic is not semisimple and that the valid but unprovable formulas are precisely the formulas whose negations determine the radical of the Lindenbaum algebra, that is the co-infinitesimals of such algebra. Hence, the valid but unprovable formulas generate the perfect skeleton of the Lindenbaum algebra. So, perfect MV-algebras, the variety generated by them and their logic are intimately related with a crucial phenomenon of first order Łukasiewicz logic.

As it is well known, MV-algebras form a category which is equivalent to the category of abelian lattice ordered groups (ℓ-groups, for short) with strong unit [28]. Let us denote by Γ the functor implementing this equivalence. In particular, each perfect MV-algebra is associated with an abelian ℓ-group with a strong unit. Moreover, the category of perfect MV-algebras is equivalent to the category of abelian ℓ-groups, see [11]. Among perfect MV-algebras the algebra C plays a very important role. Indeed, it is the generator of the variety $\mathbf{MV(C)}$, the logic L_P is complete with respect to C, and C corresponds to the Behncke-Leptin C^*-algebra $A_{1,0}$ with a two-point dual, via the composition of the functor Γ with K_0, see [30].

From above it is clear that the class of $MV(C)$-algebras, far from being a quite narrow and exotic class, deserves to be explored because of its several and fruitful links with other areas of Logic and Algebra. Now we are going to focus on the logic L_P and especially on its derivability properties.

Derivable and admissible rules were introduced by Lorenzen [26]. A rule

$$\varphi_1, ..., \varphi_n / \psi,$$

is derivable if it belongs to the consequence relation of the logic (defined semantically, or by a proof system using a set of axioms and rules); and it is admissible if the set of theorems of the logic is closed under the rule. These two notions coincide for the standard consequence relation of classical logic, but nonclassical logics often admit rules which are not derivable. A logic whose admissible rules are all derivable is called structurally complete.

Ghilardi [17, 18] discovered the connection of admissibility to projective formulas and unification, which provided another criteria for admissibility in certain modal and intermediate logics (= extensions of intuitionistic logic), and new decision procedures for admissibility in some modal and intermediate logics.

Moreover, following Ghilardi [19] defining unification problem in terms of finitely presented algebras, and having our result that finitely generated finitely presented algebras are precisely finitely generated projective algebras, we deduce that the equational class of all $MV(C)$-algebras has unitary unification type, i. e. L_P has unitary unification type.

In the present paper we prove that:

(1) For m-generated $MV(C)$-algebras to be finitely presented is equivalent to be projective.

(2) The variety **MV(C)** of $MV(C)$-algebras has unitary unification type.

(3) There exists a one-to-one correspondence between projective formulas of L_P with m-variables and the m-generated projective subalgebras of the m-generated free algebras of the variety generated by perfect MV-algebras.

(4) L_P is structurally complete.

2. The quasi variety generated by Perfect MV-algebra C

It is worth to remark that the class of perfect algebras does not form a variety, so the problem of studying the proper subvariety of the variety of all MV-algebras generated by all perfect MV-algebras arises.

Let $\mathcal{V}(Perf)$ be the variety generated by all perfect algebras, and $\mathcal{V}(C)$ be the variety generated by Chang's algebra C. Then the following theorem holds:

Theorem 1. *([11])* $\mathcal{V}(C) = \mathcal{V}(Perf)$.

Let K be a class of algebras. Then by $\mathcal{QV}(K)$ we denote the quasivariety of algebras generated by K.

Now we show that the quasivariety generated by Chang algebra C coincides with the variety generated by C.

Theorem 2. $\mathcal{V}(C) = \mathcal{QV}(C)$.

To prove this theorem we give some auxiliary assertions.

Lemma 1. $\Gamma(Z \times_{lex} Q, (1, 0)) \in \mathcal{QV}(C)$.

Proof. Let us suppose that $A = \Gamma(Z \times_{lex} Q, (1, 0))$. Suppose a quasi-identity $p(x) = 0 \to q(x) = 0$ is *false* in A. We suppose p, q are polynomials in one variable (the case of n variables is analogous). Then there is x such that $p(x) = 0$ and $q(x) \neq 0$. We can suppose $x \in Rad(A)$ and $x \neq 0$. But then x generates a copy of C. So, the quasi-identity is false also in C. ◄

Corollary 1. $\Gamma(Z \times_{lex} R, (1, 0)) \in \mathcal{QV}(C)$.

Proof. This follows by the density of the rationals in R. ◄

Corollary 2. *If U_R is an ultrapower of the reals, then*

$$\Gamma(Z \times_{lex} U_R, (1, 0)) \in \mathcal{QV}(C).$$

Proof. This follows from Los Theorem on ultraproducts. ◄

Corollary 3. *If G is any linearly ordered abelian group, then*

$$\Gamma(Z \times_{lex} G, (1, 0)) \in \mathcal{QV}(C).$$

Proof. This follows because every linearly ordered abelian group embeds in an ultrpower of the reals. ◄

Corollary 4. *Every perfect MV chain is in $\mathcal{QV}(C)$.*

Proof. This follows because every perfect MV chain has the form $\Gamma(Z \times_{lex} G, (1,0))$. ◄

Now let us conclude the proof of the theorem.

Clearly $\mathcal{QV}(C) \subseteq \mathcal{V}(C)$.

Conversely, an MV-chain belongs to $\mathcal{V}(C)$ if and only if it is perfect, so every MV-chain belonging to $\mathcal{V}(C)$ belongs to $\mathcal{QV}(C)$. But every element of $\mathcal{V}(C)$ is a subdirect product of chains of $\mathcal{V}(C)$, and $\mathcal{QV}(C)$ is closed under subdirect products. So, $\mathcal{V}(C) \subseteq \mathcal{QV}(C)$. Hence $\mathcal{V}(C) = \mathcal{QV}(C)$

3. Finitely generated projective $MV(C)$-algebras

Theorem 3. *A 1-generated free $MV(C)$-algebra $F_{\mathbf{MV(C)}}(1)$ is isomorphic to C^2 with free generator $(c, \neg c)$.*

Proof. Firstly, let us show that C^2 is generated by $(c, \neg c)$. Indeed, $2((c, \neg c)^2) = (0,1)$ and $\neg(0,1) = (1,0)$. Therefore, since c (and $\neg c$, as well) generates C, we have that $(c, \neg c)$ generates C^2.

Observe that if we have a perfect $MV(C)$-chain A, then 1-generated subalgebra of A is isomorphic to either $\Gamma(Z \times_{lex} Z, (1,0))$ or the two-element Boolean algebra S_1.

Let $\mathbf{K}$ be a variety. An m-generated free algebra A on the generators $g_1, ..., g_m$ over the variety $\mathbf{K}$ can be defined in the following way: the algebra A is a free m-generated algebra on the generators $g_1, ..., g_m$ iff for any m-variable equation $P(x_1, ..., x_m) = Q(x_1, ..., x_m)$, the equation holds in the variety $\mathbf{K}$ iff the equation $P(g_1, ..., g_m) = Q(g_1, ..., g_m)$ is true in the algebra A (on the generators $g_1, ..., g_m \in A$) [6, 21].

Now, suppose that one-variable equation $P = Q$ does not hold in the variety $\mathbf{MV(C)}$. It means that this equation does not hold in some 1-generated perfect $MV(C)$-algebra A on some element $a \in A$. Then A is isomorphic either to C or S_1 (2-element Boolean algebra). Let us suppose that A is isomorphic to C. Identify isomorphic elements. Depending on the generator of A, the one belongs to either $RadA$ or $\neg RadA$. We use the projection either $\pi_1 : C^2 \to C$ or $\pi_2 : C^2 \to C$, sending the generator $(c, \neg c)$ either to $c \in C$ or to $\neg c \in C$. From here we conclude that $P = Q$ does not hold in C^2. Now let us suppose that A is isomorphic to S_1. Notice that homomorphic image of C^2 by $Rad(C^2)$ is isomorphic to one-generated free Boolean algebra S_1^2. So, $P = Q$ does not hold in C^2. Hence, C^2 is 1-generated free $MV(C)$-algebra. ◄

Definition 1. *A subalgebra A of $F_{\mathbf{V}}(m)$ is said to be projective subalgebra if there exists an endomorphism $h : F_{\mathbf{V}}(m) \to F_{\mathbf{V}}(m)$ such that $h(F_{\mathbf{V}}(m)) = A$ and $h(x) = x$ for every $x \in A$.*

Proposition 1. *([10], [27]) Let* $\mathbf{V}$ *be a variety,* $F_{\mathbf{V}}(m)$ *be an* m*-generated free algebra of the variety* $\mathbf{V}$*, and* $g_1, \ldots, g_m$ *be its free generators. Then an* m*-generated subalgebra* A *of* $F_{\mathbf{V}}(m)$ *with the generators* $a_1, \ldots, a_m \in A$ *is projective if and only if there exist polynomials* $p_1(x_1, \ldots, x_m), \ldots, p_m(x_1, \ldots, x_m)$ *such that*

$$p_i(g_1, \ldots, g_m) = a_i$$

and

$$p_i(p_1(x_1, \ldots, x_m), \ldots, p_m(x_1, \ldots, x_m)) = p_i(x_1, \ldots, x_m),\ i = 1, \ldots, m,$$

hold in $\mathbf{V}$.

From Proposition 1 we obtain that in $F_{\mathbf{V}}(m)$ we have

$$p_i(p_1(g_1, \ldots, g_m), \ldots, p_m(g_1, \ldots, g_m)) = p_i(g_1, \ldots, g_m) = a_i,$$

$i = 1, \ldots, m,$ i. e. $p_i(a_1, \ldots, a_m) = a_i$ in A. This suggests to consider the free object $F_{\mathbf{V}}(m, \Omega)$ over the variety $\mathbf{V}$ with respect to the set of equations $\Omega = \{p_1(x_1, \ldots, x_m) = x_1, \ldots, p_1(x_1, \ldots, x_m) = x_m\}$.

Proposition 2. *([31], [10])(Lemma 2, Lemma 3) An* MV*-algebra* A *is finitely presented iff* $A \cong F_{\mathbf{MV}}(m)/[u)$*, where* $[u)$ *is a principal filter generated by some element* $u \in F_{\mathbf{MV}}(m)$.

Theorem 4. *Let* A *be an* m*-generated* $MV(C)$*-algebra. Then the following are equivalent:*

1. *A is projective.*
2. *A is finitely presented.*

Proof. $1 \Rightarrow 2$. Since A is m-generated projective $MV(C)$-algebra, A is a retract of $F_{\mathbf{MV(C)}}(m)$, i.e. there exist homomorphisms $h : F_{\mathbf{MV(C)}}(m) \to A$ and $\varepsilon : A \to F_{\mathbf{MV(C)}}(m)$ such that $h\varepsilon = Id_A$, $h(g_i) = a_i$ $(i = 1, ..., m)$, and moreover, according to Proposition 1, there exist m polynomials $p_1(x_1, \ldots, x_m)$,
$\ldots, p_m(x_1, ..., x_m)$ such that

$$p_i(g_1, \ldots, g_m) = \varepsilon(a_i) = \varepsilon h(g_i)$$

and

$$p_i(P_1(x_1, \ldots, x_m), \ldots, p_m(x_1, \ldots, x_m)) = p_i(x_1, \ldots, x_m),\ i = 1, \ldots, m,$$

where $g_1, \ldots, g_m$ are free generators of $F_{\mathbf{MV(C)}}(m)$. Observe that $h(g_1), \ldots, h(g_m)$ are generators of A which we denote by $a_1, \ldots, a_m$, respectively. Let e be the endomorphism $\varepsilon h : F_{\mathbf{MV(C)}}(m) \to F_{\mathbf{MV(C)}}(m)$. This endomorphism has the properties: $ee = e$ and $e(x) = x$ for every $x \in \varepsilon(A)$.

Let us consider the set of equations $\Omega = \{p_i(x_1, \ldots, x_m) \leftrightarrow x_i = 1 : i = 1, \ldots, m\}$ and let $u = \bigwedge_{i=1}^{n}((p_i(g_1, \ldots, g_m) \leftrightarrow g_i) \in F_{\mathbf{MV(C)}}(m)$, where $x \leftrightarrow y$ is the abbreviation of $(x \to y) \wedge (y \to x)$. Then, according to Proposition 2, $F_{\mathbf{MV(C)}}(m)/[u) \cong F_{\mathbf{MV(C)}}(m, \Omega)$.

Observe that the equations from Ω are true in A on the elements $\varepsilon(a_i) = e(g_i)$, $i = 1, \ldots, m$. Indeed, since e is an endomorphism, we have

$$e(u) = \bigwedge_{i=1}^{m} e(g_i) \leftrightarrow p_i(e(g_1), \ldots, e(g_m)).$$

But $p_i(e(g_1), \ldots, e(g_m)) = p_i(p_1(g_1, \ldots, g_m), \ldots, p_n(g_1, \ldots, g_m)) = p_i(g_1, \ldots, g_m) = = \varepsilon h(g_i) = e(g_i), i = 1, \ldots, m$. Hence $e(u) = 1$ and $u \in e^{-1}(1)$, i. e. $[u) \subseteq e^{-1}(1)$. Therefore there exists a homomorphism $f : F_{\mathbf{MV(C)}}(m)/[u) \to \varepsilon(A)$ such that the diagram

$$\begin{tikzcd}
F_{\mathbf{MV(C)}}(m) \arrow[r, "e"] \arrow[dr, "r"'] & \varepsilon(A) \\
& F_{\mathbf{MV(C)}}(m)/[u) \arrow[u, "f"']
\end{tikzcd}$$

commutes, i. e. $fr = e$, where r is a natural homomorphism sending x to $x/[u)$. Now consider the restrictions e' and r' on $\varepsilon(A) \subseteq F_{\mathbf{MV(C)}}(m)$ of e and r, respectively. Then $fr' = e'$. But $e' = Id_{\varepsilon(A)}$. Therefore $fr' = Id_{\varepsilon(A)}$. From here we conclude that r' is an injection. Moreover, r' is a surjection, since $r(\varepsilon(a_i)) = r(g_i)$. Indeed, $e(g_i) = p_i(g_1, \ldots, g_n)$ and $g_i \leftrightarrow p_i(g_1, \ldots, g_n) = g_i \leftrightarrow e(g_i)$, where $e(g_i) = \varepsilon h(g_i)$. So $g_i \leftrightarrow p_i(g_1, \ldots, g_m) \geq \bigwedge_{i=1}^{m} g_i \leftrightarrow p_i(g_1, \ldots, g_m)$, i. e. $g_i \leftrightarrow p_i(g_1, \ldots, g_m) \in [u)$. Hence, r' is an isomorphism between $\varepsilon(A)$ and $F_{\mathbf{MV(C)}}(m)/[u)$. Consequently, $A (\cong \varepsilon(A))$ is finitely presented.

$2 \Rightarrow 1$. Let A be an m-generated finitely presented $MV(C)$-algebra. Then there exists a principal filter $[u)$ of m-generated free $MV(C)$-algebra $F_{\mathbf{MV(C)}}(m)$ such that $A \cong F_{\mathbf{MV(C)}}(m)/[u)$ (Proposition 2). Since $F_{\mathbf{MV(C)}}(m)$ is a subdirect product of finitely generated chain $MV(C)$-algebras, we can represent the element $u \in F_{\mathbf{MV(C)}}(m)$ as a sequence $(u_i)_{i \in I}$. Let $J = \{i \in I : u_i \neq 1\}$. Let π_J be a natural homomorphism such that $\pi_J((a_i)_{i \in I}) = (a_i)_{i \in J}$. On the other hand, the subalgebra of $F_{\mathbf{MV(C)}}(m)$ generated by $[u)$, which is a perfect MV-algebra $[u) \cup \neg[u)$, is isomorphic to $\pi_J(F_{\mathbf{MV(C)}}(m)) \cong F_{\mathbf{MV(C)}}(m)/[u) \cong A$. Notice, that if $(x_i)_{i \in I} \in [u)$, then $x_i = 1$ for $i \in I - J$; and if $(x_i)_{i \in I} \in \neg[u)$, then $x_i = 0$ for $i \in I - J$. So, the set $A' = \{(x_i)_{i \in J} : (x_i)_{i \in I} \in [u) \cup \neg[u)\}$ forms an $MV(C)$-algebra which is isomorphic to $[u) \cup \neg[u)$. Let $\varepsilon : A' \to F_{\mathbf{MV(C)}}(m)$ be the embedding such that $\varepsilon((x_i)_{i \in J}) = (x_i)_{i \in I} \in F_{\mathbf{MV(C)}}(m)$, where $x_i = 1$ if $(x_i)_{i \in J}$ belongs to the maximal filter and $i \in I - J$; and $x_i = 0$ if $(x_i)_{i \in J}$ belongs to the maximal ideal and $i \in I - J$. Thus we conclude that $\pi_J \varepsilon = Id_{A'}$. From here we deduce that the $MV(C)$-algebra A is projective. ◄

Observe, that for ℓ-groups, Baker [1] and Beynon [2] gave the following characterization: An ℓ-group G is finitely generated projective iff it is finitely presented. For unital ℓ-groups the ($\Rightarrow$)-direction holds [29] (Proposition 5).

The algebra C is isomorphic to $\Gamma(Z \times_{lex} Z, (1, 0))$, with generator c $(=(0, 1))$. In another notation the algebra C is denoted by $S_1^{\omega} (= \Gamma(Z \times_{lex} Z, (1, 0)))$. Recall that $\mathbf{MV(C)}$ is the variety generated by perfect algebras.

Theorem 5. *The two-element Boolean algebra and the $MV(C)$-algebra C are projective.*

Proof. It is obvious that the two-element Boolean algebra is projective. Indeed, as we already stressed, the Boolean skeleton $B(C^2)$ is a retract of C^2, [11]. So, the 4-element Boolean algebra is projective. Since the 2-element Boolean algebra is a retract of the 4-element Boolean algebra, we have that the 2-element Boolean algebra is projective. As we know, C^2 is the one-generated free $MV(C)$-algebra.

Let us consider the following partition E of the algebra C^2 the classes of which are: for any $k \in \omega$

$\|(1, (\neg c)^k)\| = \{(nc, (\neg c)^k) : n \in \omega\} \cup \{((\neg c)^n, (\neg c)^k) : n \in \omega\}$,

$\|(0, kc)\| = \{(nc, kc) : n \in \omega\} \cup \{((\neg c)^n, kc) : n \in \omega\}$.

Notice that this partition is the congruence relation corresponding to the prime filter $\|(1,1)\| = \{x \in C^2 : (0,1) \leq x \leq (1,1)\}$, and $\|(0,0)\|$ is the prime ideal $\{x \in C^2 : (0,0) \leq x \leq (1,0)\}$.

Let us consider the following homomorphisms: $\pi_2 : C^2 \to C$, where $\pi_2((x,y)) = y$, and $\varepsilon : C \to C^2$, where $\varepsilon(kc) = (0, kc), \varepsilon((\neg c)^k) = (1, (\neg c)^k)$ for every $k \in \omega$. Then, it is clear that $\pi_2 \varepsilon = Id_C$. From here we conclude that C is projective. ◀

4. Projective formulas

Let us denote by $\mathcal{P}_m$ a fixed set $x_1, ..., x_m$ of propositional variables and by Φ_m the set of all propositional formulas in L_P with variables in $\mathcal{P}_m$. Notice that the m-generated free $MV(C)$-algebra $F_{\mathbf{MV(C)}}(m)$ is isomorphic to $\Phi_m / \equiv$, where $\alpha \equiv \beta$ iff $\vdash (\alpha \leftrightarrow \beta)$ and $\alpha \leftrightarrow \beta = (\alpha \to \beta) \wedge (\beta \to \alpha)$. Subsequently we do not distinguish between the formulas and their equivalence classes. Hence we simply write Φ_m for $F_{\mathbf{MV(C)}}(m)$, and $\mathcal{P}_m$ plays the role of the set of free generators. Since Φ_m is a lattice, we have an order $\leq$ on Φ_m . It follows from the definition of $\to$ that for all $\alpha, \beta \in \Phi_m$, $\alpha \leq \beta$ iff $\vdash (\alpha \to \beta)$.

Let α be a formula of the logic L_P and consider a substitution $\sigma : \mathcal{P}_m \to \Phi_m$ and extend it to all of Φ_m by $\sigma(\alpha(x_1, ..., x_m)) = \alpha(\sigma(x_1), ..., \sigma(x_m))$. We can consider the substitution as an endomorphism $\sigma : \Phi_m \to \Phi_m$ of the free algebra Φ_m.

Definition 2. *A formula $\alpha \in \Phi_m$ is called* projective *if there exists a substitution σ : $\mathcal{P}_m \to \Phi_m$ such that $\vdash \sigma(\alpha)$ and $\alpha \vdash \beta \leftrightarrow \sigma(\beta)$, for all $\beta \in \Phi_m$.*

Notice that the notion of projective formula was introduced for intuitionistic logic in [17].

Observe that we can rewrite any identity $p(x_1, ..., x_m) = q(x_1, ..., x_m)$ in the variety $\mathbf{MV(C)}$ into an equivalent one $p(x_1, ..., x_m) \leftrightarrow q(x_1, ..., x_m) = 1$. So, for $\mathbf{MV(C)}$ we can replace n identities by one

$$\bigwedge_{i=1}^{n} p_i(x_1, ..., x_m) \leftrightarrow q_i(x_1, ..., x_m) = 1.$$

Now we are ready to show a close connection between projective formulas and projective subalgebras of the free algebra Φ_m.

Theorem 6. *Let A be an m-generated projective subalgebra of the free algebra Φ_m. Then there exists a projective formula α of m variables, such that A is isomorphic to $\Phi_m/[\alpha)$, where $[\alpha)$ is the principal filter generated by $\alpha \in \Phi_m$.*

Proof. Suppose A is an m-generated projective subalgebra of Φ_m with generators $a_1, ..., a_m$. Then A is a retract of Φ_m, and there exist homomorphisms $\varepsilon : A \to \Phi_m$, $h : \Phi_m \to A$ such that $h\varepsilon = Id_A$, where $\varepsilon(x) = x$ for every $x \in A \subset \Phi_m$. Observe that εh is an endomorphism of Φ_m. We will show now that $\alpha = \bigwedge_{j=1}^{m}(x_j \leftrightarrow \varepsilon h(x_j))$ is a projective formula, namely, that $\vdash \varepsilon h(\alpha)$ and $\alpha \vdash \beta \leftrightarrow \varepsilon h(\beta)$, for all $\beta \in \Phi_m$.

Indeed, $\varepsilon h(\bigwedge_{j=1}^{m}(p_j \leftrightarrow \varepsilon h(p_j))) = \bigwedge_{j=1}^{m}(\varepsilon h(x_j) \leftrightarrow \varepsilon h \varepsilon h(x_j))$, and since $h\varepsilon = Id_A$, we have $\varepsilon h(\bigwedge_{j=1}^{m}(x_j \leftrightarrow \varepsilon h(x_j))) = \bigwedge_{j=1}^{m}(\varepsilon h(x_j) \leftrightarrow \varepsilon h(x_j))$. Thus $\vdash \varepsilon h(\alpha)$. Further, for any $\beta \in \Phi_m$, $\varepsilon h(\beta(x_1, ..., x_m)) = \beta(\varepsilon h(x_1), ..., \varepsilon h(x_m))$, and since $\alpha \vdash x_j \leftrightarrow \varepsilon h(x_j)$, $j = 1, ..., m$, we have $\alpha \vdash \beta \leftrightarrow \varepsilon h(\beta)$.

Since A is an m-generated projective $MV(C)$-algebra, according to the Proposition 1, there exist m polynomials $p_1(x_1, ..., x_m), ..., p_m(x_1, ..., x_m)$ such that

$$p_i(x_1, ..., x_m) = \varepsilon(a_i) = \varepsilon h(x_i)$$

and

$$p_i(p_1(x_1, ..., x_m), ..., p_m(x_1, ..., x_m)) = p_i(x_1, ..., x_m),\ i = 1, ..., m.$$

Observe that $h(x_i) = a_i$. Since the m-generated projective MV-algebra A is finitely presented by the equation $\bigwedge_{j=1}^{m}(x_j \leftrightarrow \varepsilon h(x_j)) = 1$, we have $A \cong \Phi_m/[\alpha)$. ◄

Theorem 7. *If α is a projective formula of m variables, then $\Phi_m/[\alpha)$ is a projective algebra which is isomorphic to a projective subalgebra of Φ_m.*

Proof. Suppose that α is a projective formula of m variables. Then there exists a substitution $\sigma : \mathcal{P}_m \to \Phi_m$ such that $\vdash \sigma(\alpha)$ and $\alpha \vdash \beta \leftrightarrow \sigma(\beta)$, for all $\beta \in \Phi_m$. Since σ is an endomorphism of Φ_m, $\sigma(\Phi_m)$ is a subalgebra of Φ_m. Now we will show that $\sigma(\Phi_m)$ is a retract of Φ_m, i. e. $\sigma^2 = \sigma$. Indeed, since α is a projective formula, $\sigma(\alpha) = 1_{\Phi_m}$, and $\alpha \leq \beta \leftrightarrow \sigma(\beta)$ for all $\beta \in \Phi_m$. But then $\sigma(\alpha) \leq \sigma(\beta) \leftrightarrow \sigma^2(\beta)$, $\sigma(\beta) \leftrightarrow \sigma^2(\beta) = 1_{\Phi_m}$, $\sigma(\beta) = \sigma^2(\beta)$, and $\sigma^2 = \sigma$. Hence $\sigma(\Phi_m)$ is a retract of Φ_m. So, $\sigma(\Phi_m)$ is isomorphic to $\Phi_m/[\alpha)$. ◄

Thus we have the following correspondence between projective formulas and projective subalgebras of Φ_m. To each m-generated projective subalgebra of m-generated free $MV(C)$-algebra, there corresponds an m-variable projective formula, and to two non-isomorphic m-generated projective subalgebras of m-generated free $MV(C)$-algebra, there correspond non-equivalent m-variable projective formulas. And to two non-equivalent m-variable projective formulas, there correspond two different m-generated projective subalgebra of m-generated free $MV(C)$-algebra (but they can be isomorphic).

Therefore we arrive at the following

Corollary 5. *There exists a one-to-one correspondence between projective formulas with m variables and m-generated projective subalgebras of Φ_m.*

5. Unification problem

Let E be an equational theory. The E-unification problem is: given two terms s, t (built from function symbols and variables), to find a unifier for them, that is, a uniform replacement of the variables occurring in s and t by other terms that makes s and t equal by modulo E. For detailed information on unification problem we refer the readers to [17, 18, 20].

Let us be more precise. Let $\mathcal{F}$ be a set of functional symbols and let V be a set of variables. Let $T_{\mathcal{F}}(V)$ be the term algebra built from $\mathcal{F}$ and V, and $T_{\mathcal{F}_m}(V)$ be the term algebra of m-variable terms. Let E be a set of identities of type $p(x_1, ..., x_m) = q(x_1, ..., x_m)$, where $p, q \in T_{\mathcal{F}_m}(V)$.

Let $\mathbf{V}$ be the variety of algebras over $\mathcal{F}$ axiomatized by the equations from E.

A unification problem modulo E is a finite set of pairs

$$\mathcal{E} = \{(s_j, t_j) : s_j, t_j \in T_{\mathcal{F}_m}(V), j \in J\},$$

for some finite set J. *A solution to* (or *a unifier for*) $\mathcal{E}$ is a substitution (or an endomorphism of the term algebra $T_{\mathcal{F}_m}(V)$) σ (which is extension of the map $s : V_m \to T_{\mathcal{F}_m}(V)$, where V_m ($= \{x_1, ..., x_m\}$) is the set of m variables) such that the identity $\sigma(s_j) = \sigma(t_j)$ holds in every algebra of the variety $\mathbf{V}$. The problem $\mathcal{E}$ is *solvable* (or *unifiable*) if it admits at least one unifier.

Let $(X, \preceq)$ be a quasi-ordered set (i. e. $\preceq$ is a reflexive and transitive relation). A μ-set [18] for $(X, \preceq)$ is a subset $M \subseteq X$ such that: (1) every $x \in X$ is less or equal to some $m \in M$; (2) all elements of M are mutually $\preceq$-incomparable. There might be no μ-set for $(X, \preceq)$ (in this case we say that $(X, \preceq)$ has *type* 0) or there might be many of them, due to the lack of antisymmetry. However, all μ-sets for $(X, \preceq)$, if any, must have the same cardinality. We say that $(X, \preceq)$ has *type* $1, \omega, \infty$ iff it has a μ-set of cardinality 1, of finite (greater than 1) cardinality or of infinite cardinality, respectively.

Substitutions are compared by instantiation in the following way: we say that $\sigma : T_{\mathcal{F}_m}(V) \to T_{\mathcal{F}_m}(V))$ is *more general than* $\tau : T_{\mathcal{F}_m}(V) \to T_{\mathcal{F}_m}(V)$ (written as $\tau \preceq \sigma$) iff there is a substitution $\eta : T_{\mathcal{F}_m}(V) \to T_{\mathcal{F}_m}(V)$ such that for all $x \in V_m$ we have $E \vdash \eta(\sigma(x)) = \tau(x)$. The relation $\preceq$ is quasi-order.

Let $U_E(\mathcal{E})$ be the set of unifiers for the unification problem $\mathcal{E}$; then $(U_E(\mathcal{E}), \preceq)$ is a quasi-ordered set.

We say that an equational theory E has:

1. Unification type 1 iff for every solvable unification problem $\mathcal{E}$, $U_E(\mathcal{E})$ has type 1;

2. Unification type ω iff for every solvable unification problem $\mathcal{E}$, $U_E(\mathcal{E})$ has type ω;

3. Unification type ∞ iff for every solvable unification problem $\mathcal{E}$, $U_E(\mathcal{E})$ has type 1 or ω or ∞ - and there is a solvable unification problem $\mathcal{E}$ such that $U_E(\mathcal{E})$ has type ∞;

4. Unification type nullary, if none of the preceding cases applies.

Following Ghilardi [17], who has introduced the relevant definitions for E-unification from an algebraic point of view, by an algebraic unification problem we mean a finitely presented algebra A of $\mathbf{V}$. In this context, an E-unification problem is simply a finitely presented algebra A, and a solution for it (also called a unifier for A) is a pair given by a projective algebra P and a homomorphism $u : A \to P$. The set of unifiers for A is denoted by $U_E(A)$. A is said to be *unifiable* or solvable iff $U_E(A)$ is not empty. Given another algebraic unifier $w : A \to Q$, we say that u is more general than w, written $w \preceq u$, if there is a homomorphism $g : P \to Q$ such that $w = gu$.

The set of all algebraic unifiers $U_E(A)$ of a finitely presented algebra A forms a quasi-ordered set with the quasi-ordering $\preceq$.

The algebraic unification type of an algebraically unifiable finitely presented algebra A in the variety $\mathbf{V}$ is now defined exactly as in the symbolic case, using the quasi-ordering set $(U_E(A), \preceq)$. If m-generated finitely presented algebra of an equational class $\mathbf{V}$ is projective, then Id_A will be most general unifier for A.

Theorem 8. *The unification type of the equational class* $\mathbf{MV}(\mathbf{C})$ *is* 1*, i. e. unitary.*

Proof. The proof of the theorem immediately follows from Theorem 4. ◄

6. Structural completeness

A logic L is structurally complete if every rule that is admissible (preserves the set of theorems) should also be derivable. In a logic, a rule of inference is admissible in a formal system if the set of theorems of the system does not change when that rule is added to the existing rules of the system.

A Tarski-style consequence relation is a relation $\vdash$ between sets of formulas, and formulas, such that

- $\alpha \vdash \alpha$,
- if $\Gamma \vdash \alpha$, then $\Gamma, \triangle \vdash \alpha$.

A consequence relation such that if $\Gamma \vdash \alpha$, then $\sigma(\Gamma) \vdash \sigma(\alpha)$ for all substitutions σ is called structural.

More precisely. If L is a logic, an *L-unifier* of a formula φ is a substitution σ such that $\vdash_L \sigma(\varphi)$. A formula which has an L-unifier is called *L-unifiable.* An inference rule is an expression of the form Γ/φ, where φ is a formula, and Γ is a finite set of formulas. An inference rule Γ/φ is *derivable* in a logic L, if $\Gamma \vdash_L \varphi$. The rule $\Gamma \vdash_L \varphi$ is *L-admissible*, if every common L-unifier of Γ is also an L-unifier of φ.

We can identify propositional formulas in terms of MV-algebras in a natural way. A *valuation* in an MV-algebra A is a homomorphism v from the term algebra to A. If φ is a k-variable formula, $(a_1, ..., a_k) \in A^k$, and v is the assignment such that $v(p_i) = a_i$, we also write $\varphi(a_1, ..., a_k) = v(\varphi)$. A valuation v satisfies a formula φ if $v(\varphi) = 1$, and it satisfies a rule Γ/φ if $v(\varphi) \neq 1$ for some $\alpha \in \Gamma$, or $v(\varphi) = 1$. A rule Γ/φ is *valid* in an MV-algebra A, written as $A \models \Gamma/\varphi$, if the rule is satisfied by every valuation in A. In other words, $A \models \Gamma/\varphi$ if and only if the open first-order formula

$$\bigwedge_{\alpha\in\Gamma} (\alpha = 1) \Rightarrow \varphi = 1,$$

is valid in A. Conversely, validity of open formulas (or equivalently, universal sentences) in A can be reduced to validity of rules. Any open formula Φ can be expressed in the conjunctive normal form as $\Phi = \bigwedge_{i<k} \Phi_i$, where each Φ_i is a clause: a disjunction of atomic formulas (i.e., equations) and their negations. Then $A \models \Phi$ iff $A \models \Phi_i$ for each $i < k$, and a clause

$$\bigvee_{i<n} (\varphi_i = \psi_i) \vee \bigvee_{i<m} (\varphi'_i \neq \psi'_i),$$

is valid in A iff the rule

$$\{\varphi'_i \leftrightarrow \psi'_i \mid i < m\}/\{\varphi_i \leftrightarrow \psi_i \mid i < n\},$$

is valid. Łukasiewicz logic L is algebraizable, and the variety of MV-algebras is its equivalent algebraic semantics, using the translation between propositional formulas and identities described above. We thus have (cf. [22]):

Claim 1. ([24]) *A rule Γ/φ is valid in all MV-algebras if and only if it is derivable in L.*

As another corollary to algebraizability of L, free MV-algebras can be described as Lindenbaum algebras of L: the Lindenbaum algebra consists of equivalence classes of formulas using elements of generators X as propositional variables modulo the equivalence relation $\varphi \sim \psi$ iff $\vdash_L \varphi \leftrightarrow \psi$, with operations defined in the natural way. Note that valuations in this Lindenbaum algebra correspond to substitutions whose range consists of formulas using variables from X, and a formula φ is satisfied under a valuation given by such a substitution σ if and only if $\vdash_L \sigma(\varphi)$. We obtain the following characterization of admissibility:

Claim 2. ([24]) *For any rule Γ/φ, the following are equivalent:*
(i) Γ/φ is admissible.
(ii) Γ/φ is valid in all free MV-algebras.
(iii) Γ/φ is valid in all free MV-algebras over finite sets of generators.

Let us note that we will have the same assertions if we change the Łukasiewicz logic L with logic L_P. Then we can reformulate the Claim 2 in the following way:

*The logic L_P is structurally complete iff the variety **MV(C)** coincides with the quasi-variety generated by all free $MV(C)$-algebras over finite sets of generators.*

Let us formulate the following property for the logic L:

(SC) $\alpha \vdash \beta \in T, \ \Leftrightarrow (\forall\varphi : Form(\mathfrak{L}) \to Form(\mathfrak{L}))[\varphi(\alpha) \in T \Rightarrow \varphi(\beta) \in T],$

where T is the set of all theorems of the logic L, φ is an endomorphism of the algebra $(F;\to,\neg,0,1)$ which is a free algebra in the class of algebras of the type (2,1,0,0). Let us note that this condition is equivalent to the notion of a structural completeness [33] in the sense of Pogorzelski, i.e. any structural admissible rule of a logic is derivable.

(SCL) $\alpha^n \to \beta \in T$, for some positive integer n, $\Leftrightarrow (\forall\varphi : F \to F)[\varphi(\alpha) \in T \Rightarrow \varphi(\beta) \in T]$,

where T is the set of all theorems of the logic L, φ is an endomorphism of the algebra $(F;\to,\neg,0,1)$ which is a free algebra in the class of algebras of the type (2,1,0,0). Let us note that, since according to deduction theorem in Łukasiewicz logic: $\alpha \vdash \beta$ if and only if $\vdash \alpha^n \to \beta$ for some positive integer n, the property is equivalent to the notion of a structural completeness in the sense of Pogorzelski, i.e. any structural admissible rule of a logic is derivable.

In algebraic terms the property has the following formulation:

- $\alpha^n \to \beta = 1$, for some positive integer $n \Leftrightarrow (\forall\varphi : Form(\mathfrak{L}) \to Form(\mathfrak{L}))[\varphi(\alpha) = 1 \Rightarrow \varphi(\beta) = 1]$,

where φ is an endomorphism of the ω-generated free algebra $(F;\to,\neg,0,1)$ in the variety of MV-algebras.

Recall that L_P is a logic corresponding to variety $\mathbf{MV(C)}$, i. e. L_P is the extension of Łukasiewicz logic by the Lukasievicz formula $\neg((\neg\alpha \to \alpha) \to \neg(\neg\alpha \to \alpha)) \leftrightarrow ((\alpha \to \neg\alpha) \to \neg(\alpha \to \neg\alpha))$, the theorems of which coincides with formulas that are valid in all $MV(C)$-algebras.

Theorem 9. *The logic L_P is structurally complete.*

Proof. Let us suppose that $\alpha \to \beta$ is m variable term. It is evident that if $\alpha^n \to \beta = 1$, then $(\forall\varphi : F \to F)[\varphi(\alpha) = 1 \Rightarrow \varphi(\beta) = 1]$.

Now suppose that $\alpha^n \to \beta \neq 1$ for all positive integers n and $\varphi : F \to F$ is an endomorphism such that $\varphi(\alpha) = 1$. Therefore, there exist m generators of $MV(C)$-algebra C where $\alpha > \beta$ on the generators $a_1, ..., a_m \in C$, i. e. $\alpha(a_1, ..., a_m) > \beta(a_1, ..., a_m)$ and $\alpha(a_1, ..., a_m)$ belongs to a prime filter, say J, and, since $\alpha^n(a_1, ..., a_m) > \beta(a_1, ..., a_m)$ for all positive integers n, $\beta(a_1, ..., a_m)$ does not belong to J. Observe that J is either the minimal prime filter $\{1\}$ or maximal filter $\{(\neg c)^k : k \in \omega\}$. Then, C/J is a chain $MV(C)$-algebra such that $\alpha(a_1/J, ..., a_m/J) = 1$ and $\beta(a_1/J, ..., a_m/J) \neq 1$. According to Theorem 7, C/J is projective, which is either two-element Boolean algebra or $MV(C)$-algebra C. Hence, there exist homomorphisms $h : F(m) \to C/J$ and $\varepsilon : C/J \to F(m)$ such that $h\varepsilon = Id_{C/J}$. Then $\varepsilon h : F(m) \to F(m)$ is an endomorphism such that $\varepsilon h(\alpha) = 1$ and $\varepsilon h(\beta) \neq 1$.

Now we give another proof of this theorem. We show that the variety $\mathbf{MV(C)}$ coincides with the quasivariety generated by all free $MV(C)$-algebras over finite sets of generators. Indeed, since C is projective, C is a subalgebra of a free $MV(C)$-algebras over finite sets of generators. But quasivariety $\mathcal{QV}(C)$ generated by C coincides with the variety $\mathcal{V}(C)$ (Theorem 2). ◄

Corollary 6. *Among the extensions of Łukasiewicz logics, only classical logic and the logic L_P are structurally complete.*

Proof. Let L_0 be a logic distinct from classical logic and the logic L_P. The rule $(3(p \wedge \neg p))^2/p$ is admissible. Indeed, there is no substitution σ such that $\vdash_{L_0} (3(\sigma(p) \wedge \neg\sigma(p)))^2$. Only in the case where $\sigma(p)$ has the value t, such that $t \leq 1/2$ and $2t \geq 1/2$, the valuation of $(3(\sigma(p) \wedge \neg\sigma(p)))^2$ has the value 1. But there is no formula which is equivalent to constant t, since we have no constant t. So, the rule $(3(p \wedge \neg p))^2/p$ is admissible. But $(3(p \wedge \neg p))^2 \to p$ is not a theorem of L_0, because it is not logically true. At the same time, the rule is derivable in classical logic and the logic L_P. ◄

Let us notice that the result of Corollary 6 was obtained by J. Gispert in [16]. Let us note that structural completeness for the logic of perfect algebras L_P was announced in [13, 14, 15]. We also mention related works on structural completeness and admissibility in MV-algebras/Łukasiewicz logic: [7, 24, 25, 35].

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