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On Mixed Vector Equilibrium Problems

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Abstract. In this paper, four kinds of the mixed vector equilibrium problems which are combinations of a vector equilibrium problem and generalized vector variational inequality problem are introduced. Some existence theorems for a solution of them, by applying KKM lemma and minimax theorem as well suitable convexity and semicontinuity on the maps in the setting of topological vector spaces are provided. Also, the solution sets of them are compared. Some examples in order to illustrate the results of the paper are given. The results of this note can be viewed as an extension and improvement of results recently obtained in [1, 2, 3, 6, 8]).

Key Words and Phrases: Generalized vector equilibrium problem, generalized vector variational inequality, upper semicontinuity, C-convexity, C-upper semicontinuity.
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1. Introduction

Let X be a topological vector space, K be a subset of X, and $f : K \times K \to \mathbb{R}$ be a mapping with $f(x, x) = 0$, for all $x \in X$. The (scalar-valued) equilibrium problem deals with the existence of $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in K.$$

The equilibrium problem was first introduced and studied by Blum and Oettli [4] as a generalization of variational inequality problem. It has been shown that the equilibrium problem provides a natural, novel and unified framework to study a wide class of problems arising in nonlinear analysis, optimization, economics, finance and game theory. It also includes many mathematical problems as particular cases such as mathematical programming problems, complementarity problems, variational inequality problems, fixed point. problems, minimax inequality problems, and Nash equilibrium problems in noncooperative games [4]. If we replace the real line by an ordered topological vector space Y with its ordering induced by a convex cone $C \subseteq Y$ (convex cone means $C + C \subseteq C$ and $\lambda C \subseteq C$ for all the nonnegative number λ), given a vector-valued mapping $f : K \times K \to Y$, then the problem of finding $\bar{x} \in K$ such that

$$f(\bar{x}, y) \notin -intC, \quad \forall y \in K,$$

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is called (weak) vector equilibrium problem and the point $\bar{x} \in K$ is said to be a vector equilibrium point (for more details, see [2, 3, 10]), where $intC$ denotes the interior of C in Y.

We recall that the convex cone C is pointed if $C \cap -C = \{0\}$.

In this paper, four kinds of mixed vector equilibrium problems and many definitions are introduced. Further, some existence theorems for a solution of them are presented. The main tools for proving the main theorems are KKM lemma and minimax theorem. The rest of this section will deal with introducing our problems.

Let Y be a topological vector space and C be a nonempty convex pointed cone subset of Y. The partial order induced on Y by C is denoted by $\preceq$ and defined as

$$x \preceq y \Longleftrightarrow y - x \in C.$$

Moreover, if C is a solid, i.e., $intC \neq \emptyset$, then we can define a pre-order (it is not reflexive) on Y by

$$y_1 \prec y_2 \Longleftrightarrow y_2 - y_1 \in intC.$$

Hence

$$y_1 \not\prec y_2 \Longleftrightarrow y_2 - y_1 \notin intC,$$

and

$$y_1 \npreceq y_2 \Longleftrightarrow y_2 - y_1 \notin C.$$

Let K be a nonempty convex subset of a topological vector space X, $f : K \times K \to Y$ be a mapping and $T : K \to 2^{L(X,Y)}$ be a multifunction, where $L(X,Y)$ is the set of all linear continuous mappings from X into Y and $2^{L(X,Y)}$ denotes all of nonempty subsets of $L(X,Y)$. Also, $< x^*, x >$ stands for the evaluation of the linear mapping x^* at x. Now, we consider the following problems which are known as mixed vector equilibrium problems:
(A) Find $\bar{x} \in K$ such that

$$\exists x^* \in T(\bar{x}),\ \forall y \in K,\ f(\bar{x}, y) + < x^*, y - \bar{x} > \not\prec 0, \tag{1}$$

(B) Find $\bar{x} \in K$ such that

$$\forall y \in K,\ \exists x^* \in T(\bar{x}),\ f(\bar{x}, y) + < x^*, y - \bar{x} > \not\prec 0, \tag{2}$$

(C) Find $\bar{x} \in K$ such that

$$\exists x^* \in T(\bar{x}),\ \forall y \in K,\ f(\bar{x}, y) + < x^*, y - \bar{x} > \npreceq 0, \tag{3}$$

(D) Find $\bar{x} \in K$ such that

$$\forall y \in K,\ \exists x^* \in T(\bar{x}),\ f(\bar{x}, y) + < x^*, y - \bar{x} > \npreceq 0. \tag{4}$$

We will denote by S_A, S_B, S_C and S_D the solution sets of the problems (A),(B),(C) and (D), respectively.

Remark 1. *(a) It is obvious that* $S_C \subseteq S_D$, $S_C \subseteq S_A \subseteq S_B$ *and if* T *is a single-valued mapping, then* $S_A = S_B$.
(b) It is worth noting that if $f = 0$, *then the problems (A),(B) and (C) are reduced to the generalized vector variational inequality problem [9]. Also, if* $T = 0$, *then the problems (A),(B) and (C) collapse to the vector equilibrium problems [3, 6, 10].*

2. Preliminaries

In this section, some definitions and preliminaries which are needed in the sequel are given. Let X and Y be topological vector spaces, K be a convex subset of X, and C be a convex (pointed) cone subset of Y.

Definition 1. *A mapping* $f : X \longrightarrow \mathbb{R}$ *is upper semicontinuous at* $x_0 \in X$ *if the following inequality, for any net* $\{x_\alpha\}$ *in* X *which converges to* x_0, *holds:*

$$lim_\alpha sup f(x_\alpha) \leq f(x_0).$$

Also, the mapping f *is lower semicontinuous at* $x_0 \in X$ *if for any net* $\{x_\alpha\}$ *in* X *which converges to* x_0, *the following inequality holds*

$$lim_\alpha inf f(x_\alpha) \geq f(x_0).$$

Definition 2 ([1])**.** *A mapping* $g : K \to Y$ *is called:*
(a) $C-$*convex, if for all* $x, x^{'} \in K$ *and* $t \in [0,1]$ *one has*

$$tg(x) + (1-t)g(x^{'}) - g(tx + (1-t)x^{'}) \in C.$$

(b) $C-$*strongly convex if for all* $x, x^{'} \in K$ *and* $t \in [0,1]$ *one has*

$$tg(x) + (1-t)g(x^{'}) - g(tx + (1-t)x^{'}) \in intC.$$

(c) $C-$*quasiconvex if for all* $x, x^{'} \in K$ *and* $t \in [0,1]$ *one has*

$$g(tx + (1-t)x^{'}) \in y - C, \quad \forall y \in C(g(x), g(x^{'})),$$

where $C(g(x), g(x^{'})$ *is the set of upper bounds of* $g(x), g(x^{'})$, *i.e.,*

$$C(g(x), g(x^{'})) = \{y \in Y : y \in g(x) + C \text{ and } y \in g(x^{'}) + C\}.$$

(d) $C-$*upper semicontinuous at a point* $x_0 \in K$, *if for any neighborhood* V *of* $g(x_0)$ *in* Y *there exists a neighborhood* U *of* x_0 *in* X *such that*

$$g(U \cap K) \subseteq V - C.$$

Also, g *is called* $C-$*lower semicontinuous at a point* $x_0 \in K$, *if*

$$g(U \cap K) \subseteq V + C.$$

Further, g *is continuous with respect to* C *at a point* $x_0 \in K$, *if it is* $C-$*lower semicontinuous and* $C-$*upper semicontinuous at* $x_0 \in K$.

Note that if in Definition 2 we take $Y = \mathbb{R}, C = [0, \infty)$ and K is a convex subset of the real line, then $C-$ convex , $C-$strongly convex and $C-$quasiconvex for the mapping $g : K \to \mathbb{R}$ convert to the usual definitions of convexity, strong convexity and quasiconvexity of the mapping g.

Also one can see that Definition 1 is equivalent to the following one:

(i) $f : X \to \mathbb{R}$ is lower semicontinuous if and only if for each real number λ the set

$$\{x \in X : f(x) > \lambda\},$$

is open.

It is clear that g is $C-$ lower semicontinuous if and only if it is $(-C)-$ upper semicontinuous.

(ii) $f : X \to \mathbb{R}$ is upper semicontinuous if and only if for each real number λ the set

$$\{x \in X : f(x) < \lambda\},$$

is open.

Also, the last definitions (that is (i) and (ii)) are equivalent to the definitions given by part (d) of Definition 2, if we take $X = \mathbb{R}, C = [0, \infty)$.

It is clear that g is $C-$ lower semicontinuous if and only if it is $-C-$ upper semicontinuous. It is also easy to check that if $g : K \to Y$ is continuous, then it is $C-$ upper semicontinuous and $C-$lower semicontinuous. Hence, $C-$ continuous. While the simple example $f(x) = [x]$ (floor function) is not continuous on the real line but it is $C = [0, \infty)-$ lower semicontinuous.

Similarly, we say that the mapping g is $C-$ concave, $C-$ strongly concave and $C-$quasiconcave, respectively, if $-g$ is $C-$convex, $C-$ strongly convex and $C-$quasiconvex, respectively.

Also, the mapping g is upper semicontinuous at x_0 if $-g$ is lower semicontinuous at x_0.

Remark 2. *If in Definition 2(c) we take $Y = \mathbb{R}, C = (-\infty, 0]$ and $g : K \longrightarrow Y$ is $C-$upper semicontinuous, then for each real number r the set $A_r = \{x \in K, f(x) \leq r\}$ is closed. Indeed, suppose the contrary: if $x \in clA_r \backslash A_r$, then there exists a net $\{x_\alpha\}_{\alpha \in I}$ in A_r such that $x_\alpha \longrightarrow x$. So since $V = (r, +\infty)$ is a neighborhood of $f(x_0)$, there exists a neighborhood U of x_0 such that $f(U \cap K) \subseteq (r, +\infty) - (-\infty, 0] \subseteq (r, +\infty)$. Since $U \cap K$ is in neighborhood of x_0, there exists α_0 such that $x_{\alpha_0} \in U \cap K$ and $f(x_{\alpha_0}) > r$, and this is a contradiction. Thus, a scalar-valued mapping g being $C-$ upper semicontinuous implies its upper semicontinuity.*

Definition 3 ([9])**.** *Let M, N be two topological spaces. A set-valued mapping $F : M \longrightarrow 2^N$ is upper semicontinuous (lower semicontinuous) at a point $x_0 \in X$, if for each open set W with $F(x_0) \subseteq W$ ($F(x_0) \cap W \neq \emptyset$) there is an open set U which contains x_0 such that*

$$F(x) \subseteq W, \ \ \forall x \in U.$$

$$(F(x) \cap W \neq \emptyset, \ \ \forall x \in U)$$

Proposition 1 ([15]). *If $F : X \longrightarrow 2^Y$ has compact values (i.e., $F(x)$ is a compact set for each $x \in X$), then F is upper semicontinuous (u.s.c. for short) at $\bar{\lambda}$ if and only if for any net $\{\lambda_i\} \subset \Lambda$ with $\lambda_i \longrightarrow \bar{\lambda}$ and for any $x_i \in F(\lambda_i)$, there exist $\bar{x} \in F(\bar{\lambda})$ and a subnet x_{i_j} of $\{x_i\}$ such that $x_{i_j} \longrightarrow \bar{x}$.*

Definition 4 ([12]). *A mapping $f : K \times K \longrightarrow Y$ is said to be $C-$monotone, if for all $x, y \in K$,*

$$f(x,y) + f(y,x) \in -C.$$

Lemma 1 ([14]). *If A, B are convex subsets of a topological vector space X and $intA \neq \emptyset$, then*

$$int(A+B) = intA + B.$$

As a consequence of Lemma 1 we have the following remark.

Remark 3. *If C is a convex cone, then $C + intC = intC$.*

The following example shows that convexity of the sets in Lemma 3 are essential. We continue by recalling the scalarization method. Let Y^* be the topological dual of Y and C be a convex cone of Y. The set

$$C^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \quad y \in C\},$$

is called the positive polar cone of C which is closed convex cone ([9, 11]). We put $C^*_+ = C^* \setminus \{0\}$.

Proposition 2 ([7, 11]). *Let C be a convex cone subset of topological vector space X. Then the following assertions are true*

$$y \in C \Longleftrightarrow [\langle y^*, y\rangle \geq 0, \ \forall y^* \in C^*], \tag{5}$$

$$y \in intC \Longleftrightarrow [\langle y^*, y\rangle > 0, \forall y^* \in C^*_+]. \tag{6}$$

The proof of the relations defined by (5) and (6) are based upon the separation theorems ([7, 11]). Proposition 1 plays a crucial role in proving main results of this note.

Proposition 3 ([9]). *Let g be a mapping from K into Y and $u^* \in C^*_+$. Let $\phi : K \longrightarrow \mathbb{R}$ be a mapping defined by $\phi(x) = \langle u^*, g(x)\rangle$ for all $x \in K$. Then the following assertions are valid:*
(a) If g is $C-$convex (resp., C-concave), then ϕ is convex (resp., concave).
(b) If g is C-upper semicontinuous (resp., C-lower semicontinuous), then ϕ is u.s.c. (resp., l.s.c.).

Lemma 2. *If $g : K \longrightarrow Y$ is a $C-$lower semicontinuous mapping, then the set $A := \{x \in K; g(x) \notin intC\}$ is closed in K.*

Proof. It suffices to prove that $g^{-1}(intC)$ is open. Let $x \in g^{-1}(intC)$. Hence $g(x) \in intC$ and by Definition 2 and Remark 3 there exists a neighborhood U of x such that

$$g(U \cap K) \subseteq intC + C \subseteq intC.$$

On the other hand, there exists a neighborhood V of x such that $V \subseteq U \cap K$, so we have $V \subseteq g^{-1}(intC)$ which implies that $g^{-1}(intC)$ is open.◄

The following theorem plays a key role in the paper.

Theorem 1 ([5, 13])**.** *Let K be a nonempty subset of a topological vector space X and $G : K \longrightarrow 2^X$ be a multifunction with closed values such that the convex hull of every finite subset $\{x_1, x_2, ..., x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^{n} G(x_i)$ If there exists $x_0 \in K$ such that $G(x_0)$ is compact, then $\bigcap_{x\in K} G(x) \neq \emptyset$.*

We recall that the multifunction G satisfying the property in Theorem 1 is called a KKM mapping. We also need the following minimax theorem (see, for instance, Jeyakumar et al. [7] and Zeidler [16]).

Theorem 2. *(Sion) Let A and B be convex subsets of some real topological vector spaces with B compact, and let $p : A\times B \longrightarrow \mathbb{R}$. If $p(.,b)$ is lower semicontinuous and quasiconvex (resp., convex) on A for all $b \in B$, and if $p(a,.)$ is upper semicontinuous and quasiconcave (resp., concave) on B for all $a \in A$, then*

$$\inf_{a\in A} \max_{b\in B} p(a,b) = \max_{b\in B} \inf_{a\in A} p(a,b).$$

3. Existence results

In this section we present sufficient conditions which guarantee that the solution sets of the problems introduced by the relations (1) and (2) are nonempty and compact. Moreover, we give some examples which show that the main theorems of this article are extensions of the well known results in this area.

The following theorem is a vector generalization of Theorem 2.3 and Corollary 2.4 of [6].

Theorem 3. *Let K be a nonempty closed convex subset of a Hausdorff topological vector space X , Y be a Hausdorff topological vector space and C be a closed convex cone in Y with $intC \neq \emptyset$. Assume $f : K \times K \longrightarrow Y$ is a mapping and $T : K \longrightarrow 2^{L(X,Y)}$ is a multifunction with compact and convex values, i.e., $T(x)$ is compact and convex for all $x \in K$. Let $u^* \in C_+^*$, $\bar{x} \in K$ and the following conditions hold:*
(i) $f(\bar{x}, \bar{x}) = 0$,
(ii) for any fixed $x \in K$; $f(x,.) : K \longrightarrow Y$ is $C-$convex and $C-$lower semicontinuous;
(iii) for all $y \in K$ the following is satisfied

$$\forall t \in (0,1), \inf_{x^*\in T(x_t)} \langle u^*, f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle\rangle \geq 0 \Longrightarrow$$

$$\Longrightarrow sup_{x^*\in T(\bar{x})}\langle u^*, f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle\rangle \geq 0,$$

where $x_t = (1-t)\bar{x} + ty$;
(iv) $\langle u^*, f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle\rangle \geq 0$ *for all* $y \in K$ *and* $x^* \in T(y)$.
Then S_A *is nonempty.*

Proof. Rewrite condition (iv) as

$$\langle u^*, f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle\rangle \geq 0 \quad \forall y \in K, \quad \forall x^* \in T(y). \tag{7}$$

Hence,we have

$$\langle u^*, f(\bar{x}, y_t) + \langle x^*, y_t - \bar{x}\rangle\rangle \geq 0, \quad \forall x^* \in T(y_t).$$

where $y_t = (1-t)\bar{x} + ty \in K$ for $t \in (0,1)$.

We claim that for any $y \in K$

$$\langle u^*, f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle\rangle \geq 0, \ \forall \ x^* \in T(y_t). \tag{8}$$

Suppose, on the contrary, that the assertion is false. Then there exist $t \in (0,1)$ and $x_t^* \in T(y_t)$ such that

$$\left\langle u^*, f(\bar{x}, y) + \langle x_t^*, y - \bar{x}\rangle\right\rangle < 0.$$

By (7), (ii) and Proposition 3 we obtain

$$\begin{aligned}
&0 \leq \left\langle u^*, f(\bar{x}, y_t) + \langle x_t^*, y_t - \bar{x}\rangle\right\rangle = \\
&= \left\langle u^*, f(\bar{x}, (1-t)\bar{x} + ty)\right\rangle + \left\langle u^*, \langle x_t^*, (1-t)\bar{x} + ty - \bar{x}\rangle\right\rangle \leq \\
&\leq (1-t)\langle u^*, f(\bar{x}, \bar{x})\rangle + t\langle u^*, f(\bar{x}, y)\rangle + t\left\langle u^*, \langle x_t^*, y - \bar{x}\rangle\right\rangle \leq \\
&\leq \left\langle u^*, f(\bar{x}, y) + \langle x_t^*, y - \bar{x}\rangle\right\rangle < 0,
\end{aligned}$$

which is a contradiction. Thus, our claim is verified. It follows from (8) that

$$\inf_{x^*\in T(y_t)} \left\langle u^*, f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle\right\rangle \geq 0.$$

By condition (iv) we have

$$\sup_{x^*\in T(\bar{x})} \left\langle u^*, f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle\right\rangle \geq 0.$$

Thus

$$\inf_{y\in K} \sup_{x^*\in T(\bar{x})} \left\langle u^*, f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle\right\rangle \geq 0. \tag{9}$$

It follows from (ii) that the mapping $P : K \times T(\bar{x}) \longrightarrow \mathbb{R}$ defined by

$$P(y, x^*) = \left\langle u^*, f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle\right\rangle,$$

is lower semicontinuous and convex on K and clearly it is upper semicontinuous and concave on $T(\bar{x})$. Then Theorem 2 and the relation (9) imply

$$\max_{x^*\in T(\bar{x})}\inf_{y\in K}\big\langle u^*, f(\bar{x},y)+\langle x^*, y-\bar{x}\rangle\big\rangle = \inf_{y\in K}\sup_{x^*\in T(\bar{x})}\big\langle u^*, f(\bar{x},y)+\langle x^*, y-\bar{x}\rangle\big\rangle \geq 0.$$

Hence there exists $x^* \in T(\bar{x})$ such that

$$\inf_{y\in K}\big\langle u^*, f(\bar{x},y)+\langle x^*, y-\bar{x}\rangle\big\rangle \geq 0.$$

Therefore

$$\big\langle u^*, f(\bar{x},y)+\langle x^*, y-\bar{x}\rangle\big\rangle \geq 0, \quad \forall y \in K. \tag{10}$$

It follows from the relation (5) that

$$f(\bar{x},y)+\langle x^*, y-\bar{x}\rangle \not< 0.$$

Because otherwise, if $\forall u^* \in C_+^*$

$$f(\bar{x},y)+\langle x^*, y-\bar{x}\rangle \in -intC,$$

then for all $u^* \in C_+^*$ we have

$$\langle u^*, f(\bar{x},y)+\langle x^*, y-\bar{x}\rangle\rangle < 0,$$

which is a contradiction to (10) and so $\bar{x} \in S_A$, and this completes the proof.◀

Remark 4. *One can check that Theorem 3 is still valid if we replace the conditions (i) and (ii) by the conditions*

$$f(\bar{x},\bar{x}) \leq f(\bar{x},y) \ \text{ and } \ C-\textit{quasiconvexity}.$$

By taking the mapping f equal to zero in Theorem 3 we obtain the following corollary which is a vector version of Theorem 15 of [8] and Corollary 2.7 of [6] by relaxing upper semiconinuity on T and coercivity condition.

Corollary 1. *Let K be a nonempty closed convex subset of a Hausdorff topological vector space X , Y be a Hausdorff topological vector space and C be a closed convex cone in Y with $intC \neq \emptyset$. Let $T : K \longrightarrow 2^{L(X,Y)}$ be a multifunction with compact and convex values. If there exists $\bar{x} \in K$ that satisfies the following conditions:*

(i)

$$\inf_{x^*\in T(x_t)}\langle u^*, \langle x^*, y-\bar{x}\rangle\rangle \geq 0, \ \forall (t,y) \in (0,1)\times K \Longrightarrow sup_{x^*\in T(\bar{x})}\langle u^*, f(\bar{x},y)+\langle x^*, y-\bar{x}\rangle\rangle \geq 0,$$

where $x_t = (1-t)\bar{x}+ty$;

(ii) $\langle u^*, f(\bar{x},y)+\langle x^*, y-\bar{x}\rangle\rangle \geq 0$ for all $y \in K$ and $x^* \in T(y)$,
then there exist $x \in K$ and $x^* \in T(x)$ such that $\langle x^*, y-x\rangle \leq 0$.

The following example satisfies all the assumptions of Corollary 1 while it is not upper semicontinuous. Hence it does not satisfy all the assumptions of Theorem 15 of [8] and Corollary 2.7 of [6] .

Example 1. *Let* $X = Y = \mathbb{R}, K = [0, 1], C = [0, +\infty)$ *and*

$$T(x) = \begin{cases} [0, \frac{1}{3}] & x = 0, \\ \{\frac{1}{2}\} & 0 < x \leq 1. \end{cases}$$

It is easy to check that T *fulfils all the conditions of Corollary 1 with* $\bar{x} = 0$.

The next result is an extension of Proposition 2.1 in [3]. Moreover, if we take $f(x, y) = 0$, then it is a vector version of Theorem 15 in [8].

Theorem 4. *Let* X *be a reflexive Banach space. Let* K *be a nonempty convex set in the* X *and* K_0 *be weakly compact subset of* K *and* Y *be a locally convex space. Let* $f : K\times K \longrightarrow Y$ *be a vector valued map and* $T : K \longrightarrow 2^{L(X,Y)}$ *be a multifunction. Assume that the following hypotheses hold:*
$(i)'$ *for all* $x \in K$, $f(x, x) = 0$;
$(ii)'$ *for all* $x \in K$, $T(x)$ *is weakly compact and convex set and* T *is u.s.c on* K;
$(iii)'$ *for all* $y \in K$, $f(., y)$ *is* $C-$*concave and* $C-$*upper semicontinuous and for all* $x \in K$, $f(x, .)$ *is* $C-$*convex.*
$(iv)'$ *for each* $x \in K \setminus K_0$ *there exists* $y \in K_0$ *such that* $f(x, y) + \langle x^*, y - x\rangle \prec 0$ *for all* $x^* \in T(x)$;
(v) for all $x, y \in K$ *and* $t \in (0, 1)$, $T(x_t) \subseteq (1 - t)T(x) - tT(y)$, *where* $x_t = (1 - t)x + ty$.
Then S_A *is nonempty and compact.*

Proof. Since K_0 is weakly compact, then it is bounded and so there exists $r > 0$ such that $K_0 \subset intB_r$, where $B_r \subset X$ is the closed ball with radius r, that is $B_r = \{x \in X : ||x|| \leq r\}$. Put $\Omega = K \cap B_r$. Define a multifunction $G : \Omega \longrightarrow 2^X$ by

$$G(y) = \{x \in \Omega|\big\langle u^*, f(x, y) + \langle x^*, y - x\rangle\big\rangle \geq 0, \ \forall x^* \in T(y)\}.$$

We claim that G is a KKM mapping. Indeed, $y \in G(y)$ for all $y \in \Omega$. Hence, $G(y) \neq \emptyset$ for all $y \in \Omega$. Let y_1 and y_2 be arbitrary elements of Ω. We show that $co\{y_1, y_2\} \subset G(y_1)\cup G(y_2)$. Let $y_t = ty_1 + (1 - t)y_2$ for all $t \in [0, 1]$. If $t = 0$ or $t = 1$, then there is nothing to prove. Suppose, on the contrary, for some $t \in (0, 1)$

$$y_t \notin G(y_i), \forall i \in \{1, 2\}.$$

Hence, $y_t \in \Omega$ since Ω is convex. Then there exist $x_1^* \in T(y_1), x_2^* \in T(y_2)$ such that

$$\big\langle u^*, f(y_t, y_i) + \langle x_i^*, y_i - y_t\rangle\big\rangle < 0, \text{ for } i = 1, 2.$$

The relation (ii) and Proposition 3(a) imply the function $\langle u^*, f(x, .)\rangle$ is convex. Also, $(i)'$ guarantees

$$0 = \big\langle u^*, f(y_t, ty_1 + (1 - t)y_2) + \langle x_1^* + x_2^*, (ty_1 + (1 - t)y_2) - y_t\rangle\big\rangle \leq$$

$$\leq t\big\langle u^*, f(y_t, y_1) + \langle x_1^*, y_1 - y_t\rangle\big\rangle + (1-t)\big\langle u^*, f(y_t, y_2) + \langle x_2^*, y_2 - y_t\rangle\big\rangle <$$
$$< 0,$$

which is a contradiction. Thus we have

$$y_t \in G(y_1) \cup G(y_2), \ \ \forall t \in [0,1],$$

and so

$$co\{y_1, y_2\} \subset G(y_1) \cup G(y_2).$$

By similar argument we can show that

$$co\{y_1, ..., y_n\} \subset \cup_{i=1}^n G(y_i).$$

Using $(iii)^{'}$ and Proposition 3 (a), we obtain that $G(y)$ is convex for all $y \in \Omega$. For each $y \in \Omega$ the set $G(y)$ is a closed set. Because, assume that $x_n \in G(y)$ is a net which converges to x. It follows from $(iii)^{'}$ and Proposition 3 (b) that, for all $x^* \in T(y)$,

$$\big\langle u^*, f(x, y) + \langle x^*, y - x\rangle\big\rangle \geq \limsup_{n \longrightarrow \infty} \big\langle u^*, f(x_n, y) + \langle x^*, y - x_n\rangle\big\rangle \geq 0,$$

and so $x \in G(y)$. which implies that $G(y)$ is a closed set for all $y \in \Omega$. Since $G(y) \subset B_r$ and X is a reflexive Banach space, $G(y)$ is a weakly compact set, for all $y \in \Omega$. Thus, G is a KKM mapping. By Theorem 1, there exists $\bar{\bar{x}} \in \Omega$ such that $\bar{\bar{x}} \in G(y)$ for all $y \in \Omega$. This implies that

$$\big\langle u^*, f(\bar{\bar{x}}, y) + \langle x^*, y - \bar{\bar{x}}\rangle\big\rangle \geq 0, \ \ \forall y \in \Omega, \ \forall x^* \in T(y). \tag{11}$$

It is straightforward to verify that $\bar{\bar{x}}$ satisfies all the assumptions of Theorem 3. It suffices to prove the condition (iii) of it. To see this, suppose that $y \in K$ is given and

$$\inf_{x^* \in T(x_t)} \big\langle u^*, f(\bar{\bar{x}}, y) + \langle x^*, y - \bar{\bar{x}}\rangle\big\rangle \geq 0, \ \ \ x_t = (1-t)\bar{\bar{x}} + ty.$$

Therefore

$$\big\langle u^*, f(\bar{\bar{x}}, y) + \langle x^*, y - \bar{\bar{x}}\rangle\big\rangle \geq 0, \ \forall x^* \in T(x_t), \ \forall t \in (0,1). \tag{12}$$

By (v) there exist $x_1^* \in T(\bar{\bar{x}}), x_2^* \in T(y)$ such that $x^* = (1-t)x_1^* - tx_2^*$. So we have

$$(1-t)\big\langle u^*, f(\bar{\bar{x}}, y) + \langle x_1^*, y - \bar{\bar{x}}\rangle\big\rangle = t\big\langle u^*, f(\bar{\bar{x}}, y) + \langle x_2^*, y - \bar{\bar{x}}\rangle\big\rangle + \big\langle u^*, f(\bar{\bar{x}}, y) + \langle x^*, y - \bar{\bar{x}}\rangle\big\rangle.$$

It follows from the relations (11) and (12) that

$$\big\langle u^*, f(\bar{\bar{x}}, y) + \langle x_1^*, y - \bar{\bar{x}}\rangle\big\rangle \geq 0.$$

Since $x_1^* \in T(\bar{\bar{x}})$, we have

$$\sup_{x^* \in T(\bar{\bar{x}})} \langle u^*, f(\bar{\bar{x}}, y) + \langle x^*, y - \bar{\bar{x}}\rangle\rangle \geq 0.$$

Hence, $\bar{\bar{x}}$ satisfies condition (iii) of Theorem 3. Then by Theorem 3 $\bar{\bar{x}}$ is a solution of the problem defined by the relation (1), that is $\bar{\bar{x}} \in S_A$. This means

$$\exists x^* \in T(\bar{\bar{x}}),\ \forall y \in \Omega \quad f(\bar{\bar{x}}, y) + \langle x^*, y - \bar{\bar{x}}\rangle \not\prec 0. \tag{13}$$

Combining this and $(iv)'$ we get $\bar{\bar{x}} \in K_0 \subset intB_r$. It remains to show that

$$\forall y \in K \quad f(\bar{\bar{x}}, y) + \langle x^*, y - \bar{\bar{x}}\rangle \not\prec 0. \tag{14}$$

Suppose on the contrary $f(\bar{\bar{x}}, y) + \langle x^*, y - \bar{\bar{x}}\rangle \prec 0$, for some $y \in K$. It is obvious that $y_t = (1-t)\bar{\bar{x}} + ty \in \Omega$ for a sufficiently small $t \in (0,1)$. Hence (13) implies that

$$f(\bar{\bar{x}}, y_t) + \langle x^*, y_t - \bar{\bar{x}}\rangle \not\prec 0. \tag{15}$$

It follows from (ii) that

$$(1-t)f(\bar{\bar{x}}, \bar{\bar{x}}) + (1-t)\langle x^*, \bar{\bar{x}} - \bar{\bar{x}}\rangle + tf(\bar{\bar{x}}, y) + t\langle x^*, y - \bar{\bar{x}}\rangle - \big(f(\bar{\bar{x}}, y_t) + \langle x^*, y_t - \bar{\bar{x}}\rangle\big) \in C.$$

Hence

$$-f(\bar{\bar{x}}, y_t) + \langle x^*, y_t - \bar{\bar{x}}\rangle \in -t\big(f(\bar{\bar{x}}, y) + \langle x^*, y - \bar{\bar{x}}\rangle\big) + C \subseteq intC + C \subseteq intC.$$

This implies that $f(\bar{\bar{x}}, y_t) + \langle x^*, y_t - \bar{\bar{x}}\rangle \prec 0$, which contradicts (15). Thus we have $f(\bar{\bar{x}}, y) + \langle x^*, y - \bar{\bar{x}}\rangle \not\prec 0$.
It is obvious from $(iv)'$ that $S_A \subseteq K_0$. Now we prove that S_A is closed.

$$S_A = \{x \in K;\ \exists x^* \in T(x),\ \forall y \in K,\ f(x,y) + \langle x^*, y - x\rangle \notin -intC\}.$$

Let $\{x_i\}$ be a net in S_A convergent to $\bar{x}$. Then there exists $x_i^* \in T(x_i)$ such that

$$f(x_i, y) + \langle x_i^*, y - x_i\rangle \notin -intC.$$

Since $T(\bar{x})$ is compact, by Proposition 1 there exists a subnet $x_{i_k}^*$ of x_i^* such that

$$x_{i_k}^* \longrightarrow x^*,$$

and it follows from $C-$upper semicontinuity of $f(.,y)$ that

$$\langle u^*, f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle\rangle \geq \limsup_{k \longrightarrow \infty} \langle u^*, f(x_{i_k}, y) + \langle x_{i_k}^*, y - x_{i_k}\rangle\rangle \geq 0,$$

which implies that $f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle \notin -intC$. Hence $\bar{x} \in S_A$. ◄

Theorem 5. *Let X be a topological vector space and Y be a locally convex space. Assume the conditions (i),(ii) and (iii) of Theorem 3 and the condition $(iv)'$ of Theorem 4 are satisfied. Then the following assertions hold:*

(a) For each $y \in K$, the set

$$\{x \in K | \max_{x^* \in T(x)} \langle u^*, f(x,y) + \langle x^*, x-y\rangle\rangle \geq 0, \forall u^* \in C^*\},$$

is a closed set,

(b) If there exist a weakly compact subset K_0 of K and $\bar{x} \in K$ such that for each $x \in K \setminus K_0$ one has

$$f(x,\bar{x}) + \langle x^*, x-\bar{x}\rangle \prec 0, \;\; x^* \in T(\bar{x}),$$

then S_B is nonempty.

Proof. Let $G : K \longrightarrow 2^K$ be defined by

$$G(y) = \{x \in K | \max_{x^* \in T(x)} \langle u^*, f(x,y) + \langle x^*, x-y\rangle\rangle \geq 0, \;\; \forall u^* \in C^*\}, \;\; y \in K.$$

We claim that $G(\bar{x}) \subset K_0$. Otherwise there exists an $x \in G(\bar{x}) \setminus K_0$. By (b),

$$f(x,\bar{x}) + \langle x^*, x-\bar{x}\rangle \in -intC, \;\; \forall x^* \in T(\bar{x}).$$

Hence it follows from the relation (5) that

$$\big\langle u^*, f(x,\bar{x}) + \langle x^*, x-\bar{x}\rangle \big\rangle < 0, \;\; \forall u^* \in C^* \backslash \{0\}.$$

Hence

$$\max_{x^* \in T(x)} \big\langle u^*, f(x,\bar{x}) + \langle x^*, x-\bar{x}\rangle \big\rangle < 0, \;\; \forall u^* \in C^* \backslash \{0\}.$$

This implies $x \notin G(\bar{x})$, which is a contradiction. Consequently, $G(\bar{x}) \subseteq K_0$ and since K_0 is weakly compact and by (a) is closed, we conclude that $G(\bar{x})$ is weakly compact. Proceeding as in the proof of Theorem 4 and using our assumptions, we get G is a KKM mapping. Therefore G satisfies all the assumptions of Theorem1 and so the set $\cap_{y \in K} G(y)$ is nonempty. Then there exists $w \in \cap_{y \in K} G(y)$, and since $S_B = \cap_{y \in K} G(y)$,we deduce that $w \in S_B$. This completes the proof.◀

The following result is a generalization of Lemma 2.1 in [6]. Also it is a vector version and generalization of Lemma 2.1 of [3] with mild assumptions. Moreover, it improves the corresponding result given in [1, 2] and the references therein.

Proposition 4. *Assume that, for all $y \in K$, the function $f(.,y)$ is $C-$convex and $C-$monotone. Then the solution set of the problem defined by* (1) *equals to the solution set of the following problem:*

Find $\bar{x} \in K$ such that

$$\exists x^* \in T(\bar{x}), \quad \forall y \in K, \quad f(y,\bar{x}) - \langle x^*, y-\bar{x}\rangle \not\succ 0. \tag{16}$$

Proof. Suppose $\bar{x} \in K$ is a solution of the problem defined by (16). Then there exists $x^* \in T(\bar{x})$ such that

$$f(y, \bar{x}) - \langle x^*, y - \bar{x}\rangle \notin C, \ \ \forall y \in K.$$

Hence for all $y \in K$

$$f(y_t, \bar{x}) - \langle x^*, y_t - \bar{x}\rangle \notin C,$$

where $y_t = ty + (1-t)\bar{x}$. Since $f(x,x) = 0$ and f is $C-$convex, we get

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, \bar{x}) \Longrightarrow tf(y_t, y) + (1-t)f(y_t, \bar{x}) \in C. \tag{17}$$

Also

$$\langle x^*, y_t - \bar{x}\rangle = t\langle x^*, y - \bar{x}\rangle \Longrightarrow (1-t)\langle x^*, y_t - \bar{x}\rangle - t(1-t)\langle x^*, y - \bar{x}\rangle = 0. \tag{18}$$

Combining (17) and (18), we obtain

$$tf(y_t, y) + (1-t)\{f(y_t, \bar{x}) - \langle x^*, y_t - \bar{x}\rangle\} + t(1-t)\langle x^*, y - \bar{x}\rangle \in C \ \ \forall t \in [0,1]. \tag{19}$$

Using (17), (18) and (5), we have

$$tf(y_t, y) + t(1-t)\langle x^*, y - \bar{x}\rangle \notin -intC \Longrightarrow f(y_t, y) + (1-t)\langle x^*, y - \bar{x}\rangle \notin -intC. \tag{20}$$

We claim that

$$f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle \notin -intC.$$

Suppose that the claim is false. Then

$$f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle \in -intC. \tag{21}$$

It follows from the $C-$convexity of $f(., y)$ that

$$\begin{aligned} &f(y_t, y) + (1-t)\langle x^*, y - \bar{x}\rangle \preceq \\ &\preceq tf(y, y) + (1-t)f(\bar{x}, y) + (1-t)\langle x^*, y - \bar{x}\rangle = \\ &= (1-t)\big(f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle\big). \end{aligned}$$

Then

$$(1-t)\big(f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle\big) - [f(y_t, y) + (1-t)\langle x^*, y - \bar{x}\rangle] \in C. \tag{22}$$

Combining (22) and (21), we obtain

$$\begin{aligned} &-\big(f(y_t, y) + (1-t)\langle x^*, y - \bar{x}\rangle\big) \in -(1-t)\big(f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle\big) + C \subseteq \\ &\subseteq intC + C = intC. \end{aligned}$$

So

$$\big(f(y_t, y) + (1-t)\langle x^*, y - \bar{x}\rangle\big) \in -intC,$$

which a contradiction to (22). Therefore $f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle \notin -intC$ and $\bar{x}$ is a solution of problem (1).

Conversely, let $\bar{x} \in K$ be a solution of the problem defined by (1). We show that $\bar{x}$ is a solution of the problem defined by (3). Indeed, suppose on the contrary that there exists a point $y \in K$ such that

$$f(y, \bar{x}) - \langle x^*, y - \bar{x}\rangle \in intC. \tag{23}$$

Since f is $C-$monotone, we have

$$f(\bar{x}, y) + f(y, \bar{x}) \in C \Longrightarrow f(y, \bar{x}) = -f(\bar{x}, y) - v, \tag{24}$$

for some $v \in C$. It follows from combining (24) and (23) that

$$\begin{aligned}&-f(\bar{x}, y) - v - \langle x^*, y - \bar{x}\rangle \in C \Longrightarrow \\ &\Longrightarrow -f(\bar{x}, y) - \langle x^*, y - \bar{x}\rangle \in v + intC \subseteq C + intC = intC \Longrightarrow \\ &\Longrightarrow f(\bar{x}, y) + \langle x^*, y - \bar{x}\rangle \in -intC,\end{aligned}$$

which is contradicted by our assumption.◄

Remark 5. *If the conditions of Proposition 4 are satisfied and $\bar{x} \in K$ is a solution of the problem defined by* (1)*, then the following equality holds:*

$$\max_{x\in K} \inf_{u^*\in C^*} \langle u^*, f(x, \bar{x}) - \langle x^*, x - \bar{x}\rangle\rangle = 0. \tag{25}$$

Indeed, by Proposition 4 we have

$$f(x, \bar{x}) - \langle x^*, x - \bar{x}\rangle \not> 0, \quad \forall x \in K,$$

and so the conclusion follows from the relation and(6) $f(\bar{x}, \bar{x}) = 0$.

Similarly, we have the following fact.

Remark 6. *If the conditions of Proposition 4 are satisfied and $\bar{x} \in K$ is a solution of the problem defined by* (1)*, then the following assertions are true:*

$$\min_{u^*\in C^*} \sup_{x\in K} \langle u^*, f(\bar{x}, x) + \langle x^*, x - \bar{x}\rangle\rangle = 0,$$

and

$$\min_{u^*\in C^*\setminus\{0\}} \sup_{x\in K} \langle u^*, f(\bar{x}, x) + \langle x^*, x - \bar{x}\rangle\rangle = 0.$$

The following example illustrates Remark 6.

Example 2. *Let* $X = Y = \mathbb{R}, K = [0,1] \subset X$ *and* $C = [0,+\infty)$. *It is obvious that* $C^* = [0,+\infty)$. *If we take* $\bar{x} = 0,\ x^* = 1$. *Define the mappings* T *and* f *by*

$$T(x) = \begin{cases} [0,1] & 0 < x \leq 1, \\ \{1\} & x = 0. \end{cases}$$

and $f(x,y) = e^x - e^y$, *respectively. Then it is easy to see that the assertions of Remark 6 hold. Indeed, we have*

$$\inf_{u^* \in C^* \setminus \{0\}} \max_{x \in K} \langle u^*, f(\bar{x},x) + \langle x^*, x - \bar{x} \rangle \rangle =$$

$$= \inf_{u^* \in C^* \setminus \{0\}} u^* \max_{x \in [0,1]} \langle f(\bar{x},x) + \langle x^*, x - \bar{x} \rangle \rangle = (\inf_{u^* \in (0,\infty)} u^*)(1 - e^0 + 0) = 0.$$

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