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Gauss Type Inequalities

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Abstract. In this paper, we obtain Gauss type inequalities for the class of convex functions. Further, we give an application of new inequalities to obtain Stolarsky type means.

Key Words and Phrases: Gauss inequality, convex function, exponential convexity, Stolarsky type means.

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1. Introduction

In [4], Gauss mentioned the following inequality which is important in statistics.

Theorem 1. *If f is a nonnegative and nonincreasing function and $k > 0$, then*

$$\int_k^\infty f(x)dx \le \frac{4}{9k^2}\int_0^\infty x^2 f(x)dx. \tag{1}$$

In [2], Alzer gave a lower bound for the Gauss' inequality (1). In fact, he proved the following theorem.

Theorem 2. *Let $g : [a,b] \to \mathbb{R}$ be increasing, convex and differentiable, and let $f : I \to \mathbb{R}$ be nonincreasing function. Then*

$$\int_a^b f(s(x))g'(x)dx \le \int_{g(a)}^{g(b)} f(x)dx \le \int_a^b f(t(x))g'(x)dx, \tag{2}$$

where

$$s(x) = \frac{g(b)-g(a)}{b-a}(x-a)+g(a), \tag{3}$$

and

$$t(x) = g'(x_0)(x-x_0)+g(x_0), \quad x_0 \in [a,b]. \tag{4}$$

(I is an interval containing $a, b, g(a), g(b), t(a)$ and $t(b)$.)

If either g is concave or f is nondecreasing, then the reversed inequalities hold.

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Remark 1. *If we consider only the left-hand side inequality in (2), interval I should only contain $a, b, g(a)$ and $g(b)$. When considering the right-hand side inequality in (2), interval I should also contain $t(a)$ and $t(b)$.*

The aim of this paper is to extend Alzer's result to the class of convex functions. Moreover, we apply Gauss type inequalities to obtain Stolarsky type means.

First, let us recall some notions: log denotes the natural logarithm function and by I° we denote the interior of interval I.

2. Main results

In [10], Pečarić and Smoljak introduced a new class of functions that extends the class of convex functions. Let us recall the definition.

Definition 1. *Let $f : I \to \mathbb{R}$ and $c \in I^\circ$. We say that f* belongs to the class $\mathcal{M}_1^c(I)$ (resp. $\mathcal{M}_2^c(I)$) *if there exists a constant A such that the function $F(x) = f(x) - Ax$ is nonincreasing (resp. nondecreasing) on $I \cap (-\infty, c]$ and nondecreasing (resp. nonincreasing) on $I \cap [c, \infty)$.*

We can describe the property from the previous definition as "convexity (concavity) at point c".

Remark 2. *If $f \in \mathcal{M}_1^c(I)$ or $f \in \mathcal{M}_2^c(I)$ and $f'(c)$ exists, then $f'(c) = A$.*
Let us show this for $f \in \mathcal{M}_1^c(I)$. Since F is nonincreasing on $I \cap (-\infty, c]$ and nondecreasing on $I \cap [c, \infty)$, for every distinct points $x_1, x_2 \in I \cap (-\infty, c]$ and $y_1, y_2 \in I \cap [c, \infty)$ we have

$$[x_1, x_2; F] = [x_1, x_2; f] - A \leq 0 \leq [y_1, y_2; f] - A = [y_1, y_2; F].$$

Therefore, since $f'_-(c)$ and $f'_+(c)$ exist, letting $x_1 = y_1 = c$, $x_2 \nearrow c$ and $y_2 \searrow c$ we get

$$f'_-(c) \leq A \leq f'_+(c). \tag{5}$$

In the following theorem we give the connection between the class of functions $\mathcal{M}_1^c(I)$ and the class of convex functions proved in [10].

Theorem 3. *The function $f : I \to \mathbb{R}$ is convex (concave) on I if and only if it is convex (concave) at every $c \in I^\circ$.*

In the following theorems we obtain Gauss type inequalities for the class of functions that are convex at point c.

Theorem 4. *Let $c \in (a, b)$ and let $g : [a, b] \to \mathbb{R}$ be increasing, convex and differentiable such that $g(c) = c$. Assume*

$$s_1(x) = \frac{g(b) - g(c)}{b - c}(x - c) + g(c), \tag{6}$$

$$t_1(x) = g'(x_0)(x - x_0) + g(x_0), \quad x_0 \in [a, c] \tag{7}$$

and

$$\int_a^c t_1(x)g'(x)dx + \int_c^b s_1(x)g'(x)dx = \frac{g^2(b) - g^2(a)}{2}. \tag{8}$$

If $f \in \mathcal{M}_1^c(I)$, *then*

$$\int_a^c f(t_1(x))g'(x)dx + \int_c^b f(s_1(x))g'(x)dx \geq \int_{g(a)}^{g(b)} f(x)dx. \tag{9}$$

If $f \in \mathcal{M}_2^c(I)$, *then the inequality in (9) is reversed.*
(I is an interval containing $a, b, g(a), g(b), t_1(a)$ *and* $t_1(c)$.*)*

Proof. From $g(c) = c$ and other conditions of theorem it follows that $g(a), t_1(a), t_1(c) \leq c$ and $g(b) \geq c$, where $g(a) \leq c$, $t_1(a) \leq c$ and $g(b) \geq c$ follow from the fact that the function g is increasing, and $t_1(c) \leq c$ follows from the convexity of the function g. Since interval I contains $a, b, g(a), g(b), t_1(a)$ and $t_1(c)$, these conditions imply $g(a), g(c), t_1(a), t_1(c) \in I \cap (-\infty, c]$ and $g(c), g(b) \in I \cap [c, \infty)$.

Let $f \in \mathcal{M}_1^c(I)$. Let A be the constant from Definition 1 and let us consider the function $F : I \to \mathbb{R}$, $F(x) = f(x) - Ax$. Since F is nonincreasing on $I \cap (-\infty, c]$ and $g(a), g(c), t_1(a), t_1(c) \in I \cap (-\infty, c]$, we can apply the right-hand side of inequality (2) to the function F, so

$$\int_{g(a)}^{g(c)} F(x)dx \leq \int_a^c F(t_1(x))g'(x)dx.$$

Hence, we obtain

$$\begin{aligned} 0 &\leq \int_a^c F(t_1(x))g'(x)dx - \int_{g(a)}^{g(c)} F(x)dx = \\ &= \int_a^c f(t_1(x))g'(x)dx - \int_{g(a)}^{g(c)} f(x)dx - A\left(\int_a^c t_1(x)g'(x)dx - \frac{g^2(c) - g^2(a)}{2}\right). \end{aligned} \tag{10}$$

Further, since F is nondecreasing on $I \cap [c, \infty)$ and $g(c), g(b) \in I \cap [c, \infty)$, the left-hand side of inequality (2) applied to the function F is reversed. So we have

$$\int_c^b F(s_1(x))g'(x)dx \geq \int_{g(c)}^{g(b)} F(x)dx.$$

Hence, we obtain

$$\begin{aligned} 0 &\leq \int_c^b F(s_1(x))g'(x)dx - \int_{g(c)}^{g(b)} F(x)dx = \\ &= \int_c^b f(s_1(x))g'(x)dx - \int_{g(c)}^{g(b)} f(x)dx - A\left(\int_c^b s_1(x)g'(x)dx - \frac{g^2(b) - g^2(c)}{2}\right). \end{aligned} \tag{11}$$

Now combining (10) and (11) we obtain

$$\int_a^c f(t_1(x))g'(x)dx + \int_c^b f(s_1(x))g'(x)dx - \int_{g(a)}^{g(b)} f(x)dx \geq$$
$$\geq A\left(\int_a^c t_1(x)g'(x)dx + \int_c^b s_1(x)g'(x)dx - \frac{g^2(b)-g^2(a)}{2}\right).$$

Hence, from (8) we conclude that (9) holds.
Proof for $f \in \mathcal{M}_2^c(I)$ is similar, so we omit the details.◀

Remark 3. *An example of a function $g : [a,b] \to \mathbb{R}$ satisfying conditions of Theorem 4 is the function*

$$g(x) = \frac{x^2}{c},$$

where $0 \leq a < c < b$.

Theorem 5. *Let $c \in (a,b)$ and let $g : [a,b] \to \mathbb{R}$ be increasing, convex and differentiable such that $g(c) = c$. Assume*

$$s_2(x) = \frac{g(c)-g(a)}{c-a}(x-a) + g(a), \tag{12}$$

$$t_2(x) = g'(x_0)(x-x_0) + g(x_0), \quad x_0 \in [c,b], \tag{13}$$

and

$$\int_a^c s_2(x)g'(x)dx + \int_c^b t_2(x)g'(x)dx = \frac{g^2(b)-g^2(a)}{2}. \tag{14}$$

If $f \in \mathcal{M}_1^c(I)$, then

$$\int_a^c f(s_2(x))g'(x)dx + \int_c^b f(t_2(x))g'(x)dx \leq \int_{g(a)}^{g(b)} f(x)dx. \tag{15}$$

If $f \in \mathcal{M}_2^c(I)$, then the inequality in (15) is reversed.
(I is an interval containing $a, b, g(a), g(b), t_2(c)$ and $t_2(b)$.)

Proof. Similar to the proof of Theorem 4, it follows that $g(a) \leq c$ and $g(b), t_2(c), t_2(b) \geq c$. Since interval I contains $a, b, g(a), g(b), t_2(c)$ and $t_2(b)$, these conditions imply $g(a), g(c) \in I \cap (-\infty, c]$ and $g(c), g(b), t_2(c), t_2(b) \in I \cap [c, \infty)$.

Let $f \in \mathcal{M}_1^c(I)$. Let A be the constant from Definition 1 and let us consider the function $F : I \to \mathbb{R}$, $F(x) = f(x) - Ax$. Since F is nonincreasing on $I \cap (-\infty, c]$ and $g(a), g(c) \in I \cap (-\infty, c]$, we can apply the left-hand side of inequality (2) to the function F. So we obtain

$$0 \leq \int_{g(a)}^{g(c)} f(x)dx - \int_a^c f(s_2(x))g'(x)dx - A\left(\frac{g^2(c)-g^2(a)}{2} - \int_a^c s_2(x)g'(x)dx\right). \tag{16}$$

Further, since F is nondecreasing on $I \cap [c, \infty)$ and $g(c), g(b), t_2(c), t_2(b) \in I \cap [c, \infty)$, the right-hand side of inequality (2) applied to the function F is reversed. So we have

$$0 \leq \int_{g(c)}^{g(b)} f(x)dx - \int_c^b f(t_2(x))g'(x)dx - A\left(\frac{g^2(b) - g^2(c)}{2} - \int_c^b t_2(x)g'(x)dx\right). \quad (17)$$

Now combining (16) and (17) we obtain

$$\int_{g(a)}^{g(b)} f(x)dx - \int_a^c f(s_2(x))g'(x)dx - \int_c^b f(t_2(x))g'(x)dx \geq$$
$$\geq A\left(\frac{g^2(b) - g^2(a)}{2} - \int_a^c s_2(x)g'(x)dx - \int_c^b t_2(x)g'(x)dx\right).$$

Hence, from (14) we conclude that (15) holds.
Proof for $f \in \mathcal{M}_2^c(I)$ is similar, so we omit the details.◄

As a consequence of Theorems 4 and 5, we obtain Gauss type inequalities that involve convex functions.

Corollary 1. *Let $c \in (a, b)$ and let $g : [a, b] \to \mathbb{R}$ be increasing, convex and differentiable such that $g(c) = c$. Assume* (6), (7) *and* (8) *hold and I is an interval as in Theorem 4. If $f : I \to \mathbb{R}$ is convex, then (9) holds. If $f : I \to \mathbb{R}$ is concave, then the inequality in (9) is reversed.*

Proof. Since the function f is convex, from Theorem 3 we have $f \in \mathcal{M}_1^c(I)$ for every $c \in (a, b) \subseteq I^\circ$. Hence, we can apply Theorem 4.◄

Corollary 2. *Let $c \in (a, b)$ and let $g : [a, b] \to \mathbb{R}$ be increasing, convex and differentiable such that $g(c) = c$. Assume* (12), (13) *and* (14) *hold and I is an interval as in Theorem 5. If $f : I \to \mathbb{R}$ is convex, then (15) holds. If $f : I \to \mathbb{R}$ is concave, then the inequality in (15) is reversed.*

Proof. Similar to the proof of Corollary 1.◄

Remark 4. *Conditions (8) and (14) can be relaxed. For $f \in \mathcal{M}_1^c(I)$ condition (8) can be replaced by the weaker condition*

$$A\left(\int_a^c t_1(x)g'(x)dx + \int_c^b s_1(x)g'(x)dx - \frac{g^2(b) - g^2(a)}{2}\right) \geq 0, \quad (18)$$

and condition (14) can be replaced by the weaker condition

$$A\left(\frac{g^2(b) - g^2(a)}{2} - \int_a^c s_2(x)g'(x)dx - \int_c^b t_2(x)g'(x)dx\right) \geq 0, \quad (19)$$

where A is the constant from Definition 1. Also, for $f \in \mathcal{M}_2^c(I)$ condition (8) (resp. (14)) can be replaced by condition (18) (resp. (19)) with the reverse inequality.

Additionaly, conditions (8) and (14) can be further weakened if the function f is monotonic. Since (5) holds, if $f \in \mathcal{M}_1^c(I)$ is nondecreasing or $f \in \mathcal{M}_2^c(I)$ is nonincreasing, from (18) we obtain that (8) can be weakened to

$$\int_a^c t_1(x)g'(x)dx + \int_c^b s_1(x)g'(x)dx \geq \frac{g^2(b) - g^2(a)}{2}, \tag{20}$$

and that (14) can be weakened to

$$\frac{g^2(b) - g^2(a)}{2} \geq \int_a^c s_2(x)g'(x)dx + \int_c^b t_2(x)g'(x)dx. \tag{21}$$

Also, if $f \in \mathcal{M}_1^c(I)$ is nonincreasing or $f \in \mathcal{M}_2^c(I)$ is nondecreasing, (8) (resp. (14)) can be weakened to (20) (resp. (21)) with the reverse inequality.

3. Mean value theorems

In this section we prove mean value theorems related to Gauss type inequalities obtained in previous section. Let us begin by defining the following linear functionals:

$$L_1(f) = \int_a^c f(t_1(x))g'(x)dx + \int_c^b f(s_1(x))g'(x)dx - \int_{g(a)}^{g(b)} f(x)dx, \tag{22}$$

and

$$L_2(f) = \int_{g(a)}^{g(b)} f(x)dx - \int_a^c f(s_2(x))g'(x)dx - \int_c^b f(t_2(x))g'(x)dx. \tag{23}$$

Remark 5. *Under assumptions of Theorems 4 and 5 we have that $L_1(f) \geq 0$ and $L_2(f) \geq 0$ for $f \in \mathcal{M}_1^c(I)$. Further, under assumptions of Corollaries 1 and 2 we have that $L_1(f) \geq 0$ and $L_2(f) \geq 0$ for any convex function f.*

First, we give the Lagrange type mean value theorems.

Theorem 6. *Let $c \in (a, b)$ and let $g : [a, b] \to \mathbb{R}$ be increasing, convex and differentiable such that $g(c) = c$. Assume (6), (7) and (8) hold and I is an interval as in Theorem 4. Then for any $f \in C^2(I)$ there exists $\xi \in I$ such that*

$$L_1(f) = \frac{f''(\xi)}{2}\left[\int_a^c t_1^2(x)g'(x)dx + \int_c^b s_1^2(x)g'(x)dx - \frac{g^3(b) - g^3(a)}{3}\right], \tag{24}$$

where L_1 is defined by (22).

Proof. Since $f \in C^2(I)$, there exist

$$m = \min_{x \in I} f''(x) \quad \text{and} \quad M = \max_{x \in I} f''(x).$$

The functions

$$\Psi_1(x) = f(x) - \frac{m}{2}x^2 \quad \text{and} \quad \Psi_2(x) = \frac{M}{2}x^2 - f(x),$$

are convex since $\Psi_i''(x) \geq 0$, $i = 1, 2$. Hence, by Remark 5 we have $L_1(\Psi_i) \geq 0$, $i = 1, 2$ and we get

$$\frac{m}{2}L_1(x^2) \leq L_1(f) \leq \frac{M}{2}L_1(x^2), \tag{25}$$

where

$$L_1(x^2) = \int_a^c t_1^2(x)g'(x)dx + \int_c^b s_1^2(x)g'(x)dx - \frac{g^3(b) - g^3(a)}{3}.$$

Since x^2 is convex, by Remark 5 we have $L_1(x^2) \geq 0$.

If $L_1(x^2) = 0$, then (25) implies $L_1(f) = 0$ and (24) holds for every $\xi \in I$. Otherwise, dividing (25) by $L_1(x^2)/2 > 0$ we get

$$m \leq \frac{2L_1(f)}{L_1(x^2)} \leq M,$$

so continuinity of f'' ensures existence of $\xi \in I$ satisfying (24).◄

Theorem 7. *Let $c \in (a, b)$ and let $g : [a, b] \to \mathbb{R}$ be increasing, convex and differentiable such that $g(c) = c$. Assume* (12), (13) *and (14) hold and I is an interval as in Theorem 5, Then for any $f \in C^2(I)$ there exists $\xi \in I$ such that*

$$L_2(f) = \frac{f''(\xi)}{2}\left[\frac{g^3(b) - g^3(a)}{3} - \int_a^c s^2(x)g'(x)dx - \int_c^b t^2(x)g'(x)dx\right], \tag{26}$$

where L_2 is defined by (23).

Proof. Similar to the proof of Theorem 6.◄

We continue with the Cauchy type mean value theorems.

Theorem 8. *Let $c \in (a, b)$ and let $g : [a, b] \to \mathbb{R}$ be increasing, convex and differentiable such that $g(c) = c$. Assume* (6), (7) *and* (8) *hold and I is an interval as in Theorem 4. Then for any $f, h \in C^2(I)$ such that $h''(x) \neq 0$ for every $x \in I$, there exists $\xi \in I$ such that*

$$\frac{L_1(f)}{L_1(h)} = \frac{f''(\xi)}{h''(\xi)} \tag{27}$$

holds, where L_1 is defined by (22).

Proof. Let us define $\Phi \in C^2(I)$ by $\Phi(x) = L_1(h)f(x) - L_1(f)h(x)$. Due to the linearity of L_1, we have $L_1(\Phi) = 0$. Now, by Theorem 6 there exist $\xi, \xi_1 \in I$ such that

$$0 \quad = L_1(\Phi) = \tfrac{\Phi''(\xi)}{2}L_1(x^2),$$

$$0 \neq L_1(h) = \frac{h''(\xi_1)}{2} L_1(x^2).$$

Therefore, $L_1(x^2) \neq 0$ and

$$0 = \Phi''(\xi) = L_1(h) f''(\xi) - L_1(f) h''(\xi),$$

which gives the claim of the theorem.◀

Theorem 9. *Let $c \in (a, b)$ and let $g : [a, b] \to \mathbb{R}$ be increasing, convex and differentiable such that $g(c) = c$. Assume* (12), (13) *and* (14) *hold and I is an interval as in Theorem 5. Then for any $f, h \in C^2(I)$ such that $h''(x) \neq 0$ for every $x \in I$, there exists $\xi \in I$ such that*

$$\frac{L_2(f)}{L_2(h)} = \frac{f''(\xi)}{h''(\xi)}, \tag{28}$$

holds, where L_2 is defined by (23).

Proof. Similar to the proof of Theorem 8.◀

Remark 6. *Conditions (8) and (14) in Theorems 6–9 can be replaced by weaker conditions given in Remark 4.*

4. Applications to Stolarsky type means

In [3] Bernstein invented exponentially convex functions, a subclass of convex functions on a given open interval. We recall some nice properties of this class of functions needed in the sequel. For other properties see [1] and [5]. Further, we use the notion of $n-$exponentially convex functions which was introduced in [6].

Let us recall definition and some results on exponential convexity.

Definition 2. *A function $\psi : J \to \mathbb{R}$ is n-*exponentially convex in the Jensen sense *on J if*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \, \psi\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

holds for all choices $\xi_1, \ldots, \xi_n \in \mathbb{R}$ and all choices $x_1, \ldots, x_n \in J$.

*A function $\psi : J \to \mathbb{R}$ is n-*exponentially convex *on J if it is n-exponentially convex in the Jensen sense and continuous on J.*

Remark 7. *It is clear from the definition that* 1*-exponentially convex functions in the Jensen sense are in fact nonnegative functions.*

Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \leq n$, $k \in \mathbb{N}$.

Definition 3. *A function $\psi : J \to \mathbb{R}$ is* exponentially convex in the Jensen sense *on J if it is n-exponentially convex in the Jensen sense on J for every $n \in \mathbb{N}$.*

A function $\psi : J \to \mathbb{R}$ is exponentially convex *on J if it is exponentially convex in the Jensen sense and continuous on J.*

Remark 8. *A function $\psi : J \to \mathbb{R}$ is log-convex in the Jensen sense, i.e.*

$$\psi\left(\frac{x+y}{2}\right)^2 \le \psi(x)\psi(y), \qquad \text{for all } x, y \in J, \tag{29}$$

if and only if

$$\alpha^2\psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \ge 0,$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in J$, i.e., if and only if ψ is 2-exponentially convex in the Jensen sense. By induction from (29) we have

$$\psi\left(\frac{1}{2^k}x + \left(1 - \frac{1}{2^k}\right)y\right) \le \psi(x)^{\frac{1}{2^k}}\psi(y)^{1-\frac{1}{2^k}}.$$

Therefore, if ψ is continuous and $\psi(x) = 0$ for some $x \in J$, then from the last inequality and nonnegativity of ψ (see Remark 7) we get

$$\psi(y) = \lim_{k\to\infty} \psi\left(\frac{1}{2^k}x + \left(1 - \frac{1}{2^k}\right)y\right) = 0 \qquad \text{for all } y \in J.$$

Hence, 2-exponentially convex function is either identically equal to zero or it is strictly positive and log-convex.

The following lemma is equivalent to the definition of convex functions (see [7]).

Lemma 1. *A function $\psi : J \to \mathbb{R}$ is convex if and only if the inequality*

$$(x_3 - x_2)\psi(x_1) + (x_1 - x_3)\psi(x_2) + (x_2 - x_1)\psi(x_3) \ge 0,$$

holds for all $x_1, x_2, x_3 \in J$ such that $x_1 < x_2 < x_3$.

We also use the following result (see [7]).

Proposition 1. *If f is a convex function on J and if $x_1 \le y_1,\ x_2 \le y_2,\ x_1 \ne x_2,\ y_1 \ne y_2$, then the following inequality holds*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(y_2) - f(y_1)}{y_2 - y_1}.$$

If the function f is concave, the inequality is reversed.

Definition 4. The second order divided difference *of a function $f : J \to \mathbb{R}$ (J is an interval in $\mathbb{R}$) at mutually different points $x_0, x_1, x_2 \in J$ is defined recursively by*

$$[x_i; f] = f(x_i), \quad i = 0, 1, 2,$$

$$[x_i, x_{i+1}; f] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}, \quad i = 0, 1,$$

$$[x_0, x_1, x_2; f] = \frac{[x_1, x_2; f] - [x_0, x_1; f]}{x_2 - x_0}. \tag{30}$$

Remark 9. *The value $[x_0, x_1, x_2; f]$ is independent of the order of the points x_0, x_1 and x_2. This definition may be extended to include the case in which some or all the points coincide. Taking the limit $x_1 \to x_0$ in (30), we get*

$$\lim_{x_1 \to x_0} [x_0, x_1, x_2; f] = [x_0, x_0, x_2; f] = \frac{f(x_2) - f(x_0) - f'(x_0)(x_2 - x_0)}{(x_2 - x_0)^2}, \quad x_2 \neq x_0,$$

provided that f' exists, and furthermore, taking the limits $x_i \to x_0$, $i = 1, 2$ in (30), we get

$$\lim_{x_2 \to x_1} \lim_{x_1 \to x_0} [x_0, x_1, x_2; f] = [x_0, x_0, x_0; f] = \frac{f''(x_0)}{2},$$

provided that f'' exists.

In the following theorem we show $n-$exponential convexity of functionals L_1 and L_2. Similar result was proved in [10], so we omit the proof. In the sequel, J, K denote intervals in $\mathbb{R}$.

Theorem 10. *Let $\Omega = \{f_p : J \to \mathbb{R} \,|\, p \in K\}$ be a family of functions such that for every mutually different points $x_0, x_1, x_2 \in J$ the mapping $p \mapsto [x_0, x_1, x_2; f_p]$ is $n-$exponentially convex in the Jensen sense on K. Let L_i, $i = 1, 2$ be linear functionals defined by* (22) *and* (23)*. Then the mapping $p \mapsto L_i(f_p)$ is $n-$exponentially convex in the Jensen sense on K. If the mapping $p \mapsto L_i(f_p)$ is continuous on K, then it is $n-$exponentially convex on K.*

If the assumptions of Theorem 10 hold for all $n \in \mathbb{N}$, then we have the following corollary.

Corollary 3. *Let $\Omega = \{f_p : J \to \mathbb{R} \,|\, p \in K\}$ be a family of functions such that for every mutually different points $x_0, x_1, x_2 \in J$ the mapping $p \mapsto [x_0, x_1, x_2; f_p]$ is exponentially convex in the Jensen sense on K. Let L_i, $i = 1, 2$ be linear functionals defined by* (22) *and* (23)*. Then the mapping $p \mapsto L_i(f_p)$ is exponentially convex in the Jensen sense on K. If the mapping $p \mapsto L_i(f_p)$ is continuous on K, then it is exponentially convex on K.*

We continue with the result which is useful for the application to Stolarsky type means. Again, similar result was obtained in [10], so we recall it without the proof.

Corollary 4. *Let $\Omega = \{f_p : J \to \mathbb{R} \,|\, p \in K\}$ be a family of functions such that for every mutually different points $x_0, x_1, x_2 \in J$ the mapping $p \mapsto [x_0, x_1, x_2; f_p]$ is* 2*-exponentially convex in the Jensen sense on K. Let L_i, $i = 1, 2$ be linear functionals defined by (22) and (23). Then the following statements hold:*

(i) If the mapping $p \mapsto L_i(f_p)$ is continuous on K, then for $r, s, t \in K$, such that $r < s < t$, we have

$$[L_i(f_s)]^{t-r} \leq [L_i(f_r)]^{t-s} [L_i(f_t)]^{s-r}, \quad i = 1, 2. \tag{31}$$

(ii) If the mapping $p \mapsto L_i(f_p)$ is strictly positive and differentiable on K, then for every $p, q, u, v \in K$ such that $p \leq u$ and $q \leq v$ we have

$$\mu_{p,q}(L_i, \Omega) \leq \mu_{u,v}(L_i, \Omega), \tag{32}$$

where

$$\mu_{p,q}(L_i, \Omega) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)} \right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp} L_i(f_p)}{L_i(f_p)} \right), & p = q. \end{cases} \tag{33}$$

Remark 10. *Results from Theorem 10, Corollaries 3 and 4 still hold when two of the points $x_0, x_1, x_2 \in J$ coincide, say $x_1 = x_0$, for a family of differentiable functions f_p such that the function $p \to [x_0, x_1, x_2; f_p]$ is $n-$exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 9 and suitable characterization of convexity.*

We continue with some families of functions $\Upsilon = \{f_p : J \to \mathbb{R} \,|\, p \in \mathbb{R}\}$ for which we use Corollaries 3 and 4 to construct exponentially convex functions and Stolarsky type means related to Gauss type inequalities. Motivation for this application was importance and intensive research of Stolarsky means [11] in various branches of mathematics.

Example 1. Let

$$\Upsilon_1 = \{f_p \colon \mathbb{R} \to [0, \infty) \,|\, p \in \mathbb{R}\},$$

be a family of functions defined by

$$f_p(x) = \begin{cases} \frac{e^{px}}{p^2}, & p \neq 0; \\ \frac{x^2}{2}, & p = 0. \end{cases}$$

For every $p \in \mathbb{R}$ we have that f_p is a convex function on $\mathbb{R}$ since $\frac{d^2}{dx^2} f_p(x) = e^{px} > 0$. Furthermore, $p \mapsto \frac{d^2}{dx^2} f_p(x)$ is exponentially convex by definition. Similar to the proof of Theorem 10, we conclude that $p \mapsto [x_0, x_1, x_2; f_p]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 3 we obtain that $p \mapsto L_i(f_p)$, $i = 1, 2$ are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous, so they are exponentially convex. For this family of functions, from Corollary 4 we have that $\mu_{p,q}(L_i, \Upsilon_1)$, $i = 1, 2$ are given by

$$\mu_{p,q}(L_i, \Upsilon_1) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)} \right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{L_i(\mathrm{id} \cdot f_p)}{L_i(f_p)} - \frac{2}{p} \right), & p = q \neq 0; \\ \exp\left(\frac{1}{3} \frac{L_i(\mathrm{id} \cdot f_0)}{L_i(f_0)} \right), & p = q = 0. \end{cases}$$

Explicitly, for $\mu_{p,q}(L_1, \Upsilon_1)$ we have the following:

* for $p \neq q$, $p, q \neq 0$:

$$\mu_{p,q}(L_1, \Upsilon_1) = \left(\frac{q^2}{p^2} \frac{\int_a^c e^{pt_1(x)} g'(x)dx + \int_c^b e^{ps_1(x)} g'(x)dx - \frac{e^{pg(b)} - e^{pg(a)}}{p}}{\int_a^c e^{qt_1(x)} g'(x)dx + \int_c^b e^{qs_1(x)} g'(x)dx - \frac{e^{qg(b)} - e^{qg(a)}}{q}} \right)^{\frac{1}{p-q}}$$

* for $p \neq q$, $q = 0$ (or $p = 0$):

$$\begin{aligned}\mu_{p,0}(L_1, \Upsilon_1) &= \left(\frac{2}{p^2} \frac{\int_a^c e^{pt_1(x)} g'(x)dx + \int_c^b e^{ps_1(x)} g'(x)dx - \frac{e^{pg(b)} - e^{pg(a)}}{p}}{\int_a^c t_1^2(x) g'(x)dx + \int_c^b s_1^2(x) g'(x)dx - \frac{g^3(b) - g^3(a)}{3}} \right)^{\frac{1}{p}} \\ &= \mu_{0,p}(L_1, \Upsilon_1)\end{aligned}$$

* for $p = q \neq 0$:

$$\mu_{p,p}(L_1, \Upsilon_1) = \exp\left(\frac{A - B}{C} - \frac{2}{p} \right),$$

where

$$\begin{aligned}A &= \int_a^c e^{pt_1(x)} t_1(x) g'(x)dx + \int_c^b e^{ps_1(x)} s_1(x) g'(x)dx, \\ B &= \frac{1}{p} \left(g(b)e^{pg(b)} - g(a)e^{pg(a)} - \frac{e^{pg(b)} - e^{pg(a)}}{p} \right), \\ C &= \int_a^c e^{pt_1(x)} g'(x)dx + \int_c^b e^{ps_1(x)} g'(x)dx - \frac{e^{pg(b)} - e^{pg(a)}}{p}.\end{aligned}$$

* for $p = q = 0$:

$$\mu_{0,0}(L_1, \Upsilon_1) = \exp\left(\frac{1}{3} \frac{\int_a^c t_1^3(x) g'(x)dx + \int_c^b s_1^3(x) g'(x)dx - \frac{g^4(b) - g^4(a)}{4}}{\int_a^c t_1^2(x) g'(x)dx + \int_c^b s_1^2(x) g'(x)dx - \frac{g^3(b) - g^3(a)}{3}} \right).$$

Applying Theorems 8 and 9 on functions $f_p, f_q \in \Upsilon_1$ and functionals L_1 and L_2, we obtain that for $i = 1, 2$

$$M_{p,q}(L_i, \Upsilon_1) = \log \mu_{p,q}(L_i, \Upsilon_1),$$

satisfy $\min I \leq M_{p,q}(L_i, \Upsilon_1) \leq \max I$. So $M_{p,q}(L_i, \Upsilon_1)$, $i = 1, 2$ are monotonic means by (32).

Example 2. Let

$$\Upsilon_2 = \{h_p \colon (0, \infty) \to \mathbb{R} \mid p \in \mathbb{R}\},$$

be a family of functions defined by

$$h_p(x) = \begin{cases} \frac{x^p}{p(p-1)}, & p \neq 0, 1; \\ -\log x, & p = 0; \\ x \log x, & p = 1. \end{cases} \tag{34}$$

We have that h_p is a convex function on $\mathbb{R}^+$, since $\frac{d^2}{dx^2}h_p(x) = x^{p-2} > 0$ for $x > 0$. Furthermore, $p \mapsto \frac{d^2}{dx^2}h_p(x)$ is exponentially convex by definition. Similar to Example 1, we obtain that $p \mapsto L_i(h_p)$, $i = 1, 2$ are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous, so they are exponentially convex. Hence, for this family of functions, from Corollary 4 we have that $\mu_{p,q}(L_i, \Upsilon_1)$, $i = 1, 2$ are given by

$$\mu_{p,q}(L_i, \Upsilon_2) = \begin{cases} \left(\frac{L_i(h_p)}{L_i(h_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{-L_i(h_p h_0)}{L_i(h_p)} - \frac{2p-1}{p(p-1)}\right), & p = q \neq 0, 1; \\ \exp\left(\frac{-L_i(h_0^2)}{2L_i(h_0)} + 1\right), & p = q = 0; \\ \exp\left(\frac{-L_i(h_0 h_1)}{2L_i(h_1)} - 1\right), & p = q = 1. \end{cases}$$

Applying Theorems 8 and 9 on functions $h_p, h_q \in \Upsilon_2$ and functionals L_1, L_2, we conclude that there exist $\xi_i \in I$ such that

$$\xi_i^{p-q} = \frac{L_i(h_p)}{L_i(h_q)}, \quad i = 1, 2.$$

Since the function $\xi \mapsto \xi^{p-q}$ is invertible, for $p \neq q$ we have

$$\min I \leq \left(\frac{L_i(h_p)}{L_i(h_q)}\right)^{\frac{1}{p-q}} \leq \max I,$$

which together with the fact that $\mu_{p,q}(L_i, \Upsilon_2)$, $i = 1, 2$ are continuous, symmetric and monotonic (by (32)) shows that $\mu_{p,q}(L_i, \Upsilon_2)$, $i = 1, 2$ are means.

Example 3. Let

$$\Upsilon_3 = \{\phi_p \colon (0, \infty) \to (0, \infty) \mid p \in (0, \infty)\},$$

be a family of functions defined by

$$\phi_p(x) = \begin{cases} \frac{p^{-x}}{\log^2 p}, & p \neq 1; \\ \frac{x^2}{2}, & p = 1. \end{cases}$$

Since $\frac{d^2}{dx^2}\phi_p(x) = p^{-x}$ is a Laplace transform of a nonnegative function (see [12]), it is exponentially convex. Similar to Example 1, we conclude that $p \mapsto L_i(\phi_p)$, $i = 1, 2$ are exponentially convex. For this family of functions, from Corollary 4 we have

$$\mu_{p,q}(L_i, \Upsilon_3) = \begin{cases} \left(\frac{L_i(\phi_p)}{L_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\frac{L_i(\mathrm{id} \cdot \phi_p)}{pL_i(\phi_p)} - \frac{2}{p \log p}\right), & p = q \neq 1; \\ \exp\left(-\frac{L_i(\mathrm{id} \cdot \phi_1)}{3L_i(\phi_1)}\right), & p = q = 1. \end{cases}$$

Applying Theorems 8 and 9 on functions $\phi_p, \phi_q \in \Upsilon_3$ and functionals L_1, L_2, we obtain that

$$M_{p,q}(L_i, \Upsilon_3) = -L(p, q) \log \mu_{p,q}(L_i, \Upsilon_3),$$

satisfy $\min I \leq M_{p,q}(L_i, \Upsilon_3) \leq \max I$, where $L(p,q)$ is logarithmic mean defined by $L(p,q) = \frac{p-q}{\log p - \log q}$. So $M_{p,q}(L_i, \Upsilon_3)$, $i = 1, 2$ are means and by (32) they are monotonic.

Example 4. Let

$$\Upsilon_4 = \{\psi_p \colon (0,\infty) \to (0,\infty) \,|\, p \in (0,\infty)\},$$

be a family of functions defined by

$$\psi_p(x) = \frac{e^{-x\sqrt{p}}}{p}.$$

Since $\frac{d^2}{dx^2}\psi_p(x) = e^{-p\sqrt{x}}$ is a Laplace transform of a nonnegative function (see [12]), it is exponentially convex. Similar to Example 1, we conclude that $p \mapsto L_i(\psi_p)$, $i = 1, 2$ are exponentially convex. For this family of functions, from Corollary 4 we have

$$\mu_{p,q}(L_i, \Upsilon_4) = \begin{cases} \left(\frac{L_i(\psi_p)}{L_i(\psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{-1}{2\sqrt{p}} \frac{L_i(\mathrm{id} \cdot \psi_p)}{L_i(\psi_p)} - \frac{1}{p}\right), & p = q. \end{cases}$$

Applying Theorems 8 and 9 on functions $\psi_p, \psi_q \in \Upsilon_4$ and functionals L_1, L_2, we obtain

$$M_{p,q}(L_i, \Upsilon_4) = -(\sqrt{p} + \sqrt{q}) \log \mu_{p,q}(L_i, \Upsilon_4),$$

satisfy $\min I \leq M_{p,q}(L_i, \Upsilon_4) \leq \max I$. So $M_{p,q}(L_i, \Upsilon_4)$, $i = 1, 2$ are monotonic means by (32).

Remark 11. *Some other Stolarsky type means related to Alzer's result given in Theorem 2 were obtained by Pečarić and Smoljak in [8] and [9].*

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