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Non-differentiability Sets for Cantor Functions with respect to Various Expansions

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Abstract. We propose here three expansions of real numbers in $[0,1)$. By using these expansions, we define three functions of Cantor type. We then determine the non-differentiability set of these functions and then we show that the dimension of these sets is all 0.

Key Words and Phrases: expansion, Cantor function, non-differentiability set, box-dimension.

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1. Introduction

The Cantor function, named after Georg Cantor, is an example of a function which has a remarkable property: Its derivative vanishes almost everywhere. Sometimes the Cantor function is referred to as the Devil's staircase. We refer to [3] for a detailed account of the Cantor function. Let us recall the definition: For $a \in [0,1)$, there exist $a_1, a_2, \dots, a_N, \dots \in \{0,1,2\}$ such that

$$a = \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{a_N}{3^N} + \cdots .$$

Denote by N_0 the smallest $N \in \mathbb{N}$ such that $a_N = 1$ when it exists. Otherwise $N_0 = \infty$. Then the Cantor function, $f : [0,1) \to \mathbb{R}$ is defined as

$$f(a) = \frac{a_1}{2 \cdot 2^1} + \frac{a_2}{2 \cdot 2^2} + \cdots + \frac{a_{N_0-1}}{2 \cdot 2^{N_0-1}} + \frac{1}{2^{N_0}}.$$

If there is no such N_0, then define

$$f(a) = \frac{a_1}{2 \cdot 2^1} + \frac{a_2}{2 \cdot 2^2} + \cdots + \frac{a_n}{2 \cdot 2^n} + \cdots .$$

Denote by C the set of all points at which the Cantor function is non-differentiable. Then we know that the (box) dimension (see Definition 9, below) is $(\log_3 2)^2 =$

$(0.699\cdots)^2$. More precisely, Darst established that the dimension of C is $(\log_3 2)^2 = (0.699\cdots)^2$, where C is defined to be

$$C = \left\{ x \in [0,1] \,:\, 0 \leq \liminf_{h \to 0} \frac{f(x+h)-f(x)}{h} < \limsup_{h \to 0} \frac{f(x+h)-f(x)}{h} \leq \infty \right\}.$$

See [1, 2] for the precise proof of this fact.

The key idea of defining the Cantor function is to use both the binary expansion and the ternary expansion. If we use similar expansions such as the p-adic expansion, the dimension of the non-differentiability set is positive. However, in this paper, we shall show that the dimension of the non-differentiability set can be zero by the use of a new expansion.

In this paper, we propose new expansions and we show, for the case of continuous increasing functions, that the dimension of the non-differentiability set can be zero.

In this paper, we are concerned with the set

$$\begin{aligned} K = & \left\{ x \in [0,1] \,:\, 0 \leq \liminf_{h \to 0} \frac{f(x+h)-f(x)}{h} < \limsup_{h \to 0} \frac{f(x+h)-f(x)}{h} \leq \infty \right\} \\ & \cup \left\{ x \in [0,1] \,:\, \liminf_{h \to 0} \frac{f(x+h)-f(x)}{h} = \limsup_{h \to 0} \frac{f(x+h)-f(x)}{h} = \infty \right\} \end{aligned}$$

and we call K the non-differentiability set.

Let us recall one more definition, the integer part of a real number.

The integer part $[a]$ of the real number a is the largest integer that does not exceed a, for example $[\sqrt{3}] = [1] = 1$ and $[-\sqrt{3}] = -2$. Furthermore, an integer b is the integer part of the real number a if $b \leq a < b+1$.

The organization of the present paper is as follows: In Section 2 we propose three types of expansions. We define three types of Cantor functions in Section 3, and we propose the notion of uniformly continuous functions of type (a_n, b_n). In Section 4, we approximate these functions. In Section 5, we specify the non-differentiability set K for each of these functions. Section 6 considers the dimension of such sets.

2. Expansions

2.1. Factorial-type expansions

We introduce expansions that are generated from factorials below. The proof is elementary and it is omitted.

Proposition 1.

(1) $\displaystyle\sum_{n=2}^{\infty} \frac{n-1}{n!} = 1.$

(2) *For all* $k \in \mathbb{N}$, $\displaystyle\sum_{n=k}^{\infty} \frac{n-1}{n!} = \frac{1}{(k-1)!}$.

On the basis of Proposition 1, we prove the following proposition.

Proposition 2. *For all* $a \in [0,1]$, *there exists a sequence* $\{a_n\}_{n \geq 2}$ *of non-negative integers such that* $a_n \leq n-1$, *and that*

$$a = \sum_{n=2}^{\infty} \frac{a_n}{n!}.$$

Proof. When $a = 1$, we may take $a_n = n - 1$, by Proposition 1. Otherwise define natural numbers $a_2, a_3, \ldots, a_n, \ldots$ in the following manner:

(i) When $0 \leq a < \frac{1}{2}$, define $a_2 = 0$. In this case, we have

$$0 \leq 6\left(a - \frac{a_2}{2!}\right) = 6a < 3. \tag{1}$$

(ii) When $\frac{1}{2} \leq a < 1$, define $a_2 = 1$. In this case, we have

$$0 \leq 6\left(a - \frac{a_2}{2!}\right) = 6a - 3 < 3. \tag{2}$$

Suppose that we have defined $a_2, a_3, \ldots, a_n$ for $n \geq 2$. Then define $S_n = \displaystyle\sum_{k=2}^{n} \frac{a_k}{k!}$, and $a_{n+1} = [(n+1)!(a - S_n)]$. Then, obviously $0 \leq a_{n+1} \leq (n+1)!(a - S_n) < a_{n+1} + 1$. ◀

Claim 1. *For all* $n \geq 2$,

$$0 \leq a - S_n < \frac{1}{n!}, \tag{3}$$

and

$$0 \leq a_{n+1} \leq n. \tag{4}$$

Proof. [Proof of the claim] We prove the claim by induction. When $n = 2$, (3) is true by (1) and (2).

Suppose that (3) is true for n. Then we have

$$\begin{aligned} a - S_{n+1} &= a - S_n - \frac{[(n+1)!(a - S_n)]}{(n+1)!} \\ &= \frac{(n+1)!(a - S_n) - [(n+1)!(a - S_n)]}{(n+1)!} \end{aligned}$$

$$< \frac{1}{(n+1)!}.$$

Once (3) is proved, (4) follows easily;

$$0 \le a - S_n \le \frac{1}{n!} \text{ and } 0 \le a_{n+1} \le (n+1)!(a - S_n),$$

which implies

$$0 \le a_{n+1} < \frac{(n+1)!}{n!} = n+1,$$

and $a_{n+1} \in N$, which implies $0 \le a_{n+1} \le n$. It follows from the sandwich theorem that:

$$\lim_{n\to\infty} \left(a - \frac{a_2}{2!} - \frac{a_3}{3!} - \cdots - \frac{a_n}{n!} \right) = 0.$$

In other words, we conclude

$$a = \sum_{n=2}^{\infty} \frac{a_n}{n!}. \quad \blacktriangleleft$$

It can happen that two different pairs of integers may yield the same real number.

Proposition 3. *Let $a \in [0,1]$ have expansion:*

$$a = \sum_{n=2}^{\infty} \frac{a_n}{n!} = \sum_{n=2}^{\infty} \frac{b_n}{n!},$$

as in Proposition 2. Then one of the following is satisfied:

(1) $a_n = b_n$, *for all* n.

(2) $a_2 = b_2, a_3 = b_3, \ldots, a_n = b_n, a_{n+1} = b_{n+1} + 1$, $a_{n+2} = 0$, $b_{n+2} = n+1$, $a_{n+3} = 0$, $b_{n+3} = n+2$, $a_{n+4} = 0$, $b_{n+4} = n+3$, $\ldots$. *In other words, we have*

$$a = \sum_{k=2}^{n+1} \frac{a_k}{k!} + \frac{1}{(n+1)!} = \sum_{m=2}^{n} \frac{b_m}{m!} + \frac{b_{n+1}}{(n+1)!} + \sum_{l=1}^{\infty} \frac{n+l}{(n+l+1)!}.$$

(3) $a_2 = b_2, a_3 = b_3, \ldots, a_n = b_n, a_{n+1} = b_{n+1} - 1$, $a_{n+2} = n+1$, $b_{n+2} = 0$, $a_{n+3} = n+2$, $b_{n+3} = 0$, $a_{n+4} = n+3$, $b_{n+4} = 0$, $\ldots$. *In other words, we have*

$$a = \sum_{k=2}^{n+1} \frac{a_k}{k!} = \sum_{m=2}^{n} \frac{b_m}{m!} + \frac{b_{n+1}-1}{(n+1)!} + \sum_{l=1}^{\infty} \frac{n+l}{(n+l+1)!}.$$

Proof. Let $a \in [0,1]$ have expansion:

$$a = \sum_{n=2}^{\infty} \frac{a_n}{n!} = \sum_{n=2}^{\infty} \frac{b_n}{n!},$$

as in Proposition 2, and, suppose that $a_n - b_n \neq 0$ for some $k \geq 2$; we let n_0 be the smallest one among such n. Then

$$\frac{|a_{n_0} - b_{n_0}|}{n_0!} \geq \frac{1}{n_0!} = \sum_{n=n_0+1}^{\infty} \frac{n-1}{n!} \geq \sum_{n=n_0+1}^{\infty} \frac{|a_n - b_n|}{n!} \geq \frac{|a_{n_0} - b_{n_0}|}{n_0!}.$$

Here, we used $|a_n - b_n| \leq n - 1$ for the second inequality, the minimality of n_0 for the second inequality, and the triangle inequality as well as $\sum_{n=2}^{\infty} \frac{a_n - b_n}{n!} = 0$ for the last inequality. In order that we have equality, the $a_n - b_n$ must satisfy either

$$a_{n_0} - b_{n_0} = 1, a_{n_0+1} - b_{n_0+1} = n_0, a_{n_0+2} - b_{n_0+2} = n_0 + 1, \dots \tag{5}$$

or

$$a_{n_0} - b_{n_0} = -1, a_{n_0+1} - b_{n_0+1} = -n_0, a_{n_0+2} - b_{n_0+2} = -n_0 - 1, \dots. \tag{6}$$

Note that (5) and (6) correspond to (2) and (3), respectively. Thus, we obtain the desired result. ◀

2.2. Odd factorial-type expansions

For $n = 1, 3, 5, 7, \dots$, recall that:

$$n!! = \begin{cases} 1, & n = 1, \\ n \cdot (n-2) \cdots 3 \cdot 1, & n \geq 3. \end{cases}$$

In analogy with Proposition 1, we have

Proposition 4.

(1) $\displaystyle\sum_{n:odd} \frac{n-1}{n!!} = 1.$

(2) *For an odd integer $k \in [-1, \infty)$,* $\displaystyle\sum_{n:odd, n\geq k+2} \frac{n-1}{n!!} = \frac{1}{k!!}.$

(1) corresponds to the case when $k = -1$ of (2).

Based on Proposition 4, we propose an expansion method of a real number $a \in [0, 1]$.

Proposition 5. *For all $a \in [0, 1]$, there exists a sequence $\{a_{2m-1}\}_{m\geq 1}$ of non-negative integers such that $a_{2m-1} < 2m - 1$ and that*

$$a = \sum_{m=1}^{\infty} \frac{a_{2m-1}}{(2m-1)!!}.$$

Proof. The proof is almost the same as that of Proposition 2: we content ourselves with outlining the proof. When $a = 1$, define $a_{2m-1} = 2m-2$. Otherwise, define $a_1 = 0$ first and then define $a_3, a_5, \ldots$ together with $S_3, S_5, \ldots$ by the following recurrence formula:

$$S_{2m-1} = \sum_{k=2}^{m} \frac{a_{2k-1}}{(2k-1)!!}, \quad a_{2m+1} = [(2m+1)!!(a - S_{2m-1})]. \blacktriangleleft$$

As before, it can happen that two different pairs of integers may yield the same real number.

Proposition 6. *Let $a \in [0,1]$ have expansion:*

$$a = \sum_{m=1}^{\infty} \frac{a_{2m+1}}{(2m+1)!!} = \sum_{m=1}^{\infty} \frac{b_{2m+1}}{(2m+1)!!},$$

as in Proposition 5, then one of the following is satisfied.

(1) $a_n = b_n$, $\forall n$.

(2) $a_1 = b_1, a_3 = b_3, \ldots, a_n = b_n, a_{n+2} = b_{n+2} + 1$, $a_{n+4} = 0$, $b_{n+4} = n+3$, $a_{n+6} = 0$, $b_{n+6} = n+5$, $a_{n+8} = 0$, $b_{n+8} = n+7$, $\ldots$. *In other words, we have*

$$\begin{aligned} a &= \sum_{m=1}^{\frac{n+1}{2}} \frac{a_{2m-1}}{(2m-1)!!} + \frac{b_{n+2}+1}{(n+2)!!} \\ &= \sum_{m=1}^{\frac{n+1}{2}} \frac{b_{2m-1}}{(2m-1)!!} + \frac{b_{n+2}}{(n+2)!!} + \sum_{m=1}^{\infty} \frac{n+2m+1}{(n+2m+2)!!}. \end{aligned}$$

(3) $a_1 = b_1, a_3 = b_3, \ldots, a_n = b_n, a_{n+2} = b_{n+2} - 1$, $a_{n+4} = n+3$, $b_{n+4} = 0$, $a_{n+6} = n+5$, $b_{n+6} = 0$, $a_{n+8} = n+7$, $b_{n+8} = 0$, $\ldots$. *In other words, we have*

$$\begin{aligned} a &= \sum_{m=1}^{\frac{n+1}{2}+1} \frac{a_{2m-1}}{(2m-1)!!} + \frac{b_{2m-1}}{(n+2)!!} \\ &= \sum_{m=1}^{\frac{n+1}{2}} \frac{b_{2m-1}}{(2m-1)!!} + \frac{b_{n+2}-1}{(n+2)!!} + \sum_{m=1}^{\infty} \frac{n+2m+1}{(n+2m+2)!!}. \end{aligned}$$

Proof. Go through the same argument as Proposition 3. $\blacktriangleleft$

2.3. Even factorial-type expansions

For $n = 0, 2, 4, 6, \ldots$, recall that:

$$n!! = \begin{cases} 1, & n = 1, \\ n \cdot (n-2) \cdots 4 \cdot 2, & n \geq 2. \end{cases}$$

Again similar to Proposition 1, we have

Proposition 7.

(1) $\displaystyle\sum_{n:even} \frac{n-1}{n!!} = 1.$

(2) *For an even number k,* $\displaystyle\sum_{n:even, n \geq k+2} \frac{n-1}{n!!} = \frac{1}{k!!}.$

(1) corresponds to the case when $k = 0$ of (2).

Based on Proposition 7, we have

Proposition 8. *For all $a \in [0,1]$, there exists a sequence $\{a_{2m}\}_{m \geq 1}$ of non-negative integers such that $a_{2m} < 2m$ and that*

$$a = \sum_{m=1}^{\infty} \frac{a_{2m}}{(2m)!!}.$$

Proof. Again the proof is almost the same as Proposition 2. When $a = 1$, we may take $a_n = n - 1$, by Proposition 4. When $0 \leq a < 1$, define the natural number $a_2, a_4, \ldots, a_n, \ldots$ in the following manner: Define $a_2 = 0$. Note that

$$0 \leq 8\left(a - \frac{a_2}{2!!}\right) = 8a < 4. \tag{7}$$

For each odd integer $n \geq 3$, define

$$S_n = \sum_{k:odd, 3 \geq k \geq n} \frac{a_k}{k!!},$$

and $a_{n+2} = [(n+2)!!(a - S_n)]$. We can check that $a_2, a_4, \ldots$, defined above does the job. ◀

As before, it can happen that two different pairs of integers may yield the same real number.

Proposition 9. *Let $a \in [0,1]$ have expansion :*

$$a = \sum_{m=1}^{\infty} \frac{a_{2m}}{(2m)!!} = \sum_{m=1}^{\infty} \frac{b_{2m}}{(2m)!!},$$

as in Proposition 8, then one of the following is satisfied.

(1) $a_n = b_n$, *for all even integers n.*

(2) $a_2 = b_2, a_4 = b_4, \ldots, a_n = b_n, a_{n+2} = b_{n+2} + 1$, $a_{n+4} = 0$, $b_{n+4} = n+3$, $a_{n+6} = 0$, $b_{n+6} = n+5$, $a_{n+8} = 0$, $b_{n+8} = n+7$, ... *In other words, we have*

$$a = \sum_{m=1}^{n/2} \frac{a_{2m}}{(2m)!!} + \frac{b_{n+2}+1}{(n+2)!!} = \sum_{m=1}^{n/2} \frac{b_{2m}}{(2m)!!} + \frac{b_{n+2}}{(n+2)!!} + \sum_{m=2}^{\infty} \frac{n+2m-1}{(n+2m)!!}.$$

(3) $a_2 = b_2, a_4 = b_4, \ldots, a_n = b_n, a_{n+2} = b_{n+2} - 1,\ a_{n+4} = n+3,\ b_{n+4} = 0,\ a_{n+6} = n+5,\ b_{n+6} = 0,\ a_{n+8} = n+7,\ b_{n+8} = 0, \ldots.$ *In other words, we have*

$$a = \sum_{m=1}^{n/2} \frac{a_{2m}}{(2m)!!} + \frac{b_{n+2}}{(n+2)!!} = \sum_{m=1}^{n/2} \frac{b_{2m}}{(2m)!!} + \frac{b_{n+2}-1}{(n+2)!!} + \sum_{m=2}^{\infty} \frac{n+2m-1}{(n+2m)!!}.$$

Proof. Go through the same argument as Proposition 3. ◀

3. On Cantor functions that are generated from the factorial

3.1. Uniformly continuous function of type (a_n, b_n)

For a function $f : [0,1] \to \mathbb{R}$ and $\delta > 0$, $x \in [0,1]$, $x + h \in [0,1]$:

$$\omega_1(f, \delta) = \sup_{x, |h| < \delta} |f(x+h) - f(x)|,$$

is called the continuous degree of f.

Proposition 10. *A function $f : [0,1] \to \mathbb{R}$ satisfies*

$$\omega_1(f, \delta) \to 0 \quad (\delta \to 0),$$

if and only if f is uniformly continuous.

Proof. If f is uniformly continuous, then for all $\varepsilon > 0$ there exists $\delta_0 > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [0,1]$ satisfy $|x - y| < \delta_0$. Thus, $\omega_1(f, \delta_0) < \varepsilon$. If $0 < \delta < \delta_0$, then $\omega_1(f, \delta) < \omega_1(f, \delta_0) < \varepsilon$, then $\omega_1(f, \delta) \to 0$ as $\delta \to 0$.

Suppose conversely, $\omega_1(f, \delta) \to 0$ as $\delta \to 0$. Then for all $\varepsilon > 0$, there exists $\delta_0 > 0$ such that $\omega_1(f, \delta) < \varepsilon$ for all $0 < \delta < \delta_0$. Thus, if $x, y \in [0,1]$ satisfy $|x - y| \le \dfrac{\delta_0}{2}$, then $|f(x) - f(y)| < \varepsilon$, which implies that f is uniformly continuous. ◀

Definition 1. *Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences decreasing to* 0. *A function f defined on $[0,1]$ is said to be of uniformly continuous function of type (a_n, b_n) if $|f(x) - f(y)| \le a_n$ for all $x, y \in [0,1]$ with $|x - y| \le b_n$.*

Remark 1. *The notion of continuity of type (a_n, b_n), which is defined below, turns out to be equivalent to the inequality $\omega_1(f, b_n) \le a_n$ for all $n \in \mathbb{N}$.*

3.2. Cantor function with respect to factorial-type expansion

To begin with, we give two kinds of rules below.

Definition 2. *Let $a \in [0,1]$. Suppose that a has an expression:*

$$0 \le a_n \le n-1 \quad (n = 2, 3, \ldots), \qquad a = \sum_{n=2}^{\infty} \frac{a_n}{n!}.$$

Define the sequence $\{a_n^*\}_{n=1}^{\infty}$ in the following manner:

Rule 1 *If there is no number n such that $1 \le a_n \le n-2$, that is, if $a_n = 0$ or $a_n = n-1$, then define $a_n^* = \dfrac{1}{n-1} a_n$. ($a^*$ assumes only the value 0 or 1.)*

Rule 2 *If there is a number n such that $1 \le a_n \le n-2$, then define n_0 as the minimum of such numbers. When $n < n_0$, define $a_n^* = \dfrac{1}{n-1} a_n$. When $n = n_0$, define $a_{n_0}^* = 1$. When $n > n_0$, define $a_n^* = 0$. (Again $a_n^* = \dfrac{1}{n-1} a_n \in \{0, 1\}$)*

For $a_2^*, a_3^*, \ldots$ determined in this way, define a^* as

$$a^* = \frac{a_2^*}{2^2} + \frac{a_3^*}{2^3} + \cdots + \frac{a_n^*}{2^n} + \cdots = \sum_{n=2}^{\infty} \frac{a_n^*}{2^n}.$$

Lemma 1. *The definition of a^* does not depend on the expression of a.*

Proof. By virtue of Proposition 1(2), when a has two different expressions, we have

$$a = \sum_{m=2}^{n-1} \frac{a_m + 1}{m!} + \frac{a_n + 1}{n!} + \sum_{l=1}^{\infty} \frac{0}{(n+l)!} = \sum_{m=2}^{n} \frac{a_n}{n!} + \sum_{l=1}^{\infty} \frac{n+l-1}{(n+l)!}.$$

Suppose that we have two different expressions as above. We suppose $a_n < n-1$. We write

$$(b_2, b_3, b_4, \ldots, b_{n-1}, b_n, \ldots) = (a_2, a_3, a_4, \ldots, a_{n-1}, a_n + 1, 0, 0, 0, \ldots),$$
$$(c_2, c_3, c_4, \ldots, c_{n-1}, c_n, \ldots) = (a_2, a_3, a_4, \ldots, a_{n-1}, a_n, n, n+1, n+2, \ldots).$$

(1) Suppose $1 \le a_i \le i-2$ for some $i = 3, 4 \ldots, n-1$. Then, among such i, choose the smallest one i_0. By applying Rule 2, we obtain

$$(b_2^*, b_3^*, b_4^*, \ldots, b_n^*, \ldots) = (a_2, a_3/2, a_4/3, \ldots, a_{i_0-1}/(i_0-2), 1, 0, 0, 0, \ldots),$$

and so

$$(c_2^*, c_3^*, c_4^*, \ldots, c_n^*, \ldots) = (a_2, a_3/2, a_4/3, \ldots, a_{i_0-1}/(i_0-2), 1, 0, 0, 0, \ldots).$$

Thus, the definition of a^* does not depend on the expression of a; no matter which expression we start from, we have

$$a^* = \sum_{m=2}^{n_0-1} \frac{a_m^*}{2^m} + \frac{1}{2^{n_0}}.$$

(2) Suppose $a_i = 0, i-1$ for all $i = 2, 3, \ldots, n-1$ and $a_n = 0$. Apply Rule 2 more than n times to get

$$(b_2^*, b_3^*, b_4^*, \ldots, b_n^*, \ldots) = (a_2, a_3/2, a_4/3, \ldots, a_{n-1}/(n-2), 1, 0, 0, 0, \ldots),$$
$$(c_1^*, c_3^*, c_5^*, \ldots, c_n^*, \ldots) = (a_2, a_3/2, a_4/3, \ldots, a_{n-1}/(n-2), 0, 1, 1, 1, \ldots).$$

Thus, the definition of a^* does not depend on the expression of a. In fact, we have

$$a = \sum_{k=2}^{n-1} \frac{b_k^*}{2^k} + \frac{1}{2^n},$$

if we start with $a = \sum_{n=2}^{\infty} \frac{b_n}{n!}$ and we have

$$\sum_{k=2}^{n-1} \frac{b_k^*}{2^k} + \frac{0}{2^n} + \sum_{k=1}^{\infty} \frac{1}{2^{n+k}},$$

if we start with $a = \sum_{n=2}^{\infty} \frac{c_n}{n!}$. Since $\sum_{k=1}^{\infty} \frac{1}{2^{n+k}} = \frac{1}{2^n}$, we see that the definition of a^* does not depend on the expression of a.

(3) Suppose $a_i = 0, i-1$ for all $i = 2, 3, \ldots, n-1$ and $1 \le a_n \le n-3$. Then

$$(b_2^*, b_3^*, b_4^*, \ldots, b_n^*, \ldots) = (a_2, a_3/2, a_4/3, \ldots, a_{n-1}/(n-2), 1, 0, 0, 0, \ldots),$$
$$(c_1^*, c_3^*, c_5^*, \ldots, c_n^*, \ldots) = (a_2, a_3/2, a_4/3, \ldots, a_{n-1}/(n-2), 1, 0, 0, 0, \ldots).$$

Thus, the definition of a^* does not depend on the expression of a.

(4) Suppose $a_i = 0, i-1$ for all $i = 2, 3, \ldots, n-1$ and $a_n = n-2$. Then

$$(b_2^*, b_3^*, b_4^*, \ldots, b_n^*, \ldots) = (a_2, a_3/2, a_4/3, \ldots, a_{n-1}/(n-2), 1, 0, 0, 0, \ldots).$$

So

$$(c_2^*, c_3^*, c_4^*, \ldots, c_n^*, \ldots) = (a_2, a_3/2, a_4/3, \ldots, a_{n-1}/(n-2), 1, 0, 0, 0, \ldots).$$

Thus, the definition of a^* does not depend on the expression of a. ◀

Let the Cantor function with respect to the factorial expansion be the function $f : [0,1] \to [0,1]$ such that $f(a) = a^*$.

Lemma 2. *f is an increasing function: Whenenver $0 \le b \le c \le 1$, $b^* \le c^*$.*

Proof. We may assume $b \ne c$; otherwise the assertion is trivial. We use the factorial expansions of b and c by using the algorithm obtained in Proposition 2:

$$b = \sum_{n=2}^{\infty} \frac{b_n}{n!}, \quad c = \sum_{n=2}^{\infty} \frac{c_n}{n!}.$$

Assuming $b < c$, we have $b_2 = c_2, b_3 = c_3, b_4 = c_4, \ldots, b_{n-1} = c_{n-1}, b_n < c_n$ for some $n \ge 2$.

(1) Suppose

$$b_2 = c_2 \in \{0,1\}, b_3 = c_3 \in \{0,2\}, \ldots, b_{n-1} = c_{n-1} \in \{0, n-2\}$$

fails. Denote by n_0 the smallest integer i such that $b_i = c_i \in \{1, 2, \ldots, n-2\}$. In this case we have $b^* = c^*$.

(2) Suppose that $b_2 = c_2 \in \{0,1\}, b_3 = c_3 \in \{0,2\}, \ldots, b_{n-1} = c_{n-1} \in \{0, n-2\}$, and that $b_n = 0$. Then, $b_n^* = 0$. Therefore, we have

$$b^* = \sum_{n=2}^{\infty} \frac{b_n^*}{2^n} \leq \sum_{n=2}^{n-1} \frac{b_n^*}{2^n} + \frac{0}{2^n} + \sum_{l=n+1}^{\infty} \frac{1}{2^l} = \sum_{n=2}^{n-1} \frac{b_n^*}{2^n} + \frac{1}{2^n}.$$

On the other hand, we have $c_n^* = 1$. Thus, we obtain

$$c^* = \sum_{n=2}^{\infty} \frac{c_n^*}{2^n} \geq \sum_{n=2}^{n} \frac{c_n^*}{2^n} = \sum_{n=2}^{n-1} \frac{c_n^*}{2^n} + \frac{1}{2^n} = \sum_{n=2}^{n-1} \frac{b_n^*}{2^n} + \frac{1}{2^n}.$$

Thus, we conclude $b^* \leq \displaystyle\sum_{n=2}^{n-1} \frac{b_{n-1}^*}{2^{n-1}} + \frac{1}{2^n} \leq c^*$.

(3) Suppose that $b_2 = c_2 \in \{0,1\}, b_3 = c_3 \in \{0,2\}, \ldots, b_{n-1} = c_{n-1} \in \{0, n-2\}$ and $b_n \geq 1$. Then $b_n < c_n \leq n-1$. In this case, we have $b_n^* = 1, b_{n+1}^* = 0, b_{n+2}^* = 0, b_{n+3}^* = 0, \ldots$. Thus, we have $b^* = \displaystyle\sum_{n=2}^{n-1} \frac{b_n^*}{2^n} + \frac{1}{2^n} \leq c^*$.

Therefore, in all cases $b^* \leq c^*$. ◄

The following theorem is one of the main theorems of this paper:

Theorem 2. *Let $a \mapsto a^*$ be the Cantor function with respect to the factorial expansion. Then it is a uniformly continuous function of type $(2^{-n}, 1/n!)$.*

Proof. The proof is made up of three large steps. The first step is a setup. Since $a \mapsto a^*$ is increasing, it suffices to show that

$$\left(a + \frac{1}{n!}\right)^* - a^* \leq \frac{1}{2^n}, \tag{8}$$

for $0 \leq a \leq 1 - 1/n!$. Representing a as

$$a = \sum_{n=2}^{\infty} \frac{a_n}{n!},$$

where $0 \leq a_n \leq n-2$ for all $n \geq 2$, in the next two large steps, we consider two cases: when

$$a_i \in \{0, i-1\} \quad (i = 2, 3, \ldots, n-1) \tag{9}$$

holds and when (9) fails. ◄

Proof. [**Case 1**] We suppose that (9) holds.

(1) Suppose first $a_n = 0$. Then

$$a = \sum_{n=2}^{n-1} \frac{a_n}{n!} + \frac{0}{n!} + \sum_{l=n+1}^{\infty} \frac{a_l}{l!} \text{ and } a + \frac{1}{n!} = \sum_{n=2}^{n-1} \frac{a_n}{n!} + \frac{1}{n!} + \sum_{l=n+1}^{\infty} \frac{a_l}{l!},$$

so

$$a^* = \sum_{k=2}^{n-1} \frac{a_k^*}{2^k} + \frac{0}{2^n} + \sum_{l=n+1}^{\infty} \frac{a_l^*}{2^l} \text{ and } \left(a + \frac{1}{n!}\right)^* = \sum_{k=2}^{n-1} \frac{a_k^*}{2^k} + \frac{1}{2^n}.$$

Hence, we have that

$$\left(a + \frac{1}{n!}\right)^* - a^* = \sum_{k=2}^{n-1} \frac{a_k^*}{2^k} + \frac{1}{2^n} - \sum_{k=2}^{n-1} \frac{a_k^*}{2^k} - \frac{0}{2^n} - \sum_{l=n+1}^{\infty} \frac{a_l^*}{2^l} \leq \frac{1}{2^n}.$$

Consequently, we have (8).

(2) Suppose, next, that $1 \leq a_n < n-2$. As we did in Lemma 1(1), we have

$$a = \sum_{k=2}^{\infty} \frac{a_k}{k!} \text{ and } a + \frac{1}{n!} = \sum_{k=2}^{\infty} \frac{a_k + 1}{k!},$$

so

$$a^* = \sum_{k=2}^{n-1} \frac{a_k^*}{2^k} + \frac{1}{2^n} \text{ and } \left(a + \frac{1}{n!}\right)^* = \sum_{k=2}^{n-1} \frac{a_k^*}{2^k} + \frac{1}{2^n}.$$

As a result, $\left(a + \frac{1}{n!}\right)^* = a^*$.

(3) Next, we suppose $a_n = n-2$. Then

$$a = \sum_{k=2}^{n-1} \frac{a_k}{k!} + \frac{n-2}{n!} + \sum_{l=n+1}^{\infty} \frac{a_l}{l!} \text{ and } a + \frac{1}{n!} = \sum_{k=2}^{n-1} \frac{a_k}{k!} + \frac{n-1}{n!} + \sum_{l=n+1}^{\infty} \frac{a_l}{l!}.$$

Hence, we have

$$a^* = \sum_{k=2}^{n-1} \frac{a_k^*}{2^k} + \frac{1}{2^n} \text{ and } \left(a + \frac{1}{n!}\right)^* = \sum_{k=2}^{n-1} \frac{a_k^*}{2^k} + \frac{1}{2^n} + \sum_{l=n+1}^{\infty} \frac{a_l^*}{2^l},$$

so

$$\left(a + \frac{1}{n!}\right)^* - a^* = \sum_{l=n+1}^{\infty} \frac{a_n^*}{2^l} \leq \sum_{l=n+1}^{\infty} \frac{1}{2^l} = \frac{1}{2^n}.$$

Consequently, we have (8).

(4) Finally, suppose $a_n = n-1$. Observe that the restriction $0 \leq a \leq 1 - 1/n!$ forces

$$a_{n+1} = a_{n+2} = \cdots = 0,$$

if $a_2 = 1, a_3 = 2, \ldots, a_n = n-1$.

(a) If $a_2 = 1$, $a_3 = 2$, …, $a_{n-1} = n-2$, then $a_{n+1} = a_{n+2} = \cdots = 0$. Thus,

$$a^* = \sum_{k=2}^{n} \frac{1}{2^k} = \frac{1}{2^2} \cdot \frac{1-2^{-n+1}}{1-2^{-1}} = \frac{1}{2} - \frac{1}{2^n}, \quad \left(a + \frac{1}{n!}\right)^* = \frac{1}{2}.$$

Consequently, $\left(a + \frac{1}{n!}\right)^* - a^* = 2^{-n}$.

(b) If

$$(a_2, a_3, \ldots, a_{n-1}) \in \{0,1\} \times \{0,2\} \times \cdots \times \{0, n-2\} \setminus \{(1,2,\ldots,n-2)\},$$

take n_0 so that $a_{n-1} = n-2$, $a_{n-2} = n-3$, …, $a_{n_0+1} = n_0$, $a_{n_0} = 0$. Using this number n_0, we have

$$a^* = \sum_{l=2}^{n_0-1} \frac{a_l^*}{2^l} + \frac{0}{2^{n_0}} + \sum_{l=n_0+1}^{n} \frac{1}{2^l} + \sum_{l=n+1}^{\infty} \frac{a_l^*}{2^l}$$

and

$$\left(a + \frac{1}{n!}\right)^* = \sum_{l=2}^{n_0-1} \frac{a_l^*}{2^l} + \frac{0}{2^{n_0}} + \frac{1}{2^{n_0}}.$$

Consequently

$$\begin{aligned}\left(a + \frac{1}{n!}\right)^* - a^* &= \frac{1}{2^{n_0}} - \frac{1}{2^{n_0+1}} - \frac{1}{2^{n_0+2}} - \cdots - \frac{1}{2^n} - \frac{a_{n+1}^*}{2^{n+1}} - \frac{a_{n+2}^*}{2^{n+2}} - \cdots \\ &= \frac{1}{2^n} - \sum_{l=n+1}^{\infty} \frac{a_l^*}{2^l} \leq \frac{1}{2^n}.\end{aligned}$$

Thus, if we assume (9), then (8) is true. ◀

Proof. [**Case 2**] Assume, instead, that (9) fails. Then, choose the smallest number n_0 such that $1 \leq a_{n_0} \leq n_0 - 2$. Let n_1 be the largest integer such that $a_{n_1} \leq n_1 - 2$.

(1) Suppose, first, $a_n < n-1$, So that n_0 is the smallest number such that $0 < a_{n_0} < n_0 - 1$. Then we have

$$a^* = \sum_{k=2}^{n_0-1} \frac{a_k^*}{2^k} + \frac{1}{2^{n_0}}, \quad \left(a + \frac{1}{n!}\right)^* = \sum_{k=2}^{n_0-1} \frac{a_k^*}{2^k} + \frac{1}{2^{n_0}} + \sum_{k=n_0+1}^{\infty} \frac{a_k^*}{2^k}.$$

Consequently, $\left(a + \frac{1}{n!}\right)^* = a^*$.

(2) Next, suppose $a_n = n-1$ and $n_0 < n_1$. Then in the same manner as Step 3 (1), we can prove $\left(a + \frac{1}{n!}\right)^* = a^*$.

(3) Next, suppose $a_n = n-1$ and $n_1 = n_0$. Then, we have $a_k \in \{0,1,\dots,k-1\}$ for $2 \le k \le n_0 - 1$, $a_{n_0} \le n_0 - 2$ and $a_{n_0+k} = n_0 + k - 1$ for all $k = 1,2,\dots,n-n_0$.

(a) Assume $a_{n_0} = 0$. Then we have

$$a^* \ge \sum_{k=2}^{n_0-1} \frac{a_k^*}{2^k} + \frac{0}{2^{n_0}} + \sum_{l=n_0+1}^{n} \frac{1}{2^l} + \frac{1}{2^n} \text{ and } \left(a + \frac{1}{n!}\right)^* = \sum_{k=2}^{n_0-1} \frac{a_k^*}{2^k} + \frac{1}{2^{n_0}}.$$

Hence, we have $\left(a + \frac{1}{n!}\right)^* - a^* \le \frac{1}{2^n}$.

(b) Assume $1 \le a_{n_0} < n_0 - 2$. Then

$$a^* = \sum_{k=2}^{n_0-1} \frac{a_k^*}{2^k} + \frac{1}{2^{n_0}} \text{ and } \left(a + \frac{1}{n!}\right)^* = \sum_{k=2}^{n_0-1} \frac{a_k^*}{2^k} + \frac{1}{2^{n_0}}.$$

Consequently, $\left(a + \frac{1}{n!}\right)^* = a^*$.

(c) Assume $a_{n_0} = n_0 - 2$. Then

$$a^* = \sum_{k=2}^{n_0-1} \frac{a_k^*}{2^k} + \frac{1}{2^{n_0}} \text{ and } \left(a + \frac{1}{n!}\right)^* = \sum_{k=2}^{n_0-1} \frac{a_k^*}{2^k} + \frac{1}{2^{n_0}} + \sum_{l=n+1}^{\infty} \frac{a_l^*}{2^l}.$$

Thus

$$\left(a + \frac{1}{n!}\right)^* - a^* = \sum_{l=n+1}^{\infty} \frac{a_n^*}{2^n} \le \sum_{l=n+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^n}. \blacktriangleleft$$

If we reexamine the proof above, we have:

Proposition 11. *Let $a \in [0, 1 - 1/n!]$ be such that $n!a$ is an integer. Write*

$$a = \sum_{k=2}^{n} \frac{a_k}{k!},$$

where $0 \le a_2 \le 1, 0 \le a_3 \le 2, \cdots, 0 \le a_{n-1} \le n-2$. Then $\left(a + \frac{1}{n!}\right)^* = a^* + \frac{1}{2^n}$, if and only if $a_2 = 0,1$, $a_3 = 0,2$, $\dots$, $a_{n-1} = 0, n-2$.

3.3. Cantor function with respect to odd factorial-type expansion

As in the case of factorial, we define a^* for $a \in [0,1]$ as follows:

Definition 3. *Let $a \in [0,1]$. Suppose that a has an expression below:*

$$0 \le a_n \le n-1 \quad (n = 1,3,\dots), \quad a = \sum_{n:odd} \frac{a_n}{n!!}.$$

Define a sequence $\{a^*_{2n-1}\}_{n=1}^{\infty}$ in the following manner.

Rule 1 *If there is no number n such that $1 \le a_n \le n-2$, that is, $a_n = 0$ or $a_n = n-1$, then define $a_n^* = \dfrac{1}{n-1} a_n$.*

Rule 2 *If there is a number n such that $1 \le a_n \le n-1$, then let The minimum number be defined as n_0. When $n < n_0$, define $a_n^* = \dfrac{1}{n-1} a_n$. When $n = n_0$, define $a_{n_0}^* = 1$. When $n > n_0$, define $a_n^* = 0$.*

For $a_1^*, a_3^*, \ldots$ defined in this way, define a^* by

$$a^* = \sum_{n:odd} \frac{a_n^*}{2^{\frac{n+1}{2}}}.$$

Below, we consider the property of the a^*.

Lemma 3. *The definition of a^* does not depend on the expression of a.*

Proof. The proof is the same as Lemma 1. ◀

As in the case of the factorial, we have

Lemma 4. *$b^* \le c^*$ whenever $0 \le b \le c \le 1$.*

Proof. The proof is the same as that of Lemma 2. ◀

As in the case of the factorial, we have

Theorem 3. *The Cantor function $a \mapsto a^*$ generated by the odd factorial is a uniformly continuous function of type $(2^{-\frac{n+1}{2}}, 1/n!!)$. Here, n runs over all odd positive integers.*

Proof. The proof is the same as Theorem 2. ◀

If we reexamine the proof above, we have:

Proposition 12. *Let n be an odd integer in $[3, \infty)$. Let $a \in [0, 1 - 1/n!!]$ be such that $n!!a$ is an integer. Write $a = \dfrac{a_1}{1!!} + \dfrac{a_3}{3!!} + \cdots + \dfrac{a_n}{n!!}$, where $a_1 = 0, 0 \le a_3 \le 2, \cdots, 0 \le a_{n-2} \le n-3$. Then $\left(a + \dfrac{1}{n!!}\right)^* - a^* = \dfrac{1}{2^{\frac{n+1}{2}}}$, if and only if $a_1 = 0$, $a_3 = 0, 2$, $\ldots$, $a_{n-2} = 0, n-3$.*

3.4. Cantor function with respect to even factorial-type expansion

As in the case of the odd factorial, we define another Cantor function.

Definition 4. *Let $a \in [0,1]$. Suppose that a has an expression:*

$$0 \le a_n \le n-1 \quad (n = 2, 4, \ldots), \quad a = \sum_{n:even} \frac{a_n}{n!!}.$$

Define the sequence $\{a^*_{2n}\}_{n=1}^{\infty}$ in the following manner:

Rule 1 *If there is no number n such that $1 \le a_n \le n-2$, that is, if $a_n = 0$ or $a_n = n-1$ define $a^*_n = \dfrac{1}{n-1} a_n$. ($a^*$ assumes only the value 0 or 1.)*

Rule 2 *If there is a number n such that $1 \le a_n \le n-2$, then n_0 as the minimum of such numbers. When $n < n_0$, define $a^*_n = \dfrac{1}{n-1} a_n$. When $n = n_0$, define $a^*_{n_0} = 1$. When $n > n_0$, define $a^*_n = 0$. (Again $a^*_n = \dfrac{1}{n-1} a_n \in \{0,1\}$.)*

For $a^*_2, a^*_4, \ldots$ which is determined in this way, define a^* as

$$a^* = \sum_{n:even} \frac{a^*_n}{2^{\frac{n}{2}}}.$$

Lemma 5. *The definition of a^* does not depend on the expression of a.*

Proof. Go through the same argument as Lemma 1. ◀

As in the case of the odd factorial, we have:

Lemma 6. *Whenenver $0 \le b \le c \le 1$, $b^* \le c^*$.*

Proof. The proof is the same as that of Lemma 2. ◀

As in the case of the odd factorial, we have:

Theorem 4. *Cantor function $a \mapsto a^*$ generated by the even factorial is a uniformly continuous function of type $(2^{-n/2}, 1/n!!)$. Here, n runs over all even integers.*

Proof. Go through the same argument as Theorem 2. ◀

If we reexamine the proof above, we have:

Proposition 13. *Let n be an odd integer in $[3, \infty)$. Let $a \in [0, 1 - 1/n!!]$ be such that $n!!a$ is an integer. Write $a = \displaystyle\sum_{l=1}^{n/2} \frac{a_{2l}}{2l!!}$, where $0 \le a_2 \le 1, 0 \le a_4 \le 3, \ldots, 0 \le a_{n-2} \le n-3$. Then $\left(a + \dfrac{1}{n!!}\right)^* - a^* = \dfrac{1}{2^{\frac{n}{2}}}$ if and only if $a_2 = 0, 1$, $a_4 = 0, 3$, $\ldots$, $a_{n-2} = 0, n-3$.*

4. An algorithm for approximating the Cantor function for some simple cases

Here, we attempt to approximate these Cantor functions.

Definition 5. *A piecewise linear function f on $[0,1]$ is a continuous function such that there exists a partition $\{t_j\}_{j=0}^N$ of $[0,1]$ such that f is affine on each $[t_{j-1}, t_j]$. In this case, the point $(t_j, f(t_j))$, $0 \le j \le N$, is called a vertex.*

4.1. Cantor function with respect to factorial-type expansion

Definition 6. *Let L be a natural number. The order L approximation of $t \mapsto t^*$ of order L is the piecewise linear function whose vertices are the points of the form (t, t^*), where $0 \le t \le 1$ and $L!t$ is an integer.*

Here, we explain the method of plotting the 5th order approximation.

(1) Calculate a_2, a_3, a_4, a_5 in the expansion of $t = \dfrac{a}{120} = \dfrac{a_2}{2!} + \dfrac{a_3}{3!} + \dfrac{a_4}{4!} + \dfrac{a_5}{5!}$ for $a = 0, 1, 2, \ldots, 119$. Define

$$a_2 = \left[\frac{a}{60}\right] = [2!t], \tag{10}$$

$$a_3 = \left[\frac{a}{20} - 3a_2\right] = \left[3!\left(t - \frac{a_2}{2!}\right)\right], \tag{11}$$

$$a_4 = \left[\frac{a}{5} - 12a_2 - 4a_3\right] = \left[4!\left(t - \frac{a_2}{2!} - \frac{a_3}{3!}\right)\right], \tag{12}$$

$$a_5 = 120\left(\frac{a}{120} - \frac{a_2}{2!} - \frac{a_3}{3!} - \frac{a_4}{4!}\right) = 5!\left(t - \frac{a_2}{2!} - \frac{a_3}{3!} - \frac{a_4}{4!}\right). \tag{13}$$

(2) We then calculate

$$t^* = \left(\frac{a}{120}\right)^* = \frac{a_2^*}{2^2} + \frac{a_3^*}{2^3} + \frac{a_4^*}{2^4} + \frac{a_5^*}{2^5},$$

for $a = 0, 1, 2, \ldots, 119$.

(3) Complementing a point $\left(1, \dfrac{1}{2}\right)$, we create an affine function using the data $\left(\dfrac{a}{120}, \left(\dfrac{a}{120}\right)^*\right)$, $(a = 0, 1, 2, \ldots, 120)$.

According to Claim 1, we have the following:

Corollary 1. *When $a = 0, 1, 2, \ldots, 119$, the integers a_2, a_3, a_4, a_5 which are determined by (10)–(13), satisfy $0 \le a_2 \le 1$, $0 \le a_3 \le 2$, $0 \le a_4 \le 3$, $0 \le a_5 \le 4$.*

4.2. Cantor function with respect to odd factorial-type expansion

Definition 7. *Let L be a natural number. The approximate function of $t \mapsto t^*$ of order L is a piecewise linear function whose vertices are the points of the form (t, t^*), where $0 \le t \le 1$ and $(2L-1)!!t$ is an integer.*

Here, we explain the method of drawing the approximate function of the 5th order.

(1) Calculate a_1, a_3, a_5, a_7, a_9 in the expansion of $t = \dfrac{a}{945} = \dfrac{a_1}{1!!} + \dfrac{a_3}{3!!} + \dfrac{a_5}{5!!} + \dfrac{a_7}{7!!} + \dfrac{a_9}{9!!}$ for $a = 0, 1, 2, \dots, 944$. Define

$$a_1 = \left[\frac{a}{945}\right] = [t], \tag{14}$$

$$a_3 = \left[\frac{a}{315} - 3a_1\right] = \left[3!!\left(t - \frac{a_1}{1!!}\right)\right], \tag{15}$$

$$a_5 = \left[\frac{a}{63} - 15a_1 - 5a_3\right] = \left[5!!\left(t - \frac{a_1}{1!!} - \frac{a_3}{3!!}\right)\right], \tag{16}$$

$$a_7 = \left[\frac{a}{9} - 105a_1 - 35a_3 - 7a_5\right] = \left[7!!\left(t - \frac{a_1}{1!!} - \frac{a_3}{3!!} - \frac{a_5}{5!!}\right)\right], \tag{17}$$

$$a_9 = 945\left(\frac{a}{945} - \frac{a_1}{1!!} - \frac{a_3}{3!!} - \frac{a_5}{5!!} - \frac{a_7}{7!!}\right) = 5!!\left(t - \frac{a_1}{1!!} - \frac{a_3}{3!!} - \frac{a_5}{5!!} - \frac{a_7}{7!!}\right). \tag{18}$$

(2) We then calculate the

$$t^* = \left(\frac{a}{945}\right)^* = \frac{a_1^*}{2^1} + \frac{a_3^*}{2^2} + \frac{a_5^*}{2^3} + \frac{a_7^*}{2^4} + \frac{a_9^*}{2^5},$$

for $a = 0, 1, 2, \dots, 944$.

(3) Complementing a point $\left(1, \dfrac{1}{2}\right)$, we create an affine function using the data $\left(\dfrac{a}{945}, \left(\dfrac{a}{945}\right)^*\right)$ $(a = 0, 1, 2, \dots, 945)$.

We shall show that this algorithm makes sense.

Corollary 2. *When $a = 0, 1, 2, \dots, 944$, the integers a_1, a_3, a_5, a_7, a_9, which are determined by (14)–(18), satisfies $a_1 = 0$, $0 \le a_3 \le 2$, $0 \le a_5 \le 4$, $0 \le a_7 \le 6$, $0 \le a_9 \le 8$.*

Proof. The proof is the same as Corollary 1. ◄

4.3. Cantor function with respect to even factorial-type expansion

Definition 8. *Let L be a natural number. The approximate function of $t \mapsto t^*$ of order L is a piecewise linear function whose vertices are the points of the form (t, t^*), where $0 \le t \le 1$ and $(2L)!!t$ is an integer.*

Here, we explain the method of drawing the approximate function of the 5th order.

(1) Calculate $a_2, a_4, a_6, a_8, a_{10}$ in the expansion of $t = \dfrac{a}{3840} = \dfrac{a_2}{2!!} + \dfrac{a_4}{4!!} + \dfrac{a_6}{6!!} + \dfrac{a_8}{8!!} + \dfrac{a_{10}}{10!!}$ for $a = 0, 1, 2, \ldots, 3839$. Define as

$$a_2 = \left[\frac{a}{1920}\right] = [2t], \tag{19}$$

$$a_4 = \left[\frac{a}{480} - 4a_2\right] = \left[4!!\left(t - \frac{a_2}{2!!}\right)\right], \tag{20}$$

$$a_6 = \left[\frac{a}{80} - 24a_2 - 6a_4\right] = \left[6!!\left(t - \frac{a_2}{2!!} - \frac{a_4}{4!!}\right)\right], \tag{21}$$

$$a_8 = \left[\frac{a}{10} - 192a_2 - 48a_4 - 8a_6\right] = \left[6!!\left(t - \frac{a_2}{2!!} - \frac{a_4}{4!!} - \frac{a_6}{6!!}\right)\right], \tag{22}$$

$$a_{10} = 3840\left(\frac{a}{3840} - \frac{a_2}{2!!} - \frac{a_4}{4!!} - \frac{a_6}{6!!} - \frac{a_8}{8!!}\right) = 10!!\left(t - \frac{a_2}{2!!} - \frac{a_4}{4!!} - \frac{a_6}{6!!} - \frac{a_8}{8!!}\right). \tag{23}$$

(2) We then calculate $t^* = \left(\dfrac{a}{945}\right)^* = \dfrac{a_2^*}{2^1} + \dfrac{a_4^*}{2^2} + \dfrac{a_6^*}{2^3} + \dfrac{a_8^*}{2^4} + \dfrac{a_{10}^*}{2^5}$ for $a = 0, 1, 2, \ldots, 3839$.

(3) Complementing $(1, 1)$, we create an affine function using data $\left(\dfrac{a}{3840}, \left(\dfrac{a}{3840}\right)^*\right)$ $(a = 0, 1, 2, \ldots, 3840)$.

We shall show that this algorithm makes sense.

Corollary 3. *When $a = 0, 1, 2, \ldots, 3839$, the integers $a_2, a_4, a_6, a_8, a_{10}$, which are determined by* (19)–(23)*, satisfies $0 \le a_2 \le 1$, $0 \le a_4 \le 3$, $0 \le a_6 \le 5$, $0 \le a_8 \le 7$, $0 \le a_{10} \le 9$.*

Proof. The proof is the same as Lemma 1. ◀

5. Nondifferentiability sets of the Cantor function

We specify the nondifferentiability set of the Cantor function with respect to the factorial-type expansion.

We consider the union K_n of the intervals of the form:

$$I_{a_2, a_3, \ldots, a_n} = \left[\frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_n}{n!}, \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_n + 1}{n!}\right],$$

where $a_2, a_3, \ldots, a_n$ run through all integers $a_2 = 0, 1, a_3 = 0, 2, \ldots, a_n = 0, n-1$.

Example 1. *($n = 3$) The endpoints of intervals that constitute K_3 and the values of f on the endpoints of the intervals are as follows:*

a_2, a_3	*endpoint of* I_{a_2,a_3}	$\min_{x \in I_{a_2,a_3}} f(x)$	$\max_{x \in I_{a_2,a_3}} f(x)$
$(0, 0)$	$0, 1/6$	0	$1/8$
$(0, 2)$	$1/3, 1/2$	$1/8$	$1/4$
$(1, 0)$	$1/2, 2/3$	$1/4$	$3/8$
$(1, 2)$	$5/6, 1$	$3/8$	$1/2$

Thus

$$K_3 = \left[0, \frac{1}{6}\right] \cup \left[\frac{1}{3}, \frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{2}{3}\right] \cup \left[\frac{5}{6}, 1\right] = \left[0, \frac{1}{6}\right] \cup \left[\frac{1}{3}, \frac{2}{3}\right] \cup \left[\frac{5}{6}, 1\right].$$

Example 2. *($n = 4$) The endpoints of intervals that constitute K_4 and the values of f on the endpoints of the intervals are as follows:*

a_2, a_3, a_4	*endpoint of* I_{a_2,a_3,a_4}	$\min_{x \in I_{a_2,a_3,a_4}} f(x)$	$\max_{x \in I_{a_2,a_3,a_4}} f(x)$
$(0,0,0)$	$0, 1/24$	0	$1/16$
$(0,0,3)$	$1/8, 1/6$	$1/16$	$1/8$
$(0,2,0)$	$1/3, 3/8$	$1/8$	$3/16$
$(0,2,3)$	$11/24, 1/2$	$3/16$	$1/4$
$(1,0,0)$	$1/2, 13/24$	$1/4$	$5/16$
$(1,0,3)$	$5/8, 2/3$	$5/16$	$3/8$
$(1,2,0)$	$5/6, 7/8$	$3/8$	$7/16$
$(1,2,3)$	$23/24, 1$	$7/16$	$1/2$

Thus

$$\begin{aligned} K_4 &= \left[0, \frac{1}{24}\right] \cup \left[\frac{1}{8}, \frac{1}{6}\right] \cup \left[\frac{1}{3}, \frac{3}{8}\right] \cup \left[\frac{11}{24}, \frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{13}{24}\right] \cup \left[\frac{5}{8}, \frac{2}{3}\right] \cup \left[\frac{5}{6}, \frac{7}{8}\right] \cup \left[\frac{23}{24}, 1\right] \\ &= \left[0, \frac{1}{24}\right] \cup \left[\frac{1}{8}, \frac{1}{6}\right] \cup \left[\frac{1}{3}, \frac{3}{8}\right] \cup \left[\frac{11}{24}, \frac{13}{24}\right] \cup \left[\frac{5}{8}, \frac{2}{3}\right] \cup \left[\frac{5}{6}, \frac{7}{8}\right] \cup \left[\frac{23}{24}, 1\right]. \end{aligned}$$

Example 3. *($n = 5$) The endpoints of intervals that constitute K_5 and the values of f on the endpoints of the intervals are as follows:*

a_2, a_3, a_4, a_5	*endpoint of* I_{a_2,a_3,a_4,a_5}	$\min_{x \in I_{a_2,a_3,a_4,a_5}} f(x)$	$\max_{x \in I_{a_2,a_3,a_4,a_5}} f(x)$
$(0,0,0,0)$	$0, 1/120$	0	$1/32$
$(0,0,0,4)$	$1/30, 1/24$	$1/32$	$1/16$
$(0,0,3,0)$	$1/8, 2/15$	$1/16$	$3/32$
$(0,0,3,4)$	$19/120, 1/6$	$3/32$	$1/8$
$(0,2,0,0)$	$1/3, 41/120$	$1/8$	$5/32$
$(0,2,0,4)$	$11/30, 3/8$	$5/32$	$3/16$
$(0,2,3,0)$	$11/24, 7/15$	$3/16$	$7/32$
$(0,2,3,4)$	$59/120, 1/2$	$7/32$	$1/4$
$(1,0,0,0)$	$1/2, 61/120$	$1/4$	$9/32$
$(1,0,0,4)$	$61/120, 13/24$	$9/32$	$5/16$
$(1,0,3,0)$	$5/8, 19/30$	$5/16$	$11/32$
$(1,0,3,4)$	$79/120, 2/3$	$11/32$	$3/8$
$(1,2,0,0)$	$5/6, 101/120$	$3/8$	$13/32$
$(1,2,0,4)$	$13/15, 7/8$	$13/32$	$7/16$
$(1,2,3,0)$	$23/24, 29/30$	$7/16$	$15/16$
$(1,2,3,4)$	$119/120, 1$	$15/32$	$1/2$

Thus

$$K_5 = \left[0, \frac{1}{120}\right] \cup \left[\frac{1}{30}, \frac{1}{24}\right] \cup \left[\frac{1}{8}, \frac{2}{15}\right] \cup \left[\frac{19}{120}, \frac{1}{6}\right] \cup \left[\frac{1}{3}, \frac{41}{120}\right] \cup \left[\frac{11}{30}, \frac{3}{8}\right]$$

$$\begin{aligned}
&\cup\left[\frac{11}{24},\frac{7}{15}\right]\cup\left[\frac{59}{120},\frac{1}{2}\right]\cup\left[\frac{1}{2},\frac{61}{120}\right]\cup\left[\frac{8}{15},\frac{13}{24}\right]\cup\left[\frac{5}{8},\frac{19}{30}\right]\\
&\cup\left[\frac{79}{120},\frac{2}{3}\right]\cup\left[\frac{5}{6},\frac{101}{120}\right]\cup\left[\frac{13}{15},\frac{7}{8}\right]\cup\left[\frac{23}{24},\frac{29}{30}\right]\cup\left[\frac{119}{120},1\right]\\
=&\left[0,\frac{1}{120}\right]\cup\left[\frac{1}{30},\frac{1}{24}\right]\cup\left[\frac{1}{8},\frac{2}{15}\right]\cup\left[\frac{19}{120},\frac{1}{6}\right]\cup\left[\frac{1}{3},\frac{41}{120}\right]\cup\left[\frac{11}{30},\frac{3}{8}\right]\\
&\cup\left[\frac{11}{24},\frac{7}{15}\right]\cup\left[\frac{59}{120},\frac{61}{120}\right]\cup\left[\frac{8}{15},\frac{13}{24}\right]\cup\left[\frac{5}{8},\frac{19}{30}\right]\\
&\cup\left[\frac{79}{120},\frac{2}{3}\right]\cup\left[\frac{5}{6},\frac{101}{120}\right]\cup\left[\frac{13}{15},\frac{7}{8}\right]\cup\left[\frac{23}{24},\frac{29}{30}\right]\cup\left[\frac{119}{120},1\right].
\end{aligned}$$

As can be seen in the above examples, we obtain:

Proposition 14. *Let $f(x) = x^*$, $x \in [0, 1]$.*

(1) *K_n consists of one long interval with length $\dfrac{1}{n!}$ and $2^{n-1}-1$ small closed intervals with length $\dfrac{2}{n!}$.*

(2) *$K_n \supset K_{n+1}$ for all $n \geq 2$.*

(3) *For $x \notin \bigcap_{n=1}^{\infty} K_n$, $f'(x) = 0$.*

Proof. We shall prove (3). Let us suppose $x \notin K_m$ for some $m \in \mathbb{N}$. Then x is between two intervals $I_{a_2,a_3,\ldots,a_m}$ and $I_{b_2,b_3,\ldots,b_m}$, where $I_{a_2,a_3,\ldots,a_m}$ lies in the left-hand side of x and $I_{b_2,b_3,\ldots,b_m}$ lies in the right-hand side of x. Denote by p the point on $I_{a_2,a_3,\ldots,a_m}$ closest to x and by q the point on $I_{b_2,b_3,\ldots,b_m}$ closest to x. Then $f(p) = f(q)$ as we have seen from the above examples. Since f is non-decreasing, $f(p) = f(x) = f(q)$, since $p < x < q$. Thus, for all $p < x' < q$, we have

$$\frac{f(x') - f(x)}{x' - x} = 0.$$

Letting $x' \to x$, we obtain

$$f'(x) = \lim_{x' \to x} \frac{f(x') - f(x)}{x' - x} = 0. \quad \blacktriangleleft$$

Contrary to Proposition 14(3) above, the function $f(x) = x^*$ is nowhere differentiable in $\bigcap_{n=1}^{\infty} K_n$.

Theorem 5. *Let $f(x) = x^*$, $x \in [0, 1]$. For all $x \in \bigcap_{n=1}^{\infty} K_n$, $f'(x)$ does not exist as a finite value.*

Proof. Let $n \in \mathbb{N}$ and $x \in \bigcap_{n=1}^{\infty} K_n$. Suppose that x is included in the interval $[a_n, b_n]$ constituting K_n. Then

$$\frac{f(b_n) - f(a_n)}{b_n - a_n} = \frac{1}{2^n} \div \frac{1}{n!} = \frac{n!}{2^n} \quad (n \to \infty).$$

Assume that $f'(x)$ is well-defined $[0,\infty]$. Then we have

$$\frac{f(b_n)-f(x)}{b_n-x}, \frac{f(x)-f(a_n)}{x-a_n} \to f'(x).$$

Since

$$\frac{f(b_n)-f(a_n)}{b_n-a_n} = \frac{b_n-x}{b_n-a_n}\cdot\frac{f(b_n)-f(x)}{b_n-x} + \frac{x-a_n}{b_n-a_n}\cdot\frac{f(x)-f(a_n)}{x-a_n}.$$

we obtain $\dfrac{f(b_n)-f(a_n)}{b_n-a_n} \to f'(x)$. It thus follows that $f(x)=\infty$. ◀

Next, going through the same argument, we specify the nondifferentiability set of the Cantor function with respect to the odd factorial-type expansion.

Example 4. *($n=3$) The endpoints of intervals that constitute K_3 and the values of f on the endpoints of the intervals are as follows:*

a_1, a_3	*endpoint of* I_{a_1,a_3}	$\min\limits_{x\in I_{a_2,a_3}} f(x)$	$\max\limits_{x\in I_{a_2,a_3}} f(x)$
$(0,0)$	$0, 1/3$	0	$1/2$
$(0,2)$	$3/2, 1$	$1/2$	1

Thus, $K_3 = \left[0,\frac{1}{3}\right] \cup \left[\frac{3}{2},1\right]$.

Example 5. *($n=5$) The endpoints of intervals that constitute K_5 and the values of f on the endpoints of the intervals are as follows:*

a_1, a_3, a_5	*endpoint of* I_{a_1,a_3,a_5}	$\min\limits_{x\in I_{a_1,a_3,a_5}} f(x)$	$\max\limits_{x\in I_{a_1,a_3,a_5}} f(x)$
$(0,0,0)$	$0, 1/15$	0	$1/4$
$(0,0,4)$	$4/15, 1/3$	$1/4$	$1/2$
$(0,2,0)$	$2/3, 11/15$	$1/2$	$3/4$
$(0,2,4)$	$14/15, 1$	$3/4$	1

Thus, $K_5 = \left[0,\frac{1}{15}\right] \cup \left[\frac{4}{15},\frac{1}{3}\right] \cup \left[\frac{2}{3},\frac{11}{15}\right] \cup \left[\frac{14}{15},1\right]$.

Example 6. *($n=7$) The endpoints of intervals that constitute K_7 and the values of f on the endpoints of the intervals are as follows:*

a_1, a_3, a_5, a_7	*endpoint of* I_{a_1,a_3,a_5,a_7}	$\min\limits_{x\in I_{a_1,a_3,a_5,a_7}} f(x)$	$\max\limits_{x\in I_{a_1,a_3,a_5,a_7}} f(x)$
$(0,0,0,0)$	$0, 1/105$	0	$1/8$
$(0,0,0,6)$	$2/35, 1/15$	$1/8$	$1/4$
$(0,0,4,0)$	$4/15, 29/105$	$1/4$	$3/8$
$(0,0,4,6)$	$34/105, 1/3$	$3/8$	$1/2$
$(0,2,0,0)$	$2/3, 71/105$	$1/2$	$5/8$
$(0,2,0,6)$	$76/105, 11/15$	$5/8$	$3/4$
$(0,2,4,0)$	$14/15, 33/35$	$3/4$	$7/8$
$(0,2,4,6)$	$104/105, 1$	$7/8$	1

Thus

$$\begin{aligned} K_7 &= \left[0, \frac{1}{105}\right] \cup \left[\frac{2}{35}, \frac{1}{15}\right] \cup \left[\frac{4}{15}, \frac{29}{105}\right] \cup \left[\frac{34}{105}, \frac{1}{3}\right] \\ &\cup \left[\frac{2}{3}, \frac{71}{105}\right] \cup \left[\frac{76}{105}, \frac{11}{15}\right] \cup \left[\frac{14}{15}, \frac{33}{35}\right] \cup \left[\frac{104}{105}, 1\right]. \end{aligned}$$

As can be seen from the above examples, we obtain;

Proposition 15. *Let* $f(x) = x^*$, $x \in [0,1]$.

(1) K_n *consists of* $2^{\frac{n-1}{2}} - 1$ *closed intervals with length* $\frac{1}{n!!}$.

(2) $K_n \supset K_{n+2}$ *for all odd* $n \geq 3$.

(3) *For* $x \notin \bigcap_{n=1}^{\infty} K_{2n+1}$, $f'(x) = 0$.

Contrary to Proposition 15(3) above, the function $f(x) = x^*$ is nowhere differentiable in $\bigcap_{n=1}^{\infty} K_{2n+1}$.

Theorem 6. *For* $x \in \bigcap_{n=1}^{\infty} K_{2n+1}$, $f'(x)$ *does not exist as a finite value.*

Proof. The proof is the same as that of Theorem 5. ◄

Finally, we specify the nondifferentiability set of the Cantor function with respect to the even factorial-type expansion.

Example 7. *(*$n = 4$*) Here, we list endpoints of intervals that constitute* K_4 *and the values of* f *on the endpoints of the intervals are as follows:*

a_2, a_4	*endpoint of* I_{a_2,a_4}	$\min\limits_{x \in I_{a_2,a_4}} f(x)$	$\max\limits_{x \in I_{a_2,a_4}} f(x)$
$(0,0)$	$0, 1/8$	0	$1/4$
$(0,3)$	$3/8, 1/2$	$1/4$	$1/2$
$(1,0)$	$1/2, 5/8$	$1/2$	$3/4$
$(1,3)$	$7/8, 1$	$3/4$	1

Thus, $K_4 = \left[0, \frac{1}{8}\right] \cup \left[\frac{3}{8}, \frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{5}{8}\right] \cup \left[\frac{7}{8}, 1\right].$

Example 8. *(*$n = 6$*) The endpoints of intervals that constitute* K_6 *and the values of* f *on the endpoints of the intervals are as follows:*

a_2, a_4, a_6	*endpoint of* I_{a_2,a_4,a_6}	$\min\limits_{x \in I_{a_2,a_4,a_6}} f(x)$	$\max\limits_{x \in I_{a_2,a_4,a_6}} f(x)$
$(0,0,0)$	$0, 1/48$	0	$1/8$
$(0,0,5)$	$5/48, 1/8$	$1/8$	$1/4$
$(0,3,0)$	$3/8, 19/48$	$1/4$	$3/8$
$(0,3,5)$	$23/48, 1/2$	$3/8$	$1/2$
$(1,0,0)$	$1/2, 25/48$	$1/2$	$5/8$
$(1,0,5)$	$29/48, 5/8$	$5/8$	$3/4$
$(1,3,0)$	$7/8, 43/48$	$3/4$	$7/8$
$(1,3,5)$	$47/48, 1$	$7/8$	1

Thus, $K_6 = \left[0, \frac{1}{48}\right] \cup \left[\frac{5}{48}, \frac{1}{8}\right] \cup \left[\frac{3}{8}, \frac{19}{48}\right] \cup \left[\frac{23}{48}, \frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{25}{48}\right] \cup \left[\frac{29}{48}, \frac{5}{8}\right] \cup \left[\frac{7}{8}, \frac{43}{48}\right] \cup \left[\frac{47}{48}, 1\right]$.

Example 9. *($n = 8$) The endpoints of intervals that constitute K_8 and the values of f on the endpoints of the intervals are as follows:*

a_2, a_4, a_6, a_8	*endpoint of* I_{a_2,a_4,a_6,a_8}	$\min_{x \in I_{a_2,a_4,a_6,a_8}} f(x)$	$\max_{x \in I_{a_2,a_4,a_6,a_8}} f(x)$
$(0,0,0,0)$	$0, 1/384$	0	$1/16$
$(0,0,0,7)$	$7/384, 1/48$	$1/16$	$1/8$
$(0,0,5,0)$	$5/48, 41/384$	$1/8$	$3/16$
$(0,0,5,7)$	$47/384, 1/8$	$3/16$	$1/4$
$(0,3,0,0)$	$3/8, 145/384$	$1/4$	$5/16$
$(0,3,0,7)$	$151/384, 19/48$	$5/16$	$3/8$
$(0,3,5,0)$	$23/48, 185/384$	$3/8$	$7/16$
$(0,3,5,7)$	$191/384, 1/2$	$7/16$	$1/2$
$(1,0,0,0)$	$1/2, 193/384$	$1/2$	$9/16$
$(1,0,0,7)$	$199/384, 25/48$	$9/16$	$5/8$
$(1,0,5,0)$	$29/48, 233/384$	$5/8$	$11/16$
$(1,0,5,7)$	$239/384, 5/8$	$11/16$	$3/4$
$(1,3,0,0)$	$7/8, 337/384$	$3/4$	$13/16$
$(1,3,0,7)$	$343/384, 43/48$	$13/16$	$7/8$
$(1,3,5,0)$	$47/48, 377/384$	$7/8$	$15/8$
$(1,3,5,7)$	$383/384, 1$	$15/16$	1

Thus

$$\begin{aligned} K_8 \;=\; & \left[0, \frac{1}{384}\right] \cup \left[\frac{7}{384}, \frac{1}{48}\right] \cup \left[\frac{5}{48}, \frac{41}{384}\right] \cup \left[\frac{47}{384}, \frac{1}{8}\right] \cup \left[\frac{3}{8}, \frac{145}{384}\right] \\ & \cup \left[\frac{151}{384}, \frac{19}{48}\right] \cup \left[\frac{23}{48}, \frac{185}{384}\right] \cup \left[\frac{191}{384}, \frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{193}{384}\right] \cup \left[\frac{199}{384}, \frac{25}{48}\right] \\ & \cup \left[\frac{29}{48}, \frac{233}{384}\right] \cup \left[\frac{239}{384}, \frac{5}{8}\right] \cup \left[\frac{7}{8}, \frac{337}{384}\right] \cup \left[\frac{343}{384}, \frac{43}{48}\right] \cup \left[\frac{47}{48}, \frac{377}{384}\right] \cup \left[\frac{383}{384}, 1\right]. \end{aligned}$$

As observed the above examples, we obtain;

Proposition 16. *Let $f(x) = x^*$, $x \in [0,1]$.*

(1) *K_n consists of one long interval with length $\frac{1}{n!!}$ and $2^{\frac{n}{2}-1}-1$ small closed intervals with length $\frac{2}{n!!}$.*

(2) *$K_n \supset K_{n+2}$ for all even $n \geq 2$.*

(3) *For $x \notin \bigcap_{n=1}^{\infty} K_{2n}$, $f'(x) = 0$.*

Contrary to Proposition 16(3) above, the function $f(x) = x^*$ is again nowhere differentiable in $\bigcap_{n=1}^{\infty} K_{2n}$.

Theorem 7. *For $x \in \bigcap_{n=1}^{\infty} K_{2n}$, $f'(x)$ does not exist as a finite value.*

Proof. The proof is the same as that of Theorem 5. ◀

6. Dimension of the nondifferentiability sets of the Cantor functions

6.1. Box dimension of the subset on the number line

Let us recall the definition of the box dimension.

Definition 9. *Let E be a subset of $[0,1]$.*

(1) *Write $N_r(E)$ for the smallest number of intervals of length r to cover E.*

(2) *The box dimension, $\dim_B E$ of E, is defined to be* $\dim_B E = \lim_{r\to 0} \dfrac{\log N_r(E)}{-\log r}$.

Example 10.

(1) *Let $E = \{0\}$. Since $N_r(E) = 1$ for any $r > 0$, we have*

$$\dim_B E = \lim_{r\to 0} \frac{\log N_r(E)}{-\log r} = \lim_{r\to 0} \frac{\log 1}{-\log r} = 0.$$

(2) *Let $E = \{a, b\}$, where $0 \leq a < b \leq 1$. When $0 < r < b - a$, then $N_r(E) = 2$, and*

$$\dim_B E = \lim_{r\to 0} \frac{\log N_r(E)}{-\log r} = \lim_{r\to 0} \frac{\log 2}{-\log r} = 0.$$

(3) *Let $E = (0, 1)$. Then it may be deduced that*

$$N_r(E) = \begin{cases} 1/r & 1/r \in \mathbb{Z}, \\ [1/r] + 1 & 1/r \notin \mathbb{Z}. \end{cases}$$

Thus, $1/r \leq N_r(E) \leq 1/r + 1$. Since

$$1 = \frac{\log 1/r}{-\log r} \leq \frac{\log N_r(E)}{-\log r} \leq \frac{\log 1/r + 1}{-\log r} = 1 - \frac{1}{\log r},$$

as $r \downarrow 0$, we obtain $\dim_B E = 1$.

6.2. Box dimension of the nondifferentiability set of the Cantor function with respect to factorial-type expansion

Theorem 8. *Let K be the nondifferentiability set of the Cantor function with respect to the expansion generated by the factorial. Then* $\dim_B K = 0$.

Proof. Let $0 < r \leq \dfrac{1}{2}$ and take $n \in \mathbb{N}$ so that $\dfrac{1}{n!} < r \leq \dfrac{1}{(n-1)!}$. Recall that K_{n-1} is composed of $2^{n-2}-1$ intervals. We denote by L_n the union of the one-point set $\{1/2\}$ and the set of all endpoints of these intervals. Thus L_n is made up of 2^{n-2} points and the distance between them is at least $\dfrac{1}{(n-1)!}$. Therefore, $N_r(L_n) = 2^{n-2}$. Since $L_n \subset K$, $N_r(K) \geq N_r(L_n) = 2^{n-2}$.

On the other hand, $K \subset K_n$, and K_n is made up of $2^{n-1}-1$ intervals. Decompose one large interval equally to obtain $[a_j^n, b_j^n]$ for $j = 1, 2, \ldots, 2^{n-1}$. Considering

$$\left[\frac{a_j^n + b_j^n}{2} - r, \frac{a_j^n + b_j^n}{2} + r\right] \quad j = 1, 2, \ldots, 2^{n-1},$$

we obtain $N_r(K) \leq N_r(K_n) \leq 2^{n-1}$. Therefore

$$\frac{\log 2^{n-2}}{\log n!} \leq \frac{\log N_r(K)}{-\log r} \leq \frac{\log 2^{n-1}}{\log(n-1)!}.$$

Since $n \to \infty$ as $r \to 0$, we obtain

$$\dim_B K = \lim_{r \downarrow 0} \frac{\log N_r(K)}{-\log r} = 0. \quad ◀$$

6.3. Box dimension of the nondifferentiability set of the Cantor function with respect to odd factorial-type expansion

Theorem 9. *Let K be the nondifferentiability set of the Cantor function with respect to the expansion generated by the odd factorial. Then* $\dim_B K = 0$.

Proof. Although the proof is almost the same as Theorem 8, we give the details since the intervals we need to consider are a little different from other cases. Let $0 < r \leq \dfrac{1}{3}$.

We take $n \in \mathbb{N}$ so that $\dfrac{1}{(n+2)!!} < r \leq \dfrac{1}{n!!}$. Recall that K_{n-2} is composed of $2^{\frac{n-3}{2}}$ intervals. We denote by L_n the set of all endpoints of these intervals. Note that L_n is made up of $2^{\frac{n-1}{2}}$ points. Note that these points are torn apart: the distance of any distinct points is at least $\dfrac{1}{n!!}$. Therefore, $N_r(L_n) = 2^{\frac{n-1}{2}}$. Since $L_n \subset K$, $N_r(K) \geq N_r(L_n) = 2^{\frac{n-1}{2}}$.

On the other hand, $K \subset K_n$, and K_n is made up of $2^{\frac{n-1}{2}}$ intervals, which we label $[a_j^n, b_j^n]$ for $j = 1, 2, \ldots, 2^{\frac{n-1}{2}}$. Considering

$$\left[\frac{a_j^n + b_j^n}{2} - r, \frac{a_j^n + b_j^n}{2} + r\right] \quad j = 1, 2, \ldots, 2^{\frac{n-1}{2}},$$

we obtain $N_r(K) \leq N_r(K_n) \leq 2^{\frac{n-1}{2}}$. Therefore

$$\frac{\log 2^{\frac{n-1}{2}}}{\log(n+2)!!} \leq \frac{\log N_r(K)}{-\log r} \leq \frac{\log 2^{\frac{n-1}{2}}}{\log n!!}.$$

Since $n \to \infty$ as $r \to 0$, we obtain

$$\dim_B K = \lim_{r \downarrow 0} \frac{\log N_r(K)}{-\log r} = 0. \blacktriangleleft$$

6.4. Box dimension of the nondifferentiability set of the Cantor function with respect to even factorial-type expansion

Theorem 10. *Let K be the nondifferentiability set of the Cantor function with respect to the expansion generated by the even factorial. Then* $\dim_B K = 0$.

Proof. The proof is the same as Theorem 8. ◄

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