Azerbaijan Journal of Mathematics
V. 6, No 2, 2016, July
ISSN 2218-6816

Blow-up of Solutions for a Viscoelastic Equation with Nonlinear Boundary Damping and Interior Source

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Abstract. In this paper we consider the viscoelastic wave equation with nonlinear boundary damping $-|y_t(L,t)|^{m-1}y_t(L,t)$ and interior source $|y(t)|^{p-1}y(t)$. We obtain two blow-up results for the solution with negative initial energy as well as nonnegative initial energy.

Key Words and Phrases: boundary damping term, interior source term, viscoelastic wave equation.

2010 Mathematics Subject Classifications: 35B44, 35L10, 35L71

1. Introduction

In this paper, we shall study the following viscoelastic equation with nonlinear boundary damping and interior source terms:

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) + \displaystyle\int_0^t g(t-s)y_{xx}\mathrm{d}s = |y(x,t)|^{p-1}y(x,t), & (x,t) \in (0,L)\times(0,T), \\ y(0,t) = 0, \quad y_x(L,t) - \displaystyle\int_0^t g(t-s)y_x(L,s)\mathrm{d}s = -|y_t(L,t)|^{m-1}y_t(L,t), & t \in [0,T), \\ y(x,0) = y^0(x), \quad y_t(x,0) = y^1(x), & x \in [0,L], \end{cases} \tag{1}$$

where $(0,L)$ is a bounded open interval in $\mathbb{R}$, $m \geq 1$, $p > 1$.

In the absence of the viscoelastic term ($g = 0$), the wave equation with interior damping term has been extensively studied and several results concerning existence, asymptotic behavior and blow-up have been established. When $m = 1$, Levine [11, 12] proved that the solution blows up in finite time with negative initial energy. When $m > 1$, Georgiev and Todorova [10] extended this result and established a global existence result if $m \geq p$ and a blow-up result if $m < p$ for sufficiently large initial data. Later Messaoudi [20] improved [10] by considering only negative initial energy. The wave equation with boundary source term has also been extensively studied. Vitillaro [24] proved a global existence result when

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$p \leq m$ or the initial data are inside the potential well. In [28], Zhang and Hu proved the decay result when the initial data are inside a stable set, and the blow-up result when $p > m$ and the initial data is inside an unstable set. For other related equations with various source, we can also refer the reader to [1, 3, 4, 5, 7, 9, 13, 14, 15, 16, 17, 19, 22, 25, 26, 27] and references therein.

In the presence of the viscoelastic term ($g \neq 0$), Messaoudi [21] studied the following system:

$$\begin{cases} u_{tt}(x,t) - \triangle u + \displaystyle\int_0^t g(t-s)\triangle u(\tau)\mathrm{d}\tau + a u_t|u_t|^{m-2} = bu|u|^{p-2}, & in \quad \Omega \times (0,\infty), \\ u = 0, & on \quad \Gamma \times (0,\infty), \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), & x \in \Omega, \end{cases} \tag{2}$$

and proved that any weak solution with negative initial energy blows up in finite time if $p > m$, while the solutions exist globally for any initial data, in the appropriate space, provided that $m \geq p$. Cavalcanti et al. [6] studied (2) for $m = 2$ and a localized damping $a(x)u_t$. They obtained an exponential rate of decay by assuming that the kernel g is of exponential decay. This work extended the result of Zuazua [29] in which the author considered (2) with $g = 0$ and the linear damping is localized.

Recently, Feng et al. [8] considered problem (1) without the viscoelastic term and obtained the blow-up results with one of the following conditions: (A) $2m < p+1$ and $E(0) < 0$; (B) $2m \geq p+1$, $E(0) < 0$, and $L > \frac{4p}{(p-1)(p+1)}$. Later, Liu et al. [18] improved theses results for the solution with nonnegative initial energy.

In this paper we consider problem (1) which contains the viscoelastic term. We are interested in the interaction among the viscoelastic term $\int_0^t g(t-s)y_{xx}\mathrm{d}s$, the boundary damping $-|y_t(L,t)|^{m-1}y_t(L,t)$ and the interior source $|y(t)|^{p-1}y(t)$. We obtain two blow-up results for the solution with negative initial energy as well as nonnegative initial energy.

This paper is organized as follows. In Section 2, we present some notations needed for our work and state our main blow-up results (see Theorem 2 and Theorem 3 below). In Section 3, we give the proof of Theorem 2. Section 4 is dedicated to the proof of Theorem 3.

2. Notations and main results

As in [8], we introduce the notation $\|\cdot\|_q = \|\cdot\|_{L^q(0,L)}$ and the Hilbert space

$$H^1_{\text{left}}(0,L) := \{u \in H^1(0,L) : \ u(0) = 0\}.$$

We define the following functionals:

$$E(t) = \frac{1}{2}\|y_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|y_x(t)\|_2^2 + \frac{1}{2}(g \circ y_x)(t) - \frac{1}{p+1}\|y(t)\|_{p+1}^{p+1}, \tag{3}$$

$$I(t) = \left(1 - \int_0^t g(s)\mathrm{d}s\right)\|y_x(t)\|_2^2 + (g \circ y_x)(t) - \|y(t)\|_{p+1}^{p+1}, \tag{4}$$

where

$$(g \circ v)(t) = \int_0^t g(t-\tau)\|v(t) - v(\tau)\|_2^2 \mathrm{d}\tau.$$

Then, we make the following assumptions on g:

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)\mathrm{d}s = l > 0, \quad g(t) \geq 0, \quad g'(t) \leq 0, \quad \forall t \geq 0. \tag{5}$$

By taking a derivative of (3), we get

$$E'(t) = -|y_t(L,t)|^{m+1} + \frac{1}{2}(g' \circ y_x)(t) - \frac{1}{2}g(t)\|y_x(t)\|_2^2 \leq 0, \quad \forall t \geq 0.$$

Next, we state the local existence theorem, which can be proved by adopting the arguments of [2, 10].

Theorem 1. *Assume that* $(y^0, y^1) \in H^1_{left}(0,L) \times L^2(0,L)$ *and* g *is a* C^1 *function satisfying* (5)*. Then problem* (1) *has a unique local solution* $y(x,t)$ *satisfying*

$$y(x,t) \in C(0,T_m; H^1_{left}(0,L)), \quad y_t(x,t) \in C(0,T_m; L^2(0,L)), \quad y_t(L,t) \in L^{m+1}(0,T_m)$$

for some $T_m > 0$.

Now, we state our main results as follows.

Theorem 2. *Let* $y(x,t)$ *be a solution of problem* (1)*. Assume that* $2m < p+1$, (5) *and*

$$\int_0^\infty g(s)\mathrm{d}s < \frac{p-1}{p-1+[1/(p+1)]}, \tag{6}$$

hold, and the initial energy satisfies $E(0) < 0$*, where*

$$E(0) = \frac{1}{2}\|y^1\|_2^2 + \frac{1}{2}\|y_x^0\|_2^2 - \frac{1}{p+1}\|y^0\|_{p+1}^{p+1}.$$

Then the solution blows up in finite time.

Set

$$E_1 := \left(\frac{1}{2} - \frac{1}{p+1}\right)\alpha_0 l, \quad \alpha_0 := \left(\frac{l}{C_*^{p+1}}\right)^{\frac{2}{p-1}}, \tag{7}$$

where C_* is the optimal constant of the Sobolev embedding $\|y\|_{p+1} \leq C_*\|y_x\|_2$, for any $y \in H^1_{\text{left}}(0,L)$. We have the following result.

Theorem 3. *Let* $y(x,t)$ *be a solution of problem* (1)*. Assume that* $2m < p+1$, $I(0) < 0$, (5) *and*

$$\int_0^\infty g(s)\mathrm{d}s < \frac{p-1}{p-1+1/[(1-\eta\theta)^2(p-1)+2(1-\eta\theta)]}, \tag{8}$$

hold, where $\eta = \frac{2}{(p+1)-\theta(p-1)} \in (0,1)$*. Suppose that for any fixed* $0 < \theta < 1$, $0 \leq E(0) < \theta E_1$*. Then the solution blows up in finite time.*

3. Proof of Theorem 2

In this section, we consider the blow-up result in the case of $E(0) < 0$. Our proof technique follows the arguments of [8, 21], with some modifications needed for our problems.

Lemma 1. *Let $y(x,t)$ be a solution of problem (1). Assume that $E(0) < 0$ and $2 \le s \le p+1$. Then there exists a positive constant $C > 0$ such that*

$$\|y\|_{p+1}^{s} \le C(\|y_x\|_2^2 + \|y\|_{p+1}^{p+1}). \tag{9}$$

Proof. If $\|y\|_{p+1} \le 1$, then it follows from Sobolev embedding and the fact $E(t) \le E(0) < 0$ that

$$\|y\|_{p+1}^{s} \le \|y\|_{p+1}^{2} \le C\|y_x\|_2^2 \le C\|y\|_{p+1}^{p+1}.$$

If $\|y\|_{p+1} > 1$, then

$$\|y\|_{p+1}^{s} \le \|y\|_{p+1}^{p+1}. \tag{10}$$

Therefore (9) follows.◀

Set

$$H(t) := -E(t). \tag{11}$$

As a result of (3) and (9), we have

Lemma 2. ([21]) *Let the assumptions of Lemma 1 hold. Then we have*

$$\|y\|_{p+1}^{s} \le C(-H(t) - \|y_t\|_2^2 - (g \circ y_x)(t) + \|y\|_{p+1}^{p+1}), \quad \textit{for all } t \in [0,T),$$

for any $y(\cdot,t) \in H^1_{left}(0,L)$.

Taking the conditions $m \ge 1$, $p > 1$ and $2m < p+1$ into account, it follows that $\frac{2}{p+1}, \frac{m}{p+1-m} < 1$. As in [8], we can choose a constant r such that

$$0 < \max\left\{\frac{2}{p+1}, \frac{m}{p+1-m}\right\} < r < 1. \tag{12}$$

Then we infer that

$$2 \le m+1, m\frac{r+1}{r}, \frac{p+1}{2}(1+r) < p+1. \tag{13}$$

Lemma 3 ([8])**.** *Let $y(x,t)$ be a solution of problem (1) with $E(0) < 0$. Then there exists a positive constant C such that*

$$|y(L,t)|^{m+1} \le C\left[\|y\|_{p+1}^{m+1} + \|y\|_{p+1}^{m\frac{r+1}{r}} + \|y\|_{p+1}^{\frac{p+1}{2}(1+r)}\right], \tag{14}$$

for all $t \in [0,+\infty)$.

Define a functional $L(t)$ as

$$L(t) := H^{1-\sigma}(t) + \varepsilon \int_0^L y(t)y_t(t)\mathrm{d}x, \tag{15}$$

for ε small to be chosen later and

$$0 < \sigma < \min\left\{\frac{p-1}{2(p+1)}, \frac{p-m}{m(p+1)}, \frac{1}{m} - \frac{1+r}{r(p+1)}, \frac{1}{m} - \frac{1+r}{2m}\right\}, \tag{16}$$

where r is defined in (12). Then we have the following lemma.

Lemma 4 ([8])**.** *Let $y(x,t)$ be a solution of problem* (1) *with $E(0) < 0$ and $2m < p+1$. Then there exists a positive constant C such that*

$$H^{\sigma m}(t)|y(L,t)|^{m+1} \leq C\|y(t)\|_{p+1}^{p+1}, \tag{17}$$

for all $t \in [0, +\infty)$.

Now we are ready to prove our first result.

Proof. (Proof of Theorem 2) Taking a derivative of (15) yields

$$\begin{aligned} L'(t) \geq &(1-\sigma)H^{-\sigma}(t)|y_t(L,t)|^{m+1} + \varepsilon \int_0^L [y_t^2 - |y_x|^2](x,t)\mathrm{d}x + \varepsilon \int_0^L |y(t)|^{p+1}\mathrm{d}x - \\ &- \varepsilon|y_t(L,t)|^m|y(L,t)| + \varepsilon \int_0^t g(t-\tau)\int_0^L y_x(t)[y_x(\tau) - y_x(t)]\mathrm{d}x\mathrm{d}\tau + \\ &+ \varepsilon \int_0^t g(t-\tau)\|y_x(t)\|_2^2\mathrm{d}\tau. \end{aligned}$$

Using Schwarz inequality, we obtain

$$\begin{aligned} L'(t) \geq &(1-\sigma)H^{-\sigma}(t)|y_t(L,t)|^{m+1} + \varepsilon \int_0^L [y_t^2 - |y_x|^2](x,t)\mathrm{d}x + \varepsilon \int_0^L |y(t)|^{p+1}\mathrm{d}x - \\ &- \varepsilon|y_t(L,t)|^m|y(L,t)| - \varepsilon \int_0^t g(t-\tau)\|y_x(t)\|_2\|y_x(\tau) - y_x(t)\|_2\mathrm{d}\tau + \\ &+ \varepsilon \int_0^t g(t-\tau)\|y_x(t)\|_2^2\mathrm{d}\tau. \end{aligned} \tag{18}$$

By using Young's inequality, (3) and (11), we obtain from (18)

$$\begin{aligned} L'(t) \geq &(1-\sigma)H^{-\sigma}(t)|y_t(L,t)|^{m+1} + \varepsilon \int_0^L y_t^2(x,t)\mathrm{d}x - \varepsilon\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|y_x(t)\|_2^2 + \\ &+ \varepsilon\left[(p+1)H(t) + \frac{p+1}{2}\|y_t(t)\|_2^2 + \frac{p+1}{2}\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|y_x(t)\|_2^2 + \right. \\ &\left. + \frac{p+1}{2}(g \circ y_x)(t)\right] - \varepsilon|y_t(L,t)|^m|y(L,t)| - \varepsilon\beta(g \circ y_x)(t) - \frac{\varepsilon}{4\beta}\int_0^t g(s)\mathrm{d}s\|y_x(t)\|_2^2 \geq \end{aligned}$$

$$\begin{aligned}\geq&(1-\sigma)H^{-\sigma}(t)|y_t(L,t)|^{m+1}+\varepsilon\left(1+\frac{p+1}{2}\right)\int_0^L y_t^2(x,t)\mathrm{d}x+\varepsilon(p+1)H(t)+\\&+\varepsilon\left(\frac{p+1}{2}-\beta\right)(g\circ y_x)(t)-\varepsilon|y_t(L,t)|^m|y(L,t)|+\\&+\varepsilon\left[\left(\frac{p+1}{2}-1\right)-\left(\frac{p+1}{2}-1+\frac{1}{4\beta}\right)\int_0^t g(s)\mathrm{d}s\right]\|y_x(t)\|_2^2,\end{aligned}\tag{19}$$

for some number β with $0<\beta<\frac{p+1}{2}$. Since (6) holds, (19) reduces to

$$\begin{aligned}L'(t)\geq&(1-\sigma)H^{-\sigma}(t)|y_t(L,t)|^{m+1}+\varepsilon\left(1+\frac{p+1}{2}\right)\int_0^L y_t^2(x,t)\mathrm{d}x+\varepsilon(p+1)H(t)+\\&+\varepsilon a_1(g\circ y_x)(t)+\varepsilon a_2\|y_x(t)\|_2^2-\varepsilon|y_t(L,t)|^m|y(L,t)|,\end{aligned}$$

where

$$a_1=\frac{p+1}{2}-\beta>0,\quad a_2=\frac{p+1}{2}-1-\left(\frac{p+1}{2}-1+\frac{1}{4\beta}\right)\int_0^t g(s)\mathrm{d}s>0.$$

Using Young's inequality, we have

$$\begin{aligned}L'(t)\geq&(1-\sigma)H^{-\sigma}(t)|y_t(L,t)|^{m+1}+\varepsilon\left(1+\frac{p+1}{2}\right)\int_0^L y_t^2(x,t)\mathrm{d}x+\varepsilon(p+1)H(t)+\\&+\varepsilon a_1(g\circ y_x)(t)+\varepsilon a_2\|y_x(t)\|_2^2-\frac{m\varepsilon}{m+1}\delta^{-\frac{m+1}{m}}|y_t(L,t)|^{m+1}-\\&-\frac{\varepsilon}{m+1}\delta^{m+1}|y(L,t)|^{m+1}.\end{aligned}$$

If we let $\delta^{m+1}=k^{-m}H^{\sigma m}$, i.e., $\delta^{-\frac{m+1}{m}}=kH^{-\sigma}$, $k>0$ to be determined later, then

$$\begin{aligned}L'(t)\geq&\left(1-\sigma-\frac{km\varepsilon}{m+1}\right)H^{-\sigma}(t)|y_t(L,t)|^{m+1}+\varepsilon\left(1+\frac{p+1}{2}\right)\int_0^L y_t^2(x,t)\mathrm{d}x+\\&+\varepsilon(p+1)H(t)+\varepsilon a_1(g\circ y_x)(t)+\varepsilon a_2\|y_x(t)\|_2^2-\frac{k^{-m}\varepsilon}{m+1}H^{\sigma m}(t)|y(L,t)|^{m+1}.\end{aligned}$$

It follows from Lemma 4 that

$$\begin{aligned}L'(t)\geq&\left(1-\sigma-\frac{km\varepsilon}{m+1}\right)H^{-\sigma}(t)|y_t(L,t)|^{m+1}+\varepsilon\left(1+\frac{p+1}{2}\right)\int_0^L y_t^2(x,t)\mathrm{d}x+\\&+\varepsilon(p+1)H(t)+\varepsilon a_1(g\circ y_x)(t)+\varepsilon a_2\|y_x(t)\|_2^2-\frac{Ck^{-m}\varepsilon}{m+1}\|y(t)\|_{p+1}^{p+1}.\end{aligned}\tag{20}$$

By noting that

$$H(t)\geq\frac{1}{p+1}\|y(t)\|_{p+1}^{p+1}-\frac{1}{2}\|y_t(t)\|_2^2-\frac{1}{2}\|y_x(t)\|_2^2-\frac{1}{2}(g\circ y_x)(t),$$

and writing $p+1 = 2a_3 + (p+1-2a_3)$, where $a_3 = \min\{a_1, a_2\}$, (20) becomes

$$\begin{aligned} L'(t) \geq &\left(1-\sigma-\frac{km\varepsilon}{m+1}\right) H^{-\sigma}(t)|y_t(L,t)|^{m+1} + \varepsilon\left(1+\frac{p+1}{2}-a_3\right)\|y_t(t)\|_2^2 + \\ &+\varepsilon(a_1-a_3)(g\circ y_x)(t) + \varepsilon(a_2-a_3)\|y_x(t)\|_2^2 + \varepsilon[(p+1)-2a_3]H(t) + \\ &+\varepsilon\left(\frac{2a_3}{p+1}-\frac{Ck^{-m}}{m+1}\right)\|y(t)\|_{p+1}^{p+1}. \end{aligned} \tag{21}$$

At this point, choosing k large enough, we have (since $1+\frac{p+1}{2}-a_3>0$ and $(p+1)-2a_3>0$)

$$L'(t) \geq \left(1-\sigma-\frac{km\varepsilon}{m+1}\right) H^{-\sigma}(t)|y_t(L,t)|^{m+1} + \varepsilon\gamma[H(t) + \|y_t(t)\|_2^2 + \|y(t)\|_{p+1}^{p+1}], \tag{22}$$

where $\gamma > 0$ is the minimum of the coefficients of $H(t)$, $\|y_t(t)\|_2^2$, and $\|y(t)\|_{p+1}^{p+1}$ in (21). Once k is fixed, we pick ε small enough so that

$$1-\sigma-\frac{km\varepsilon}{m+1} \geq 0,$$

and

$$L(0) = H^{1-\sigma}(0) + \varepsilon\int_0^L y^0(x)y^1(x)\mathrm{d}x > 0.$$

Therefore (22) takes the form

$$L'(t) \geq \varepsilon\gamma[H(t) + \|y_t(t)\|_2^2 + \|y(t)\|_{p+1}^{p+1}].$$

Consequently, we have

$$L(t) \geq L(0) > 0, \quad for \quad all \quad t \geq 0.$$

We now estimate

$$\left|\int_0^L y(t)y_t(x,t)\mathrm{d}x\right| \leq \|y(t)\|_2\|y_t(t)\|_2 \leq C\|y(t)\|_{p+1}\|y_t(t)\|_2.$$

Again Young's inequality gives us

$$\left|\int_0^L y(t)y_t(x,t)\mathrm{d}x\right|^{1/(1-\sigma)} \leq C[\|y(t)\|_{p+1}^{\mu/(1-\sigma)} + \|y_t(t)\|_2^{\theta/(1-\sigma)}], \tag{23}$$

for $1/\mu + 1/\theta = 1$. We take $\theta = 2(1-\sigma)$ to get $\mu/(1-\sigma) = 2/(1-2\sigma) \leq p+1$. Therefore (23) becomes

$$\left|\int_0^L y(t)y_t(x,t)\mathrm{d}x\right|^{1/(1-\sigma)} \leq C[\|y(t)\|_{p+1}^{s} + \|y_t(t)\|_2^2],$$

where $s = 2/(1-2\sigma) \leq p+1$. By using Lemma 2 we obtain

$$\left|\int_0^L y(t)y_t(x,t)\mathrm{d}x\right|^{1/(1-\sigma)} \leq C[H(t) + \|y(t)\|_{p+1}^{p+1} + \|y_t(t)\|_2^2],$$

for all $t \geq 0$. Therefore we have

$$L^{1/(1-\sigma)}(t) \leq 2^{1/(1-\sigma)}\left(H(t) + \left|\int_0^L y(t)y_t(x,t)\mathrm{d}x)\right|^{1/(1-\sigma)}\right) \leq$$
$$\leq C[H(t) + \|y(t)\|_{p+1}^{p+1} + \|y_t(t)\|_2^2], \tag{24}$$

for all $t \geq 0$.

By combining (22) and (24) we arrive at

$$L'(t) \geq \Gamma L^{1/(1-\sigma)}(t), \quad for \quad all \quad t \geq 0, \tag{25}$$

where Γ is a positive constant depending only on $\varepsilon\gamma$. A simple integration of (25) over $(0,t)$ then yields

$$L^{\sigma/(1-\sigma)}(t) \geq \frac{1}{L^{-\sigma/(1-\sigma)}(0) - \Gamma t\sigma/(1-\sigma)}. \tag{26}$$

Therefore (26) shows that $L(t)$ blows up in time

$$T^* \leq \frac{1-\sigma}{\Gamma\sigma[L(0)]^{\sigma/(1-\sigma)}}.$$

This completes the proof.◄

4. Proof of Theorem 3

In this section, we shall combine the frameworks of [18] to improve the former blow-up result to the case of $0 \leq E(0) < \theta E_1$.

Lemma 5. *Let $y(x,t)$ be a solution of problem (1) with $0 \leq E(0) < \theta E_1$ and $I(0) < 0$. Then there exists a positive constant $0 < \eta < 1$ such that*

$$E_1 < \eta\frac{p-1}{2(p+1)}\|y(x,t)\|_{p+1}^{p+1}, \quad \forall\ t > 0. \tag{27}$$

Proof. We adopt the manner which was first introduced in [23]. From (3) and Sobolev embedding, we have

$$E(t) \geq \frac{l}{2}\|y_x\|_2^2 - \frac{1}{p+1}\|y\|_{p+1}^{p+1} \geq \frac{l}{2}\|y_x\|_2^2 - \frac{C_*^{p+1}}{p+1}\|y_x\|_2^{p+1}.$$

Let $h(\xi) = \frac{l}{2}\xi - \frac{C_*^{p+1}}{p+1}\xi^{\frac{p+1}{2}}$. Then

$$E(t) \geq h(\xi) \quad \text{with} \quad \xi = \|y_x\|_2^2.$$

It is easy to see that $h(\xi)$ is strictly increasing on $[0, \alpha_0)$, strictly decreasing on $(\alpha_0, +\infty)$ and takes its maximum value E_1 at α_0.

Since $I(0) < 0$, we have

$$l\|y_x^0\|_2^2 < \|y_x^0\|_2^2 < \|y^0\|_{p+1}^{p+1} \leq C_*^{p+1}\|y_x^0\|_2^{p+1},$$

which leads to

$$\|y_x^0\|_2^2 > \alpha_0, \quad \text{for } \alpha_0 \ \text{ defined in (7).}$$

Furthermore, since

$$E_1 > E(0) \geq E(t) \geq h(\|y_x\|_2^2), \quad \forall\, t \geq 0,$$

we can get that there exists no time t^* such that $\|y_x(t^*)\|_2^2 = \alpha_0$. By the continuity of $\|y_x\|_2^2$, we obtain

$$\|y_x\|_2^2 > \alpha_0, \quad \forall\, t \geq 0.$$

On the other hand, we have

$$\frac{1}{p+1}\|y\|_{p+1}^{p+1} \geq -E(0) + \frac{l}{2}\|y_x\|_2^2 > -\theta E_1 + \frac{l}{2}\alpha_0 = \left(\frac{p+1}{p-1} - \theta\right) E_1,$$

which gives

$$E_1 < \frac{p-1}{2(p+1)} \frac{2}{(p+1) - \theta(p-1)}\|y\|_{p+1}^{p+1}.$$

Taking $\eta = \frac{2}{(p+1)-\theta(p-1)} \in (0,1)$, we get the validity of (27).◀

Set

$$H_1(t) = \theta E_1 - E(t).$$

Then it is clear that $H_1(t)$ is increasing, $H_1(t) \geq H_1(0) > 0$ and

$$H_1(t) \leq \frac{\theta\eta(p-1) + 2}{2(p+1)}\|y\|_{p+1}^{p+1}. \tag{28}$$

Lemma 6. *Under the assumptions of Lemma 5, there exists a positive constant C such that*

$$\|y\|_{p+1}^{s} \leq C\|y\|_{p+1}^{p+1},$$

for any $2 \leq s \leq p+1$.

Proof. If $\|y\|_{p+1}^{p+1} \geq 1$, then $\|y\|_{p+1}^{s} \leq C\|y\|_{p+1}^{p+1}$, since $s \leq p+1$. If $\|y\|_{p+1}^{p+1} < 1$, then $\|y\|_{p+1}^{s} \leq \|y\|_{p+1}^{2}$, since $2 \leq s$. By using Sobolev embedding inequality, (3), and Lemma 5, we have

$$\|y\|_{p+1}^2 \leq C_*\|y_x\|_2^2 \leq C\left(E(t) + \|y\|_{p+1}^{p+1}\right) \leq C\left(E_1 + \|y\|_{p+1}^{p+1}\right) \leq C\|y\|_{p+1}^{p+1}.$$

This finishes the proof.◀

Similar to the proof of Theorem 2, we can prove the following lemmas.

Lemma 7. *Let the assumptions of Lemma 5 hold. Then we have*

$$\|y\|_{p+1}^{s} \leq C(-H_1(t) - \|y_t\|_2^2 - (g \circ y_x)(t) + \|y\|_{p+1}^{p+1}), \quad \textit{for all } t \in [0,T),$$

for any $y(\cdot,t) \in H^1_{left}(0,L)$.

We choose a constant r such that

$$0 < \max\left\{\frac{2}{p+1}, \frac{m}{p+1-m}\right\} < r < 1.$$

Then we infer that

$$2 \leq m+1, m\frac{r+1}{r}, \frac{p+1}{2}(1+r) < p+1.$$

Lemma 8. *Under the assumptions of Lemma 5, there exists a positive constant C such that*

$$|y(L,t)|^{m+1} \leq C\left[\|y\|_{p+1}^{m+1} + \|y\|_{p+1}^{m\frac{r+1}{r}} + \|y\|_{p+1}^{\frac{p+1}{2}(1+r)}\right].$$

Set

$$L_1(t) = H_1^{1-\sigma}(t) + \varepsilon\int_0^L y(t)y_t(t)\mathrm{d}x, \tag{29}$$

for ε small to be chosen later and

$$0 < \sigma < \min\left\{\frac{p-1}{2(p+1)}, \frac{p-m}{m(p+1)}, \frac{1}{m} - \frac{1+r}{r(p+1)}, \frac{1}{m} - \frac{1+r}{2m}\right\}.$$

Then we have the following lemma.

Lemma 9. *Under the assumptions of Lemma 5, there exists a positive constant C such that*

$$H_1^{\sigma m}(t)|y(L,t)|^{m+1} \leq C\|y\|_{p+1}^{p+1},$$

for any $2m < p+1$.

Now we are ready to prove our second main result.

Proof. (Proof of Theorem 3) Taking a derivative of (29) yields

$$\begin{aligned} L_1'(t) \geq &(1-\sigma)H_1^{-\sigma}(t)|y_t(L,t)|^{m+1} + \varepsilon\int_0^L [y_t^2 - |y_x|^2](x,t)\mathrm{d}x + \varepsilon\int_0^L |y(t)|^{p+1}\mathrm{d}x - \\ &- \varepsilon|y_t(L,t)|^m|y(L,t)| + \varepsilon\int_0^t g(t-\tau)\int_0^L y_x(t)[y_x(\tau) - y_x(t)]\mathrm{d}x\mathrm{d}\tau + \\ &+ \varepsilon\int_0^t g(t-\tau)\|y_x(t)\|_2^2\mathrm{d}\tau. \end{aligned}$$

By using Schwarz and Young's inequality, we have

$$L_1'(t) \geq (1-\sigma)H_1^{-\sigma}(t)|y_t(L,t)|^{m+1} + \varepsilon\int_0^L y_t^2(x,t)\mathrm{d}x - \varepsilon\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|y_x(t)\|_2^2 +$$

$$
\begin{aligned}
&+\varepsilon\left[(p+1)H_1(t)+\frac{p+1}{2}\|y_t(t)\|_2^2+\frac{p+1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)\|y_x(t)\|_2^2+\right.\\
&\left.+\frac{p+1}{2}(g\circ y_x)(t)-(p+1)\theta E_1\right]-\varepsilon|y_t(L,t)|^m|y(L,t)|-\varepsilon\beta_1(g\circ y_x)(t)-\\
&-\frac{\varepsilon}{4\beta_1}\int_0^t g(s)\mathrm{d}s\|y_x(t)\|_2^2\geq\\
\geq&(1-\sigma)H_1^{-\sigma}(t)|y_t(L,t)|^{m+1}+\varepsilon\left(1+\frac{p+1}{2}\right)\int_0^L y_t^2(x,t)\mathrm{d}x+\varepsilon(p+1)H_1(t)-\\
&-\varepsilon(p+1)\theta E_1+\varepsilon\left[\left(\frac{p+1}{2}-1\right)-\left(\frac{p+1}{2}-1+\frac{1}{4\beta_1}\right)\int_0^t g(s)\mathrm{d}s\right]\|y_x(t)\|_2^2+\\
&+\varepsilon\left(\frac{p+1}{2}-\beta_1\right)(g\circ y_x)(t)-\varepsilon|y_t(L,t)|^m|y(L,t)|\geq\\
\geq&(1-\sigma)H_1^{-\sigma}(t)|y_t(L,t)|^{m+1}+\varepsilon\left(1+\frac{p+1}{2}\right)\int_0^L y_t^2(x,t)\mathrm{d}x+\varepsilon(p+1)H_1(t)+\\
&+\varepsilon\left[(1-\eta\theta)\left(\frac{p+1}{2}-1\right)-\left((1-\eta\theta)\left(\frac{p+1}{2}-1\right)+\frac{1}{4\beta_1}\right)\int_0^t g(s)\mathrm{d}s\right]\|y_x(t)\|_2^2+\\
&+\varepsilon\left[(1-\eta\theta)\left(\frac{p+1}{2}-1\right)+(1-\beta_1)\right](g\circ y_x)(t)-\varepsilon|y_t(L,t)|^m|y(L,t)|, \qquad (30)
\end{aligned}
$$

for some number β_1 with $0<\beta_1<(1-\eta\theta)(\frac{p+1}{2}-1)+1$.

Since (8) holds, (30) reduces to

$$
\begin{aligned}
L_1'(t)\geq&(1-\sigma)H_1^{-\sigma}(t)|y_t(L,t)|^{m+1}+\varepsilon\left(1+\frac{p+1}{2}\right)\int_0^L y_t^2(x,t)\mathrm{d}x+\varepsilon(p+1)H_1(t)+\\
&+\varepsilon b_1(g\circ y_x)(t)+\varepsilon b_2\|y_x(t)\|_2^2-\varepsilon|y_t(L,t)|^m|y(L,t)|,
\end{aligned}
$$

where

$$
\begin{aligned}
b_1&=(1-\eta\theta)\left(\frac{p+1}{2}-1\right)+(1-\beta_1)>0,\\
b_2&=(1-\eta\theta)\left(\frac{p+1}{2}-1\right)-\left((1-\eta\theta)\left(\frac{p+1}{2}-1\right)+\frac{1}{4\beta_1}\right)\int_0^t g(s)\mathrm{d}s>0.
\end{aligned}
$$

Following the steps in the proof of Theorem 2, we get

$$
\begin{aligned}
L_1'(t)\geq&\left(1-\sigma-\frac{km\varepsilon}{m+1}\right)H_1^{-\sigma}(t)|y_t(L,t)|^{m+1}+\varepsilon\left(1+\frac{p+1}{2}\right)\int_0^L y_t^2(x,t)\mathrm{d}x+\\
&+\varepsilon(p+1)H_1(t)+\varepsilon b_1(g\circ y_x)(t)+\varepsilon b_2\|y_x(t)\|_2^2-\frac{Ck^{-m}\varepsilon}{m+1}\|y(t)\|_{p+1}^{p+1}.
\end{aligned}
$$

The remaining part is similar to the proof of Theorem 2, so we omit it.◄

Acknowledgements

This work was supported by the project of Chinese Ministry of Finance (Grant No. GYHY200906006), the National Natural Science Foundation of China (Grant No. 11301277), and the Natural Science Foundation of Jiangsu Province (Grant No. BK20151523).

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Received 14 April 2015
Accepted 27 April 2016