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Impossibility of Power Series Expansion for Continuous Functions

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Abstract. It is well known that if $\varphi(t) \equiv t$, then the system $\{\varphi^n(t)\}_{n=0}^{\infty}$ is not a Schauder basis for $C[0,1]$ space. It has recently been shown that $\{\varphi^n(t)\}_{n=0}^{\infty}$ is not a Schauder basis for any continuous function $\varphi(t)$. The aim of this short note is to prove that, in the most general case of any continuous function $\varphi(t)$ defined on $[a,b]$, the system of powers $\{\varphi^n(t)\}_{n=0}^{\infty}$ can not even be a pseudo-basis for $C[a,b]$.

Key Words and Phrases: Schauder basis, pseudo-basis, system of powers, space of continuous functions.
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1. Introduction

We begin by recalling some notions.

Definition 1 ([1], [2]). *A sequence $\{x_n\}_{n=0}^{\infty}$ in a Banach space X is said to be a Schauder basis for X if to each vector x in X there corresponds a unique sequence of scalars $\{\alpha_n\}_{n=0}^{\infty}$ such that*

$$x = \alpha_0 x_0 + ... + \alpha_n x_n + ...$$

Definition 2 ([2]). *A sequence $\{x_n\}_{n=0}^{\infty}$ in a Banach space X with $x_n \neq 0$ is said to be a pseudo-basis (or a system of representation) for X if to each vector x in X there corresponds a sequence of scalars $\{\alpha_n\}_{n=0}^{\infty}$ such that*

$$x = \alpha_0 x_0 + ... + \alpha_n x_n + ...$$

It is evident that every Schauder basis is a pseudo-basis, but the converse statement is not true. In particular, it is known that [2, p. 150] every separable Banach space contains a pseudo-basis; but it is also known that not every separable Banach space has a Schauder basis.

It is well known that if $\varphi(t) \equiv t$, then the system $\{\varphi^n(t)\}_{n=0}^{\infty}$ is not a Schauder basis for $C[0,1]$ space (see, for example, [3, p. 51]). In [5] this fact is generalized to the case of any continuous function $\varphi(t)$:

Theorem 1 ([5]). *Let $\varphi(t)$ be any (real or complex valued) continuous function on $[a,b]$. Then $\{\varphi^n(t)\}_{n=0}^{\infty}$ is not a basis for $C[a,b]$.*

But this result does not answer the following question:

Is there a continuous function $\varphi(t)$ such that $\{\varphi^n(t)\}_{n=0}^{\infty}$ is a pseudo-basis for $C[a,b]$?

It turns out that the answer to this question is also negative. The aim of this short note is to prove this fact.

2. Main result

Theorem 2. *Let $\varphi(t)$ be any (real or complex valued) continuous function on $[a,b]$. Then $\{\varphi^n(t)\}_{n=0}^{\infty}$ is not a pseudo-basis for $C[a,b]$.*

Proof. Assume that $\{\varphi^n(t)\}_{n=0}^{\infty}$ is a pseudo-basis for $C[a,b]$. Then, $\varphi(t_1) \neq \varphi(t_2)$, if $t_1 \neq t_2$ (otherwise, every $f \in C[a,b]$ is periodic).

First, we will prove that every $f \in C[a,b]$ may have only a unique representation of the form

$$f(t) = \alpha_0 + \alpha_1\varphi(t) + ... + \alpha_n\varphi^n(t) + ..., \tag{1}$$

where the series converges uniformly on $[a,b]$.

Assume the contrary: there is a continuous function with at least two different representations of the form (1). It is equivalent to saying that the zero function has a representation

$$0 = \alpha_0 + \alpha_1\varphi(t) + ... + \alpha_n\varphi^n(t) + ..., \tag{2}$$

with at least one nonzero coefficient α_n.

We will show that such a representation is impossible. Consider the power series

$$P(z) = \alpha_0 + \alpha_1 z + ... + \alpha_n z^n + ...$$

It is evident that the radius of convergence R of this series is not less than $max_{t\in[a,b]}|\varphi(t)|$. Therefore, there are only two possibilities:

1) $R > max_{t\in[a,b]}|\varphi(t)|$;

or

2) $R = max_{t\in[a,b]}|\varphi(t)|$.

In the first case, applying the uniqueness theorem for analytic functions [4, p. 201], we obtain that $P(z) \equiv 0$ on $\{z : |z| < R\}$. This implies that $\alpha_0 = ... = \alpha_n = ... = 0$ which contradicts our assumption.

Consider the second case. If $min_{t\in[a,b]}|\varphi(t)| < max_{t\in[a,b]}|\varphi(t)|$, then again the usual uniqueness theorem for analytic functions is applicable; application of this uniqueness theorem yields $\alpha_0 = ... = \alpha_n = ... = 0$.

Now, let $R = min_{t\in[a,b]}|\varphi(t)| = max_{t\in[a,b]}|\varphi(t)| \neq 0$. Then $|\varphi(t)| \equiv const = R$ on $[a, b]$. It is easy to see that the usual uniqueness theorem for analytic functions is not applicable in this case.

The continuity of the function $\varphi(t)$ implies that the image $\varphi([a, b])$ contains some arc of the circle $\{z : |z| = R\}$. Besides, the equality (2) and the Abel's second theorem on power series (see, for example [4, p. 66]) shows that the limit of the function $P(z)$ along the radius is equal to zero at each point of $\varphi([a, b])$ and in particular, at each point of the above-mentioned arc. Therefore, application of the Lusin-Privalov radial uniqueness theorem (see, for example [4, p. 371]) shows that $P(z) \equiv 0$ on $\{z : |z| < R\}$. This yields $\alpha_0 = ... = \alpha_n = ... = 0$.

The arguments given above show that there is only a trivial representation of the form (2). Hence, under the hypothesis of the theorem, every $f \in C[a, b]$ has a unique representation of the form (1); this means that the system $\{\varphi^n(t)\}_{n=0}^{\infty}$ is a Schauder basis for $C[a, b]$. This contradicts Theorem 1 and proves the theorem. ◄

It should be noted that some questions of basicity of double systems of powers have been previously considered in [6, 7, 8].

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