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Some Spaces of Double Sequences, Their Duals and Matrix Transformations

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Abstract. In this paper we define certain generalized double difference sequence spaces by means of an Orlicz function in 2-normed spaces. We prove that these spaces are Banach spaces and establish some inclusion relations. We also determine the $\alpha-$ and $\beta(t)$-dual of spaces $\mathcal{M}_u(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ and $\mathcal{C}_t(M, \Delta^n, u, w, \|\cdot,\cdot\|)$, respectively. Finally, we characterize the classes $(\mu : \mathcal{C}_\vartheta(M, \Delta^n, u, w, \|\cdot,\cdot\|))$ for $\vartheta \in \{p, bp, t\}$ of matrix transformations where μ is any given space of double sequences.

Key Words and Phrases: double sequence space, Orlicz function, difference sequence space, 2-normed space, duals, matrix transformation.
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1. Introduction and Preliminaries

In [8] Gähler introduced the notion of 2-normed spaces as a generalization of a normed linear spaces, which was used to study sequence spaces and summability (see ([11],[31]), [12], [26]).

Let X be a real vector space of dimension d, where $2 \le d < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to \mathbb{R}$ which satisfies the following:

1. $\|x, y\| = 0$ if and only if x and y are linearly dependent,
2. $\|x, y\| = \|y, x\|$,
3. $\|\alpha x, y\| = |\alpha|\|x, y\|, \alpha \in \mathbb{R}$,
4. $\|x, y + z\| \le \|x, y\| + \|x, z\|$, for all $x, y, z \in X$.

The pair $(X, \|.,.\|)$ is called a 2-normed space (see [11]). For example, we may take $X = \mathbb{R}^2$ equipped with the 2-norm defined as $\|x, y\| =$ the area of the parallelogram spanned by the vectors x and y which may be given explicitly by the formula

$$\|x_1, x_2\|_E = \left(\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \right).$$

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Then, clearly $(X, \|\cdot,\cdot,\cdot\|)$ is a 2-normed space. Recall that $(X, \|\cdot,\cdot,\cdot\|)$ is a 2-Banach space if every Cauchy sequence in X is convergent to some x in X.

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [14] used the idea of Orlicz function to define the following sequence space:

$$\ell_M = \Big\{x \in \omega : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\},$$

which is called an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [14] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. In the later stage, different Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [27], Mursaleen [17] and many others.

The notion of difference sequence spaces was introduced by Kızmaz [13], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [7] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$.

Let n, υ be non-negative integers. Then for $Z = c, c_0$ and l_∞, we have sequence spaces

$$Z(\Delta^n_\upsilon) = \{x = (x_k) \in w : (\Delta^n_\upsilon x_m) \in Z\},$$

where $\Delta^n_\upsilon x = (\Delta^n_\upsilon x_m) = (\Delta^{n-1}_\upsilon x_m - \Delta^{n-1}_\upsilon x_{m+1})$ and $\Delta^0 x_m = x_m$ for all $m \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta^n_\upsilon x_m = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} x_{m+\upsilon\nu}.$$

Taking $n = \upsilon = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ studied by Kızmaz[13]. Taking $\upsilon = 1$, we get the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$ studied by Et and Colak [7]. Similarly, we can define difference operators on double sequence spaces as:

$$\begin{aligned} \Delta x_{m,v} &= (x_{m,v} - x_{m,v+1}) - (x_{m+1,v} - x_{m+1,v+1}) \\ &= x_{m,v} - x_{m,v+1} - x_{m+1,v} + x_{m+1,v+1}, \end{aligned}$$

and

$$\Delta^n x_{m,v} = \Delta^{n-1} x_{m,v} - \Delta^{n-1} x_{m,v+1} - \Delta^{n-1} x_{m+1,v} + \Delta^{n-1} x_{m+1,v+1}.$$

For more details about sequence spaces see [18], [21], [22], [23], [25], [28], [29], [30]), and for double sequence spaces one can refer to [15], [19], [24]. By ω and Ω we denote the sets of all real valued single and double sequences which are the vector spaces with coordinatewise addition and scalar multiplication. Any vector subspaces of w and Ω are called the single sequence space and the double sequence space, respectively. By $\mathcal{M}_u$ we denote the space of all bounded double sequences, that is

$$\mathcal{M}_u = \Big\{x = (x_{mv}) \in \Omega : \|x\|_\infty = \sup_{m,v\in\mathbb{N}} |x_{mv}| < \infty\Big\},$$

which is a Banach space with the norm $\|x\|_\infty$, where $\mathbb{N}$ denotes the set of all positive integers. Consider a sequence $x = (x_{mv}) \in \Omega$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $l \in \mathbb{R}$ such that $|x_{mv} - l| < \varepsilon$ for all $m, v > n_0$, then we say that the double sequence x is convergent in the Pringsheim's sense to the limit l and write $p - \lim x_{mv} = l$, where $\mathbb{R}$ denotes the real field. By $\mathcal{C}_p$ we denote the space of all convergent double sequences in the Pringsheim's sense. It is well-known that there are such sequences in the space $\mathcal{C}_p$ but not in the space $\mathcal{M}_u$. Indeed, following Boos [4], if we define the sequence $x = (x_{mv})$ by

$$x_{mv} = \begin{cases} v, & m = 1,\ v \in \mathbb{N}, \\ 0, & m \geq 2,\ v \in \mathbb{N}, \end{cases}$$

then it is trivial that $x \in \mathcal{C}_p \backslash \mathcal{M}_u$, since $p-\lim x_{mv} = 0$ but $\|x\|_\infty = \infty$. So, we can consider the space $\mathcal{C}_{bp}$ of the double sequences which are both convergent in the Pringsheim's sense and bounded, i.e., $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$. A sequence in the space $\mathcal{C}_p$ is said to be regularly convergent if it is a single convergent sequence with respect to each index. We denote the set of all such sequences by $\mathcal{C}_t$. Also, by $\mathcal{C}_{bp0}$ and $\mathcal{C}_{t0}$ we denote the spaces of all double sequences converging to 0 contained in the sequence spaces $\mathcal{C}_{bp}$ and $\mathcal{C}_t$, respectively. Moricz [16] proved that $\mathcal{C}_{bp}, \mathcal{C}_{b0p}, \mathcal{C}_t$ and $\mathcal{C}_{t0}$ are Banach spaces with the norm $\|\cdot\|_\infty$. Let us consider the isomorphism T defined by

$$\begin{aligned} T : \Omega &\to w, \qquad (1) \\ x \mapsto z &= (z_r) := (x_{\varphi^{-1}(r)}), \end{aligned}$$

where $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection defined by

$$\varphi[(1,1)] = 1,$$

$$\varphi[(1,2)] = 2, \varphi[(2,2)] = 3, \varphi[(2,1)] = 4,$$

$$\vdots$$

$$\varphi[(1,v)] = (v-1)^2 + 1, \varphi[(2,v)] = (v-1)^2 + 2, \cdots,$$

$$\varphi[(v,v)] = (v-1)^2 + v, \varphi[(v,v-1)] = v^2 - v + 2, \cdots, \varphi[(v,1)] = v^2,$$

$$\vdots$$

Let us consider a double sequence $x = (x_{mv})$ and define the sequence $h = (h_{mv})$ which will be used throughout via x by

$$h_{mv} = \sum_{r}^{m}\sum_{s}^{v} x_{rs}, \tag{2}$$

for all $m, v \in \mathbb{N}$. For the sake of brevity, here and in what follows, we abbreviate the summation $\sum_{r}^{m}\sum_{s}^{v}$ by $\sum_{rs}$ and we use this abbreviation with other letters. Let λ be a space of a double sequences, converging with respect to some linear convergence rule $v-\lim : \lambda \to \mathbb{R}$. The sum of a double series $\sum_{rs} x_{rs}$ with respect to this rule is defined by $v - \sum_{rs} x_{rs} = v - \lim_{mv\to\infty} h_{mv}$. Let $\lambda,\ \mu$ be two spaces of double sequences, converging with respect to the linear convergence rules $v_1 - \lim$ and $v_2 - \lim$, respectively, and $A = (a_{mvkl})$ also be a four dimensional infinite matrix over the real or complex field.

The $\alpha-$dual λ^{α}, $\beta(v)-$dual $\lambda^{\beta(v)}$ with respect to the $v-$convergence for $b \in \{p, bp, t\}$ and the $\gamma-$dual λ^{γ} of a double sequence space λ are respectively defined by

$$\lambda^{\alpha} = \Big\{(a_{rs}) \in \Omega : \sum_{r,s} |a_{rs}x_{rs}| < \infty \text{ for all } (x_{rs}) \in \lambda\Big\},$$

$$\lambda^{\beta(v)} = \Big\{(a_{rs}) \in \Omega : v - \sum_{r,s} a_{rs}x_{rs} \text{ exists for all } (x_{rs}) \in \lambda\Big\},$$

$$\lambda^{\gamma} = \Big\{(a_{rs}) \in \Omega : \sup_{k,l}\Big|\sum_{r,s}^{k,l} a_{rs}x_{rs}\Big| < \infty \text{ for all } (x_{rs}) \in \lambda\Big\}.$$

It is easy to see for any two spaces $\lambda,\ \mu$ of double sequences that $\mu^{\alpha} \subset \lambda^{\alpha}$ whenever $\lambda \subset \mu$ and $\lambda^{\alpha} \subset \lambda^{\gamma}$. Additionally, it is known that the inclusion $\lambda^{\alpha} \subset \lambda^{\beta(v)}$, holds while the inclusion $\lambda^{\beta(v)} \subset \lambda^{\gamma}$ does not hold, since the $v-$convergence of the sequence of partial sums of a double series does not imply its boundedness. The $v-$summability domain $\lambda_A^{(v)}$ of a four dimensional infinite matrix $A = (a_{mnkl})$ in a space λ of a double sequences is defined by

$$\lambda_A^{(v)} = \Big\{x = (x_{kl}) \in \Omega : Ax = \Big(v - \sum_{k,l} a_{mvkl}x_{kl}\Big)_{m,v\in\mathbb{N}} \text{ exists and is in } \lambda\Big\}. \tag{3}$$

We say, with the notataion (3), that A maps the space λ into the space μ if and only if Ax exists and is in μ for all $x \in \lambda$ and denote the set of all four dimensional matrices, transforming the space λ into the space μ, by $(\lambda : \mu)$. It is trivial that for any matrix $A \in (\lambda : \mu)$, $(a_{mvkl})_{k,l\in\mathbb{N}}$ is in the $\beta(v)-$dual $\lambda^{\beta(v)}$ of the space λ for all $m, v \in \mathbb{N}$. An infinite matrix A is said to be $\mathcal{C}_v-$ conservative if $\mathcal{C}_v \subset (\mathcal{C}_v)_A$. Also by $(\lambda : \mu : p)$, we denote the class of all four dimensional matrices $A = (a_{mnkl})$ in the class $(\lambda : \mu)$ such that $v_2 - \lim Ax = v_1 - \lim x$ for all $x \in \lambda$. Now, following Zeltser [32], we note the terminology for double sequence spaces. A locally convex double sequence space λ is called a $DK-$space, if all of the seminorms $t_{kl} : \lambda \to \mathbb{R}, \quad x = (x_{kl}) \mapsto |x_{kl}|$ for all $k, l \in \mathbb{N}$ are continuous. A $DK-$space with a Fréchet topology is called an $FDK-$space. A normed $FDK-$space is called a $BDK-$space. We record that $\mathcal{C}_t$ endowed with the norm $\|\cdot\|_\infty : \mathcal{C}_t \to \mathbb{R}, \quad x = (x_{kl}) \mapsto \sup_{k,l\in\mathbb{N}} |x_{kl}|$ is a $BDK-$space. Let us define the following sets of double sequences:

$$\mathcal{M}_u(g) = \Big\{(x_{mv}) \in \Omega : \sup_{m,v\in\mathbb{N}} |x_{mv}|^{g_{mv}} < \infty\Big\},$$

$$\mathcal{C}_p(g) = \Big\{(x_{mv}) \in \Omega : \exists l \in \mathbb{C} \ni p - \lim_{m,v\in\mathbb{N}} |x_{mv} - l|^{g_{mv}} = 0\Big\},$$

$$\mathcal{C}_{0p}(g) = \Big\{(x_{mv}) \in \Omega : p - \lim_{m,v\in\mathbb{N}} |x_{mv}|^{g_{mv}} = 0\Big\},$$

$$\mathcal{L}_u(g) = \Big\{(x_{mv}) \in \Omega : \sum_{m,v} |x_{mv}|^{g_{mv}} < \infty\Big\},$$

$$\mathcal{C}_{bp}(g) = \mathcal{C}_p(g) \cap \mathcal{M}_u(g) \text{ and } \mathcal{C}_{0bp}(g) = \mathcal{C}_{0p}(g) \cap \mathcal{M}_u(g),$$

where $g = (g_{mv})$ is the sequence of strictly positive reals g_{mv} for all $m, v \in \mathbb{N}$. In the case $g_{mv} = 1$ for all $m, v \in \mathbb{N}; \mathcal{M}_u(g), \mathcal{C}_p(g), \mathcal{C}_{0p}(g), \mathcal{L}_u(g), \mathcal{C}_{bp}(g)$ and $\mathcal{C}_{0bp}(g)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$, respectively. Now, we can summarize the knowledge given in some previous works related to the double sequence spaces. Gökhan and Çolak [9, 10] have proved that $\mathcal{M}_u(g), \mathcal{C}_p(g)$ and $\mathcal{C}_{bp}(g)$ are complete paranormed spaces of double sequences and gave the alpha-, beta-, gamma-duals of the spaces $\mathcal{M}_u(g)$ and $\mathcal{C}_{bp}(g)$. Mursaleen and Edely [20] have introduced the statistical convergence and statistical Cauchy for double sequences, and gave the relation between statistically convergent and strongly Cesàro summable double sequences. For recent on statistical convergence, we refer to [5]. In [1], Altay and Başar have defined the spaces $\mathcal{BS}, \mathcal{BS}(g), \mathcal{CS}_{bp}, \mathcal{CS}_t$ and $\mathcal{BV}$ of double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(g), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_t$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the alpha-duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(v)-$duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_t$ of double series. Quite recently, Başar and Sever [2] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space l_q of absolutely $q-$summable single sequences and examined some properties of the space $\mathcal{L}_q$. Furthermore, they determined the $\beta(v)-$dual of the space and established that the alpha- and gamma-duals of the space $\mathcal{L}_q$ coincide with the $\beta(v)-$dual, where

$$\mathcal{L}_q = \left\{(x_{rs}) \in \Omega : \sum_{r,s} |x_{rs}|^q < \infty\right\}, \quad (1 \le q \le< \infty),$$

$$\mathcal{CS}_v = \Big\{(x_{rs}) \in \Omega : (h_{mv}) \in \mathcal{C}_v\Big\}.$$

Here and after we assume that $v \in p, bp, t$.
Demiriz and Duyar [6] introduced the new double difference sequence spaces $\mathcal{M}_u(\Delta)$,

$\mathcal{C}_p(\Delta)$, $\mathcal{C}_{0p}(\Delta)$ and $\mathcal{L}_q(\Delta)$, where $\Delta = \delta_{mvkl}$ is the double difference matrix of order one defined by

$$\delta_{mvkl} = \left\{ \begin{array}{ll} (-1)^{m+v-k-l}, & m-1 \le k \le m, \quad v-1 \le l \le v; \\ 0, & otherwise \end{array} \right\},$$

for all $m, v, k, l \in \mathbb{N}$. Additionally, a direct calculation gives the inverse $\Delta^{-1} = H = (h_{mvkl})$ of matrix Δ as follows:

$$h_{mvkl} = \left\{ \begin{array}{ll} 1, & 0 \le k \le m, \quad 0 \le l \le v; \\ 0, & otherwise \end{array} \right\},$$

for all $m, v, k, l \in \mathbb{N}$. By $\mathcal{C}_{bp}(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ and $\mathcal{C}_t(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ we denote the set of all bounded and convergent and regularly convergent double sequence spaces, respectively (see [3]). Let M be an Orlicz function, $(X, \|\cdot,\cdot\|)$ be a 2-normed space, $w = (w_{mv})$ be a bounded sequence of strictly positive real numbers and $u = (u_{mv})$ be a sequence of positive real numbers. By $\Omega(2-x)$ we denote the space of all double sequences defined over $(X, \|\cdot,\cdot\|)$. In the present paper we define new generalized double difference classes of sequences as follows:

$$\mathcal{M}_u(M, \Delta^n, u, w, \|\cdot,\cdot\|) = \left\{ (x_{mv}) \in \Omega(2-x) : \sup_{m,v \in \mathbb{N}} M\Big(\Big\| \frac{u_{mv}\Delta^n x_{mv}}{\rho}, z \Big\| \Big)^{w_{mv}} < \infty \right\},$$

$$\mathcal{C}_p(M, \Delta^n, u, w, \|\cdot,\cdot\|) =$$

$$= \left\{ (x_{mv}) \in \Omega(2-x) : \exists l \in \mathbb{C} \ni p - \lim_{m,v \to \infty} M\Big(\Big\| \frac{u_{mv}\Delta^n x_{mv} - l}{\rho}, z \Big\| \Big)^{w_{mv}} = 0 \right\},$$

$$\mathcal{C}_{0p}(M, \Delta^n, u, w, \|\cdot,\cdot\|) = \left\{ (x_{mv}) \in \Omega(2-x) : p - \lim_{m,v \to \infty} M\Big(\Big\| \frac{u_{mv}\Delta^n x_{mv}}{\rho}, z \Big\| \Big)^{w_{mv}} = 0 \right\},$$

$$\mathcal{L}_q(M, \Delta^n, u, w, \|\cdot,\cdot\|) = \left\{ (x_{mv}) \in \Omega(2-x) : \Big[\sum_{m,v} M\Big(\Big\| \frac{u_{mv}\Delta^n x_{mv}}{\rho}, z \Big\| \Big)^{w_{mv}} \Big]^q < \infty, \right.$$

$$\left. 1 \le q < \infty \right\},$$

where $\Delta^n x_{mv} = \Delta^{n-1} x_{mv} - \Delta^{n-1} x_{m,v+1} - \Delta^{n-1} x_{m+1,v} + \Delta^{n-1} x_{m+1,v+1}$.

Define the sequence $y = (y_{mv})$ as the difference transform of the sequence $x = (x_{mv})$, and an Orlicz function over 2-normed space:

$$y_{mv} = (\Delta^n x)_{mv} = M\Big(\Big\| \frac{u_{mv}\Delta^n x_{mv}}{\rho}, z \Big\| \Big)^{w_{mv}}, \tag{4}$$

for all $m, v \in \mathbb{N}$.

The main purpose of this paper is to study some generalized double difference sequence spaces via Orlicz function over 2-normed spaces. In the beginning we establish some topological properties and prove some inclusion relations between above defined sequence spaces. Further we also determine the duals of some spaces. In the end we characterize the matrix transformation from the space $\mathcal{C}_t(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ to the double sequence space $\mathcal{C}_\vartheta$.

2. Some topological properties

In this section we make an effort to prove some topological properties and inclusion relations between above defined sequence spaces.

Theorem 1. *Let M be an Orlicz function, $w = (w_{mv})$ be a bounded sequence of strictly positive real numbers and $u = (u_{mv})$ be a sequence of positive real numbers. Then the classes of sequences $\mathcal{M}_u(M, \Delta^n, u, w, \|\cdot,\cdot\|)$, $\mathcal{C}_p(M, \Delta^n, u, w, \|\cdot,\cdot\|)$, $\mathcal{C}_{0p}(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ and $\mathcal{L}_q(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ are linear spaces with the coordinate wise addition and scalar multiplication and $\mathcal{M}_u(M, \Delta^n, u, w, \|\cdot,\cdot\|)$, $\mathcal{C}_p(M, \Delta^n, u, w, \|\cdot,\cdot\|)$, $\mathcal{C}_{0p}(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ and $\mathcal{L}_q(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ are Banach spaces with the norms*

$$\|x\|_{\mathcal{M}_u(M\Delta^n, u, w, \|\cdot,\cdot\|)} = \sup_{m,v\in\mathbb{N}} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}}, \tag{5}$$

$$\|x\|_{\mathcal{L}_q(M,\Delta^n, u, w, \|\cdot,\cdot\|)} = \left[\sum_{m,v}\left[M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}}\right]^q\right]^{\frac{1}{q}} \quad (1 \le q \le \infty). \tag{6}$$

Proof. The first part of the theorem is obvious, so we omit the details. Since the proof may be given for the spaces $\mathcal{M}_u(M, \Delta^n, u, w\|\cdot,\cdot\|)$, $\mathcal{C}_p(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ and $\mathcal{C}_{0p}(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ to avoid the repetition of the similar statements, we prove the theorem only for the space $\mathcal{L}_q(M, \Delta^n, u, w, \|\cdot,\cdot\|)$.

It is obvious that $\|x\|_{\mathcal{L}_q(M,\Delta^n,u,w,\|\cdot,\cdot\|)} = \|y\|_{\mathcal{L}_q}$, where $\|\cdot\|_q$ is the norm on the space $\mathcal{L}_q$. Let $\{x^{(i)}\}$ is a Cauchy sequence in $\mathcal{L}_q(M, \Delta^n, u, w, \|\cdot,\cdot\|)$. Then $\{y^{(i)}\}_{i\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}_q$, where $y^{(i)} = \{y^{(i)}_{mv}\}_{m,v=0}^{\infty}$ with

$$y^{(i)}_{mv} = M\Big(\Big\|\frac{u_{mv}\Delta^n x^i_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}},$$

for all $m, v \in \mathbb{N}$. Then for a given $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that

$$\begin{aligned}\|y^i - y^j\|_q &= \Big\{\sum_{m,v}|y^i_{mv} - y^j_{mv}|^q\Big\}^{\frac{1}{q}}\\ &= \Big\{\sum_{m,v}\Big[M\Big(\Big\|\frac{u_{mv}\Delta^n x^i_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} - M\Big(\Big\|\frac{u_{mv}\Delta^n x^j_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}}\Big]^q\Big\}^{\frac{1}{q}}\end{aligned}$$

$$< \quad \varepsilon \quad \text{for all} \quad i,j \in \mathbb{N}, \tag{7}$$

which lead us to the fact that $\{y_{mv}^{(i)}\}_{m,v\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$. As $\mathbb{C}$ is complete, it converges, say

$$\lim_{i\to\infty} y_{mv}^{(i)} = y_{mv}. \tag{8}$$

Using these infinitely many limits, we define the sequence $y = (y_{mv})_{m,v=0}^{\infty}$. Then we get by (8) that

$$\lim_{i\to\infty} \|y_{mv}^{i} - y_{mv}\|_{\mathcal{L}_q(M,\Delta^n,u,w,\|\cdot,\cdot\|)}$$

$$= \lim_{i\to\infty} \Big\{ \sum_{m,v} \Big[M\Big(\Big\| \frac{u_{mv}\Delta^n x_{mv}}{\rho}, z \Big\| \Big)^{w_{mv}} - M\Big(\Big\| \frac{u_{mv}\Delta^n x_{mv}^{i}}{\rho}, z \Big\| \Big)^{w_{mv}} \Big]^q \Big\}^{\frac{1}{q}}$$

$$= 0. \tag{9}$$

Now we have to show that $y \in \mathcal{L}_q$. Since $y^i = \{y_{mv}^{i}\}_{m,v=0}^{\infty} \in \mathcal{L}_q$, by (9)

$$\Big\{ \sum_{m,v} \Big[M\Big(\Big\| \frac{u_{mv}\Delta^n x_{mv}}{\rho}, z \Big\| \Big)^{w_{mv}} \Big]^q \Big\}^{\frac{1}{q}}$$

$$\begin{aligned} \leq \quad & \Big\{ \sum_{m,v} \Big[M\Big(\Big\| \frac{u_{mv}\Delta^n x_{mv}}{\rho}, z \Big\| \Big)^{w_{mv}} - M\Big(\Big\| \frac{u_{mv}\Delta^n x_{mv}^{i}}{\rho}, z \Big\| \Big)^{w_{mv}} \Big]^q \Big\}^{\frac{1}{q}} \\ + \quad & \sum_{m,v} \Big[M\Big(\Big\| \frac{u_{mv}\Delta^n x_{mv}^{i}}{\rho}, z \Big\| \Big)^{w_{mv}} \Big]^q \Big\}^{\frac{1}{q}} < \infty, \end{aligned}$$

which shows that the sequence $\mathcal{L}_q(M,\Delta^n,u,w,\|\cdot,\cdot\|)$ belongs to $\mathcal{L}_q$. As $\{x_{mv}^{(i)}\}_{i\in\mathbb{N}}$ was arbitrary Cauchy sequence in $\mathcal{L}_q(M,\Delta^n,u,w,\|\cdot,\cdot\|)$, the space is complete. This completes the proof. ◄

Theorem 2. *Let M be an Orlicz function, $w = (w_{mv})$ be a bounded sequence of strictly positive real numbers and $u = (u_{mv})$ be a sequence of positive real numbers. Then the space $\lambda(M,\Delta^n,u,w,\|\cdot,\cdot\|)$ is linear isomorphic to λ, where λ denotes any of the spaces $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{0p}$ and $\mathcal{L}_q$.*

Proof. We show here that $\mathcal{M}_u(M,\Delta^n,u,w,\|\cdot,\cdot\|)$ is linear isomorphic to $\mathcal{M}_u$. Consider the transformation T from $\mathcal{M}_u(M,\Delta^n,u,w,\|\cdot,\cdot\|)$ to $\mathcal{M}_u$ defined by

$$x_{rs} = \sum_{m=0}^{r}\sum_{v=0}^{s} y_{mv} = \sum_{m=0}^{r}\sum_{v=0}^{s} M\Big(\Big\| \frac{u_{mv}\Delta^n x_{mv}}{\rho}, z \Big\| \Big)^{w_{mv}}, \tag{10}$$

for all $r,s \in \mathbb{N}$. Suppose $y \in \mathcal{M}_u$. Then, since

$$\|x\|_{\mathcal{M}_u(M,\Delta^n,u,w,\|\cdot,\cdot\|)} \quad = \quad \sup_{r,s\in\mathbb{N}} \Big| \sum_{m=0}^{r}\sum_{v=0}^{s} M\Big(\Big\| \frac{u_{mv}\Delta^n x_{mv}}{\rho}, z \Big\| \Big)^{w_{mv}} \Big|$$

$$= \sup_{r,s\in\mathbb{N}} |y_{rs}| = \|y\|_\infty < \infty,$$

$x = (x_{rs})$ defined by (10) is in the space $\mathcal{M}_u(M, \Delta^n, u, w, \|\cdot,\cdot\|)$. Hence T is surjective and norm preserving. This completes the proof. ◄

Theorem 3. *Let M be an Orlicz function, $w = (w_{mv})$ be a bounded sequence of strictly positive real numbers and $u = (u_{mv})$ be a sequence of positive real numbers. Then $\mathcal{M}_u$ is a subspace of the space $\mathcal{M}_u(M, \Delta^n, u, w, \|\cdot,\cdot\|)$.*

Proof. Let us take $x = (x_{mv}) \in \mathcal{M}_u$. Then there exists a K such that $\sup_{m,v} |x_{mv}| \leq K$ for all $m, v \in \mathbb{N}$. If we take $M = I$(Identity), $u = (u_{mv}) = 1$, $w = (w_{mv}) = 1$ for all $m, v \in \mathbb{N}$, $n = 1, \rho = 1$, and replace 2-norm by 1-norm($\|x\| = \sum_{i=1}^{n} |x_i|$), then one can observe that

$$\begin{aligned} |\Delta x_{mv}| &= |x_{mv} - x_{m,v+1} - x_{m+1,v} + x_{m+1,v+1}| \\ &\leq |x_{mv}| + |x_{m,v+1}| + |x_{m+1,v}| + |x_{m+1,v+1}|. \end{aligned} \tag{11}$$

Then we see by taking supremum over $m, v \in \mathbb{N}$ in (11), that $\|x\|_\infty \leq 4K$ that is $x \in \mathcal{M}_u(M, \Delta^n, u, w, \|\cdot,\cdot\|)$.

Now we see that inclusion is strict. Let $x = (x_{mv})$ be defined by $x_{mv} = mv$, for all $m, v \in \mathbb{N}$. Then $x \in \mathcal{M}_u(M, \Delta^n, u, w, \|\cdot,\cdot\|) \backslash \mathcal{M}_u$. This completes the proof. ◄

Theorem 4. *Let M be an Orlicz function, $w = (w_{mv})$ be a bounded sequence of strictly positive real numbers and $u = (u_{mv})$ be a sequence of positive real numbers. The inclusion $\mathcal{L}_q \subset \mathcal{L}_q(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ strictly holds, where $1 \leq q < \infty$.*

Proof. To prove the validity of the inclusion $\mathcal{L}_q \subset \mathcal{L}_q(M, \Delta^n, u, w, \|\cdot,\cdot\|)$, it suffices to show the existence of a number $K > 0$ such that

$$\|x\|_{\mathcal{L}_q(M,\Delta^n,u,w,\|\cdot,\cdot\|)} \leq K \|x\|_{\mathcal{L}_q},$$

for every $x \in \mathcal{L}_q$. Let $x \in \mathcal{L}_q$, $1 \leq q < \infty$, take $M = I$(Identity), $u = (u_{mv}) = 1$, $w = (w_{mv}) = 1$ for all $m, v \in \mathbb{N}$, $n = 1, \rho = 1$, and replace 2-norm by 1-norm. Then we obtain

$$\begin{aligned} \|x\|_{\mathcal{L}_q(M,\Delta^n,u,w,\|\cdot,\cdot\|)} &= \left\{ \sum_{m,v} |x_{mv} - x_{m,v+1} - x_{m+1,v} + x_{m+1,v+1}|^q \right\}^{\frac{1}{q}} \\ &\leq 4\|x\|_{\mathcal{L}_q}. \end{aligned}$$

This shows that the inclusion $\mathcal{L}_q \subset \mathcal{L}_q(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ holds. Additionally, the sequence $x = (x_{mv})$ defined by

$$x_{mv} = \begin{cases} 1, & v = 0, \\ 0, & \textit{otherwise}, \end{cases}$$

for all $m, v \in \mathbb{N}$ is in $\mathcal{L}_q(M, \Delta^n, u, w, \|\cdot,\cdot\|)$, but not in $\mathcal{L}_q$, as asserted. This completes the proof. ◄

Theorem 5. *Let M be an Orlicz function, $w = (w_{mv})$ be a bounded sequence of strictly positive real numbers and $u = (u_{mv})$ be a sequence of positive real numbers. Then the following statements hold:*
(i) $\mathcal{C}_p$ is a subspace of $\mathcal{C}_p(M, \Delta^n, u, w, \|\cdot,\cdot\|)$;
(ii) $\mathcal{C}_{0p}$ is a subspace of $\mathcal{C}_{0p}(M, \Delta^n, u, w, \|\cdot,\cdot\|)$.

Proof. We only prove the inclusion $\mathcal{C}_p \subset \mathcal{C}_p(M, \Delta^n, u, w, \|\cdot,\cdot\|)$. Let us take $x \in \mathcal{C}_p$. Then for given $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$|x_{mv} - l| < \frac{\varepsilon}{4},$$

for all $m, v \in n(\varepsilon)$. In particular, by taking $M = I$(Identity), $u = (u_{mv}) = 1$ $w = (w_{mv}) =$ 1for all $m, v \in \mathbb{N}$, $n = 1, \rho = 1$, and replacing 2-norm by 1-norm, we obtain

$$\begin{aligned} |\Delta x_{mv}| &= |x_{mv} - x_{m,v+1} - x_{m+1,v} + x_{m+1,v+1}| \\ &\leq |x_{mv} - l| + |x_{m,v+1} - l| + |x_{m+1,v} - l| + |x_{m+1,v+1} - l| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon, \end{aligned}$$

for sufficiently large m, v, which means $p - \lim_{m,v\to\infty} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} = 0$. Hence $x \in \mathcal{C}_p(M, \Delta^n, u, w, \|\cdot,\cdot\|)$. This is to say that $\mathcal{C}_p \subset \mathcal{C}_p(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ holds.
Now we show that the inclusion is strict. Let $x = (x_{mv})$ be defined by $x_{mv} = (m+1)(v+1)$ for all $m, v \in \mathbb{N}$. It is easy to see that

$$p - \lim_{m,v\to\infty} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} = 1.$$

But $\lim_{m,v\to\infty}(m+1)(v+1)$ does not tend to a finite limit. Hence $x \notin \mathcal{C}_p$. This completes the proof. ◄

3. The $\alpha-$ and $\beta(t)-$ duals

In this section, we determine the alpha-dual of the space $\mathcal{M}_u(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ and the $\beta(t)-$ dual of the space $\mathcal{C}_t(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ and $\beta(\vartheta)-$ dual of the space $\mathcal{C}_\eta(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ of double sequences, $\vartheta, \eta \in \{p, bp, t\}$. Although the α-dual of a space of double sequences is unique, its β-dual may be more than one with respect to $\vartheta-$convergence.

Theorem 6. *Let M be an Orlicz function, $w = (w_{mv})$ be a bounded sequence of strictly positive real numbers and $u = (u_{mv})$ be a sequence of positive real numbers. Then the $\alpha-$dual of space $\mathcal{M}_u(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ is $\mathcal{L}_u$.*

Proof. Let $x = (x_{rs}) \in \mathcal{M}_u(M, \Delta^n, u, w, \|\cdot, \cdot\|)$ and $z = (z_{rs}) \in \mathcal{L}_u$. Hence, by Theorem 2, there is a sequence $y = (y_{mv}) \in \mathcal{M}_u$, and by taking $M = I$(Identity), $u = (u_{mv}) = 1$, $w = (w_{mv}) = 1$, for all $m, v \in \mathbb{N}$, $n = 1, \rho = 1$, and replacing 2-norm by 1-norm, we obtain that there exists a positive real number K such that $|y_{mv}| = \frac{K}{(m+1)(v+1)}$ for all $m, v \in \mathbb{N}$. So we use the relation (10) to have

$$\begin{aligned}\sum_{r,s} |z_{rs}x_{rs}| &= \sum_{r,s} \left| z_{rs} \sum_{m=0}^{r} \sum_{v=0}^{s} M\left(\left\| \frac{u_{mv}\Delta^n x_{mv}}{\rho}, z \right\|\right)^{w_{mv}} \right| \\ &= \sum_{r,s} \left| z_{rs} \sum_{m=0}^{r} \sum_{v=0}^{s} y_{mv} \right| \\ &\leq K \sum_{r,s} |z_{rs}| < \infty.\end{aligned}$$

So $z \in \{\mathcal{M}_u(M, \Delta^n, u, w, \|\cdot, \cdot\|)\}^\alpha$, that is

$$\mathcal{L}_u \subset \{\mathcal{M}_u(M, \Delta^n, u, w, \|\cdot, \cdot\|)\}^\alpha. \tag{12}$$

Conversely, suppose $z = (z_{rs}) \in \{\mathcal{M}_u(M, \Delta^n, u, w, \|\cdot, \cdot\|)\}^\alpha$. By taking $M = I$(Identity), $u = (u_{mv}) = 1$, $w = (w_{mv}) = 1$, for all $m, v \in \mathbb{N}$, $n = 1, \rho = 1$, and replacing 2-norm by 1-norm, we have $\sum_{r,s} |z_{rs}x_{rs}| < \infty$, where $x = (x_{rs}) \in \mathcal{M}_u(M, \Delta^n, u, w, \|\cdot, \cdot\|)$. If $z = (z_{rs}) \notin \mathcal{L}_u$, then $\sum_{r,s} |z_{rs}| = \infty$,. Further if, we choose $y = (y_{mv})$ such that

$$y_{mv} = \begin{cases} \frac{1}{(m+1)(v+1)}, & 0 \leq r \leq m, \quad 0 \leq s \leq v; \\ 0, & \textit{otherwise}. \end{cases}$$

for all $m, v \in \mathbb{N}$, then $y \in \mathcal{M}_u$, but

$$\sum_{r,s} |z_{rs}x_{rs}| = \sum_{r,s} \left| z_{rs} \sum_{m=0}^{r} \sum_{v=0}^{s} \frac{1}{(m+1)(v+1)} \right| = \sum_{r,s} |z_{rs}| = \infty.$$

Hence $z \notin \{\mathcal{M}_u(M, \Delta^n, u, w, \|\cdot, \cdot\|)\}^\alpha$, which is a contradiction. So we have the following inclusion:

$$\{\mathcal{M}_u(M, \Delta^n, u, w, \|\cdot, \cdot\|)\}^\alpha \subset \mathcal{L}_u. \tag{13}$$

Hence from the inclusions (12) and (13) we have

$$\{\mathcal{M}_u(M, \Delta^n, u, w, \|\cdot, \cdot\|)\}^\alpha = \mathcal{L}_u. \quad \blacktriangleleft$$

Lemma 1. *The matrix $A = (a_{mvrs})$ is in $(\mathcal{C}_t : \mathcal{C}_\vartheta)$ if and only if the conditions hold:*

$$\sup_{m,v} \sum_{r,s} |a_{mnrs}| < \infty, \tag{14}$$

$$\exists \upsilon \in \mathbb{C} \ni \vartheta - \lim_{m,v\to\infty} \sum_{r,s} a_{mvrs} = \upsilon, \tag{15}$$

$$\exists (a_{rs}) \in \Omega \ni \vartheta - \lim_{m,v\to\infty} \sum_{r,s} a_{mvrs} = a_{rs} \ \ \textit{for all} \ \ r,s \in \mathbb{N}, \tag{16}$$

$$\exists u^{s_0} \in \mathbb{C} \ni \vartheta - \lim_{m,v\to\infty} \sum_{r} a_{mvrs_0} = u^{s_0} \ \ \textit{for fixed} \ \ s_0 \in \mathbb{N}, \tag{17}$$

$$\exists v_{r_0} \in \mathbb{C} \ni \vartheta - \lim_{m,v\to\infty} \sum_{r} a_{mvr_0 s} = v_{r_0} \ \ \textit{for fixed} \ \ r_0 \in \mathbb{N}. \tag{18}$$

Lemma 2. *The matrix* $A = (a_{mvrs})$ *is in* $(\mathcal{C}_{bp} : \mathcal{C}_{\vartheta})$ *if and only if the following conditions* (14)-(16) *of Lemma 1 hold, and*

$$\vartheta - \lim_{m,v\to\infty} \sum_{r} |a_{mvrs_0} - a_{rs_0}| = 0 \ \ \textit{for fixed} \ \ s_0 \in \mathbb{N}, \tag{19}$$

$$\vartheta - \lim_{m,v\to\infty} \sum_{r} |a_{mvr_0 s} - a_{r_0 s}| = 0 \ \ \textit{for fixed} \ \ r_0 \in \mathbb{N}. \tag{20}$$

Lemma 3. *The matrix* $A = (a_{mvrs})$ *is in* $(\mathcal{C}_{bp} : \mathcal{C}_{\vartheta})$ *if and only if the conditions* (14)-(16) *of Lemma 1 hold, and*

$$\forall \, r \in \mathbb{N} \, \exists \, R \in \mathbb{N} \ \ni \ a_{mvrs} = 0 \ \ \textit{for} \ \ r > R \ \ \textit{for all} \ \ \ m,v \in \mathbb{N}, \tag{21}$$

$$\forall \, s \in \mathbb{N} \, \exists \, S \in \mathbb{N} \ \ni \ a_{mvrs} = 0 \ \ \textit{for} \ \ s > S \ \ \textit{for all} \ \ \ m,v \in \mathbb{N}. \tag{22}$$

Theorem 7. *Define the sets*

$$F_1 = \Big\{ a = (a_{rs}) \in \Omega(2-X) : \sum_{rs} (r+1)(s+1) \Big| M\Big(\Big\| \frac{u_{mv}\Delta^n x_{mv}}{\rho}, z \Big\| a_{rs} \Big)^{w_{mv}} \Big| < \infty \Big\},$$

$$F_2 = \Big\{ a = (a_{rs}) \in \Omega(2-X) : t - \lim_{m,v\to\infty} \sum_{r} \sum_{p=r}^{m} \sum_{q=s_0}^{v} M\Big(\Big\| \frac{u_{mv}\Delta^n x_{mv}}{\rho}, z \Big\| \Big)^{w_{mv}} a_{pq},$$

$$\textit{exists for each fixed} \ \ s_0 \Big\},$$

$$F_3 = \Big\{ a = (a_{rs}) \in \Omega(2-X) : t - \lim_{m,v\to\infty} \sum_{s} \sum_{p=r_0}^{m} \sum_{q=s_0}^{v} M\Big(\Big\| \frac{u_{mv}\Delta^n x_{mv}}{\rho}, z \Big\| \Big)^{w_{mv}} a_{pq},$$

$$\textit{exists for each fixed} \ \ r_0 \Big\}.$$

Then $\{\mathcal{C}_t(M, \Delta^n, u, w, \|\cdot,\cdot\|)\}^{\beta(t)} = F_1 \cap F_2 \cap F_3$.

Proof. Let $x = (x_{rs}) \in \mathcal{C}_t(M, \Delta^n, u, w, \|\cdot, \cdot\|)$. Then there exists a sequence $y = (y_{mv}) \in \mathcal{C}_t$. Consider the inequality

$$\begin{aligned}
z_{mv} &= \lim_{m,v\to\infty} \sum_{r=0}^{m}\sum_{s=0}^{v} a_{rs}x_{rs} \\
&= \lim_{m,v\to\infty} \sum_{r=0}^{m}\sum_{s=0}^{v} a_{rs}\left(\sum_{p=0}^{r}\sum_{q=0}^{s}\left(M\left(\left\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\right\|\right)^{w_{mv}} y_{pq}\right)\right. \\
&= \lim_{m,v\to\infty} \sum_{r=0}^{m}\sum_{s=0}^{v} \left(\sum_{p=0}^{r}\sum_{q=0}^{s} M\left(\left\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\right\|\right)^{w_{mv}} a_{pq}\right) y_{rs} \\
&= \sum_{r=0}^{m}\sum_{s=0}^{v} b_{mvrs}y_{rs} \\
&= (By)_{rs},
\end{aligned}$$

for all $m, v \in \mathbb{N}$. Hence we define the four dimensional matrix $B = (b_{mvrs})$ as follows:

$$b_{mvrs} = \begin{cases} \lim\limits_{m,v\to\infty} \sum\limits_{p=0}^{r}\sum\limits_{q=0}^{s} M\left(\left\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\right\|\right)^{w_{mv}} a_{pq}, & 0 \le r \le m,\ 0 \le s \le v; \\ 0, & otherwise. \end{cases} \tag{23}$$

Thus we see that $ax = (a_{mv}x_{mv}) \in \mathcal{CS}_{\sqcup}$ whenever $x = (x_{mv}) \in \mathcal{C}_t(M, \Delta^n, u, w, \|\cdot, \cdot\|)$ if and only if $z = (z_{mv}) \in \mathcal{C}_t$ whenever $y = (y_{mv}) \in \mathcal{C}_t$. This means that $a = (a_{mv}) \in \{\mathcal{C}_t(M, \Delta^n, u, w, \|\cdot, \cdot\|)\}^{\beta(t)}$ if and only if $B \in (\mathcal{C}_t : \mathcal{C}_t)$. Therefore, we consider the following equality and equation:

$$\begin{aligned}
\sup_{m,v}\sum_{r=0}^{m}\sum_{s=0}^{v} |b_{mvrs}| &\le \sup_{m,v}\sum_{r=0}^{m}\sum_{s=0}^{v}\left(\sum_{p=0}^{r}\sum_{q=0}^{s} M\left(\left\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\right\|\right)^{w_{mv}} |a_{pq}|\right) \\
&= \sup_{m,v}\sum_{r=0}^{m}\sum_{s=0}^{v}\left(\sum_{p=0}^{r}\sum_{q=0}^{s} M\left(\left\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\right\|\right)^{w_{mv}} |a_{rs}|\right)
\end{aligned}$$

$$= \sup_{r,s}\sum_{r=0}^{m}\sum_{s=0}^{v}(r+1)(s+1)M\left(\left\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\right\|\right)^{w_{mv}} |a_{rs}|, \tag{24}$$

$$t - \lim_{m,v}\sum_{r,s} b_{mvrs} = t - \lim_{m,v}\sum_{r=0}^{m}\sum_{s=0}^{v}\left(\sum_{p=0}^{r}\sum_{q=0}^{s} M\left(\left\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\right\|\right)^{w_{mv}} a_{pq}\right)$$

$$= \sup_{r,s}\left(\sum_{p=0}^{r}\sum_{q=0}^{s} M\left(\left\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\right\|\right)^{w_{mv}} a_{pq}\right). \tag{25}$$

Then we derive from the condition (14)-(16) that

$$\sup_{r,s}(r+1)(s+1)M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho},z\Big\|\Big)^{w_{mv}}|a_{rs}|<\infty. \tag{26}$$

From Lemma 1, conditions (17) and (18) it follows that

$$t-\lim_{m,v\to\infty}\sum_{r}^{m} b_{mvrs_0}=t-\lim_{m,v\to\infty}\sum_{r}^{m}\sum_{p=r}^{m}\sum_{q=s_0}^{v} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho},z\Big\|\Big)^{w_{mv}}a_{pq}, \tag{27}$$

exists for each fixed $s_0\in\mathbb{N}$ and

$$t-\lim_{m,v\to\infty}\sum_{s}^{m} b_{mvr_0s}=t-\lim_{m,v\to\infty}\sum_{s}^{m}\sum_{p=r_0}^{m}\sum_{q=s}^{v} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho},z\Big\|\Big)^{w_{mv}}a_{pq}, \tag{28}$$

exists for each fixed $r_0\in\mathbb{N}$. This shows that $\{\mathcal{C}_t(M,\Delta^n,u,w,\|\cdot,\cdot\|)\}^{\beta(t)}=F_1\cap F_2\cap F_3$, which completes the proof. ◄

Now, we may give our theorem exhibiting the $\beta(\vartheta)$-dual of the series space $\mathcal{C}_\eta(M,\Delta^n,u,w,\|\cdot,\cdot\|)$ in the case $\eta,\ \vartheta\in\{p,bp,t\}$ without proof.

Theorem 8. $\{\mathcal{C}_\eta(M,\Delta^n,u,w,\|\cdot,\cdot\|)\}^{\beta(\vartheta)}=\{a=(a_{mv})\in\Omega : B=(b_{mvrs})\in(\mathcal{C}_\eta:\mathcal{C}_\vartheta)\}$, *where* $B=(b_{mvrs})$ *is defined by* (23).

4. Characterization of some four dimensional matrices

Theorem 9. *The matrix* $A=(a_{mvrs})$ *is in* $(\mathcal{C}_t(M,\Delta^n,u,w,\|\cdot,\cdot\|):\mathcal{C}_\vartheta)$ *if and only if the following conditions hold:*

$$\sup_{m,v}\sum_{k,l}\Big|\sum_{p=k}^{\infty}\sum_{q=l}^{\infty} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho},z\Big\|\Big)^{w_{mv}}a_{mvpq}\Big|<\infty, \tag{29}$$

$$\vartheta-\lim_{b,c\to\infty}\sum_{r=0}^{b}\sum_{p=r}^{b}\sum_{q=s_0}^{c} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho},z\Big\|\Big)^{w_{mv}}a_{mvpq}\ \textit{exists for fixed}\ s_0, \tag{30}$$

$$\vartheta-\lim_{b,c\to\infty}\sum_{s=0}^{c}\sum_{p=r_0}^{b}\sum_{q=s}^{c} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho},z\Big\|\Big)^{w_{mv}}a_{mvpq}\ \textit{exists for fixed}\ r_0, \tag{31}$$

$$\vartheta-\lim_{mv}\sum_{p=k}^{\infty}\sum_{q=l}^{\infty} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho},z\Big\|\Big)^{w_{mv}}a_{mvpq}=a_{kl}\ \textit{for all}\ k,l\in\mathbb{N}, \tag{32}$$

$$\exists u^{l_0}\in\mathbb{C}\ni\vartheta-\lim_{m,v}\sum_{k}\sum_{p=k}^{\infty}\sum_{q=l_0}^{\infty} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho},z\Big\|\Big)^{w_{mv}}a_{mvpq}=u^{l_0}\ \ \textit{for fixed}\ \ l_0\in\mathbb{N}, \tag{33}$$

$$\exists v_{k_0} \in \mathbb{C} \ni \vartheta - \lim_{m,v} \sum_{l} \sum_{p=k_0}^{\infty} \sum_{q=l}^{\infty} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} a_{mvpq} = v_{k_0} \ \ \textit{for fixed} \ \ k_0 \in \mathbb{N}, \tag{34}$$

$$\exists v \in \mathbb{C} \ni \vartheta - \lim_{m,v} \sum_{k,l} \sum_{l} \sum_{p=k}^{\infty} \sum_{q=l}^{\infty} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} a_{mvpq} = v. \tag{35}$$

Proof. Let $x = (x_{mv}) \in \mathcal{C}_t(M, \Delta^n, u, w, \|\cdot,\cdot\|)$. If we take $M = I$(Identity), $u = (u_{mv}) = 1$, $w = (w_{mv}) = 1$ for all $m, v \in \mathbb{N}$, $n = 1, \rho = 1$, and replace 2-norm by 1-norm, then we define the sequence $y = (y_{k,l})$ by

$$y_{kl} = x_{kl} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1} \ \ (k,l) \in \mathbb{N}.$$

Then $y = (y_{kl}) \in \mathcal{C}_t$ by Theorem 2. Now for the (b,c)th rectangular partial sum of the series $\sum_{r,s} a_{mvrs}x_{rs}$, we derive that

$$\begin{aligned}(Ax)_{mv}^{[b,c]} &= \sum_{r=0}^{b}\sum_{s=0}^{c} a_{mvrs}x_{rs}\\ &= \sum_{r=0}^{b}\sum_{s=0}^{c}\left(\sum_{p=0}^{r}\sum_{q=0}^{s} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} y_{pq}\right) a_{mvrs}\end{aligned}$$

$$= \sum_{r=0}^{b}\sum_{s=0}^{c}\left(\sum_{p=0}^{r}\sum_{q=0}^{s} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} a_{mvrs}\right) y_{pq}, \tag{36}$$

for all $m, v, b, c \in \mathbb{N}$. Define the matrix $D_{mv} = (d_{mvrs}^{[b,c]})$ by

$$(d_{mvrs}^{[b,c]}) = \begin{cases} \sum_{p=r}^{b}\sum_{q=s}^{c} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} a_{mvpq}, & 0 \le r \le b, \ 0 \le s \le c; \\ 0, & \textit{otherwise}. \end{cases} \tag{37}$$

Then the inequality (36) may be rewritten as

$$(Ax)_{mv}^{[b,c]} = (D_{mv}y)_{[b,c]}. \tag{38}$$

Then the convergence of the rectangular partial sums $(Ax)_{mv}^{[b,c]}$ in the regular sense for all $m, v \in \mathbb{N}$ and for all $x \in \mathcal{C}_t(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ is equivalent to saying that $D_{mv} \in (\mathcal{C}_t : \mathcal{C}_\vartheta)$. Hence the following conditions

$$\sum_{k,l}(k+1)(l+1)\Big|M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} a_{mvkl}\Big| < \infty, \tag{39}$$

$$\vartheta - \lim_{b,c\to\infty} \sum_{r=0}^{b} \sum_{p=r}^{b} \sum_{q=s_0}^{c} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} a_{mvpq} \text{ exists for fixed } s_0, \tag{40}$$

$$\vartheta - \lim_{b,c\to\infty} \sum_{s=0}^{c} \sum_{p=r_0}^{b} \sum_{q=s}^{c} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} a_{mvpq} \text{ exists for fixed } r_0, \tag{41}$$

must be satisfied for every fixed $m, v \in \mathbb{N}$. In this case,

$$\vartheta - \lim_{b,c\to\infty} (d^{[b,c]}_{mvrs}) = \sum_{p=r}^{b} \sum_{q=s}^{c} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} a_{mvpq},$$

$$\vartheta - (Ax)^{[b,c]}_{mv} = t - \lim(D_{mv}y),$$

hold. Thus we derive from the two sided implication that Ax is in $\mathcal{C}_t$ whenever $x \in \mathcal{C}_t(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ if and only if

$$D = \left(\sum_{p=r}^{\infty} \sum_{q=s}^{\infty} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} a_{mvpq} \right) \in (\mathcal{C}_t : \mathcal{C}_\vartheta)''.$$

We have

$$\sup_{m,v} \sum_{k,l} \Big| \sum_{p=k}^{\infty} \sum_{q=l}^{\infty} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} a_{mvpq} \Big| < \infty, \tag{42}$$

$$\vartheta - \lim_{mv} \sum_{p=k}^{\infty} \sum_{q=l}^{\infty} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} a_{mvpq} = a_{kl} \text{ for all } k, l \in \mathbb{N}, \tag{43}$$

$$\exists u^{l_0} \in \mathbb{C} \ni \vartheta - \lim_{m,v} \sum_{k} \sum_{p=k}^{\infty} \sum_{q=l_0}^{\infty} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} a_{mvpq} = u^{l_0} \text{ for fixed } l_0 \in \mathbb{N}, \tag{44}$$

$$\exists v_{k_0} \in \mathbb{C} \ni \vartheta - \lim_{m,v} \sum_{l} \sum_{p=k_0}^{\infty} \sum_{q=l}^{\infty} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} a_{mvpq} = v_{k_0} \text{ for fixed } k_0 \in \mathbb{N}, \tag{45}$$

$$\exists v \in \mathbb{C} \ni \vartheta - \lim_{m,v} \sum_{k,l} \sum_{l} \sum_{p=k}^{\infty} \sum_{q=l}^{\infty} M\Big(\Big\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\Big\|\Big)^{w_{mv}} a_{mvpq} = v. \tag{46}$$

Now, from the conditions (39)-(46) we have that $A = (a_{mvrs})$ is in $(\mathcal{C}_t(M, \Delta^n, u, \|\cdot,\cdot\|) : \mathcal{C}_\vartheta)$ if and only if the conditions (29)-(35) hold. This completes the proof. ◄

Theorem 10. *Suppose that the elements of the four dimensional infinite matrices* $E = (e_{mvkl})$ *and* $F = (f_{mvkl})$ *are concentrated with the relation*

$$f_{mvkl} = \sum_{r=m-1}^{m} \sum_{s=v-1}^{v} (-1)^{m+v-r-s} M\left(\left\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\right\|\right)^{w_{mv}} e_{rskl}, \tag{47}$$

for all $k, l, m, v \in \mathbb{N}$ *and* μ *is any given space of double sequences. Then* $E \in (\mu : \mathcal{C}_\vartheta(M, \Delta^n, u, w, \|\cdot,\cdot\|))$ *if and only* $F \in (\mu : \mathcal{C}_\vartheta)$.

Proof. Let $x = (x_{kl}) \in \mu$ and consider the following inequality with (47):

$$\sum_{r=m-1}^{m} \sum_{s=v-1}^{v} \sum_{k=b-1}^{b} \sum_{l=c-1}^{c} (-1)^{m+v-r-s} M\left(\left\|\frac{u_{mv}\Delta^n x_{mv}}{\rho}, z\right\|\right)^{w_{mv}} \times$$

$$\times e_{rskl} x_{kl} = \sum_{k=b-1}^{b} \sum_{l=c-1}^{c} f_{mvkl} x_{kl}, \tag{48}$$

for all $m, v, b, c \in \mathbb{N}$. By letting $b, c \to \infty$ in (48), one can derive that

$$\sum_{r=m-1}^{m} \sum_{s=v-1}^{v} (-1)^{m+v-r-s} (Ex)_{rs} = (Fx)_{mv}, \tag{49}$$

for all $m, v \in \mathbb{N}$. Therefore $Ex \in \mathcal{C}_\vartheta(M, \Delta^n, u, w, \|\cdot,\cdot\|)$ if and only if $Fx \in \mathcal{C}_\vartheta$ whenever $x \in \mu$. This step completes the proof. ◀

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