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On Conservative and Dissipative Finite Difference Scheme of Barotropic Gas Dynamics and Its Application to Problems of Calculation of Large-scale Sea Currents Using Shallow Water Model

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Abstract. In this paper we consider the approximation of a differential continuity equation by conservative difference scheme with non-negative solution. The grid analogue of energy inequality is obtained for nonlinear difference scheme for barotropic gas. The scheme allows to calculate the solution in the case when density is equal to zero. The obtained scheme is applied to the calculations of large-scale currents of the Black and Caspian seas on shallow water model.

Key Words and Phrases: Euler equations, finite difference scheme, shallow water model.

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1. Introduction

In continuum mechanics, the model of the dynamics of non-viscous compressible gas, consisting of the law of conservation of mass and Euler's equation, appears to be most simple. At the same time, to date the conditions of existence of the solution of the initial-boundary value problems are not fully clarified for such a model. Moreover, it remains an open question what is the functional space to which the solution of such problems should belong to. It even turned out to be easier to consider problems with viscosity elements in Euler equations of motion and with additional regularization of the continuity equation. The construction of difference schemes for such problems is much easier of course, but also the solutions obtained by numerical methods in such schemes may be quite different than described by not simplified system of equations. In this paper we consider difference schemes for the original conservation laws of continuum mechanics, constructed using the principle of "difference against the flow", which managed to get a number of useful properties.

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2. Main Results

We consider the following boundary value problem (compressible Euler equations [5]) for $t > 0, x \in \Omega$:

$$\begin{gathered}\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0,\\ \frac{\partial \rho \mathbf{u}}{\partial t} + \operatorname{div}(\rho \mathbf{u} \times \mathbf{u}) + \operatorname{grad} p = 0,\\ p = \rho^{\gamma},\end{gathered} \tag{1}$$

where ρ is the gas density, $\mathbf{u}$ is a velocity, p is a pressure, and $\gamma = \mathrm{const} > 1$ is a given constant.

The conditions of impermeability are defined on the boundary of the domain. In one-dimensional case $\Omega = [0, 1]$:

$$u(t, 0) = u(t, 1) = 0.$$

In case of two dimensions $\Omega = [0, 1] \times [0, 1], \mathbf{u} = (u_1, u_2)$:

$$u_1(t, 0, x_2) = u_1(t, 1, x_2) = 0,\ u_2(t, x_1, 0) = u_2(t, x_1, 1) = 0.$$

And the initial conditions are

$$\rho|_{t=0} = \rho^0(x) \geqslant 0, \mathbf{u}|_{t=0} = \mathbf{u}^0(x), x \in \Omega.$$

For the above problem under the conditions of existence, uniqueness and differentiability of functions ρ, $\mathbf{u}$ one can formally obtain the integral identity

$$\int\limits_{\Omega} \left(\rho(t, x) \frac{\mathbf{u}^2(t, x)}{2} + \frac{p(t, x)}{\gamma - 1} \right) dx = \int\limits_{\Omega} \left(\rho(0, x) \frac{\mathbf{u}^2(0, x)}{2} + \frac{p(0, x)}{\gamma - 1} \right) dx,$$

and if the problem has a viscosity

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \operatorname{div}(\rho \mathbf{u} \times \mathbf{u}) + L\mathbf{u} + \operatorname{grad} p = 0,$$

with some linear symmetric positive definite operator L (supplemented if necessary with boundary conditions), then, assuming non-negativity of the function $\rho(t, x)$, it is possible to obtain the energy inequality

$$\int\limits_{\Omega} \left(\rho(t, x) \frac{\mathbf{u}^2(t, x)}{2} + \frac{p(t, x)}{\gamma - 1} \right) dx \leqslant \int\limits_{\Omega} \left(\rho(0, x) \frac{\mathbf{u}^2(0, x)}{2} + \frac{p(0, x)}{\gamma - 1} \right) dx. \tag{2}$$

Next, we construct a difference scheme for this problem. The purpose of the work is to get the difference analogues of the non-negativity condition for the density function, conditions of conservatism

$$\int\limits_{\Omega} \rho(t, x) dx = \int\limits_{\Omega} \rho(0, x) dx,\ \forall t > 0$$

and energy inequality (2).

We introduce a uniform grid on Ω. In one-dimensional case $x_i = ih, i = 0, ..., N, Nh = 1$, in two-dimensional case $x_{1,i} = ih_1, x_{2,j} = jh_2, i = 0, ..., N_1, j = 0, ..., N_2, N_1 h_1 = 1, N_2 h_2 = 1$. We use a constant step in the time variable τ.

We define grid functions $\rho_i^n, 0 \leqslant i \leqslant N-1, u_i^n, 0 \leqslant i \leqslant N, u_0^n = u_N^n = 0$ in one-dimensional case; and in case of two spatial directions $\rho_{ij}^n, 0 \leqslant i \leqslant N_1 - 1, 0 \leqslant j \leqslant N_2 - 1, u_{1,ij}^n, 0 \leqslant i \leqslant N_1, 0 \leqslant j \leqslant N_2 - 1, u_{1,ij}^n = 0 \,\text{at}\, i = 0 \,\text{or}\, i = N_1, u_{2,ij}^n, 0 \leqslant i \leqslant N_1 - 1, 0 \leqslant j \leqslant N_2, u_{2,ij}^n = 0 \,\text{at}\, j = 0 \,\text{or}\, j = N_2$. We assume that the grid functions are zero outside the boundaries of the above indices. As usual, not shifted indices n, i will be omitted. We use standard notations of the difference schemes theory [6]: $y = y_i^n$, $y_{t,i} = (y_i^{n+1} - y_i^n)/\tau, y_{\bar{x},i} = (y_i^n - y_{i-1}^n)/h$.

In one-dimensional case, the approximation of continuity equation is

$$\rho_t + (\{\rho^{n+1}\}u)_x = 0,$$

$$\{\rho^{n+1}\} = \rho^{n+1} - h\rho_{\bar{x}}^{n+1}\max(0, \operatorname{sign}(u)) = \frac{\rho_i^{n+1} + \rho_{i-1}^{n+1}}{2} - \frac{h}{2}\rho_{\bar{x}}^{n+1}\operatorname{sign}(u),$$

or

$$\{\rho^{n+1}\} = [\rho_i^{n+1}] - \frac{h}{2}\rho_{\bar{x}}^{n+1}\operatorname{sign}(u), \text{ where } [\rho_i^{n+1}] = \frac{\rho_i^{n+1} + \rho_{i-1}^{n+1}}{2}.$$

Operator braces $\{.\}$ depend on the grid function u.

In two-dimensional case

$$\rho_t + (\{\rho^{n+1}\}_1 u_1)_{x_1} + (\{\rho^{n+1}\}_2 u_2)_{x_2} = 0,$$

$$\{\rho^{n+1}\}_1 = \rho^{n+1} - h\rho_{\bar{x}_1}^{n+1}\max(0, \operatorname{sign}(u_1)) = [\rho^{n+1}]_1 - \frac{h}{2}\rho_{\bar{x}_1}^{n+1}\operatorname{sign}(u_1),$$
$$\{\rho^{n+1}\}_2 = \rho^{n+1} - h\rho_{\bar{x}_2}^{n+1}\max(0, \operatorname{sign}(u_2)) = [\rho^{n+1}]_2 - \frac{h}{2}\rho_{\bar{x}_2}^{n+1}\operatorname{sign}(u_2).$$

Operators braces $\{.\}_1, \{.\}_2$ depend on the grid functions u_1, u_2.

The difference equation for the one-dimensional case in indices

$$\frac{\rho_i^{n+1} - \rho_i^n}{\tau} +$$
$$+ \frac{-\rho_{i+1}^{n+1}(-u_{i+1} + |u_{i+1}|) + \rho_i^{n+1}(u_{i+1} + |u_{i+1}| - u_i + |u_i|) - \rho_{i-1}^{n+1}(u_i + |u_i|)}{2h} = 0,$$
$$0 \leqslant i \leqslant N-1, n \geqslant 0.$$

It is assumed that the grid function ρ_i^{n+1} is equal to zero for $i = -1, i = N$. The matrix of the grid operator $A\rho^{n+1} = (\{\rho^{n+1}\}u)_x$ is

$$\frac{1}{2h}\begin{pmatrix} u_1 + |u_1| & u_1 - |u_1| & 0 & \dots & 0 \\ -u_1 - |u_1| & u_2 + |u_2| - u_1 + |u_1| & u_2 - |u_2| & 0 & \dots \\ 0 & -u_2 - |u_2| & u_3 + |u_3| - u_2 + |u_2| & u_3 - |u_3| & 0 \\ 0 & \dots & -u_3 - |u_3| & \ddots & 0 \\ \dots & \dots & \dots & \ddots & A_{N-2,N-1} \\ 0 & \dots & 0 & A_{N-1,N-2} & A_{N-1,N-1} \end{pmatrix}.$$

It has the following properties: the diagonal consists of non-negative numbers, out off the diagonal there are only non-positive numbers; the sum of elements in every column is equal to zero. The matrix is tridiagonal, and if for some $i : A_{i,i+1} \neq 0$, then the $A_{i+1,i} = 0$, and if the matrix has three non-zero elements in the row for some i, then the i-th column consists of only one nonzero diagonal element.

In one-dimensional case, the matrix $E + \tau A$ can be inverted by Gauss-Jordan elimination and the elements of the inverse matrix $(E + \tau A)^{-1}$ are non-negative numbers. The same conclusion can to be reached even under easier arguments that are presented below, including the case of two spatial variables.

In case of two spatial directions, the matrix $A\rho^{n+1} = (\{\rho^{n+1}\}_1 u_1)_{x_1} + (\{\rho^{n+1}\}_2 u_2)_{x_2}$ has the same properties: on the diagonal there are non-negative numbers, out off diagonal there are non-positive numbers; the sum of the elements in a column is zero.

Let's consider the matrix $B = (E + \tau A)^T$, $\dim B = M$. It has non-negative numbers on the diagonal non-positive numbers out of the diagonal, and there is a strict diagonally dominance by row. For a matrix with these properties iterative methods of Jacobi and Seidel converge with any initial condition. Let SLAE $Bx = b$ be solved by Jacobi method with initial condition $x^0 = 0$, and the right-hand side vector b consist of non-negative numbers. Let's consider the computational scheme of Jacobi method

$$x_i^{n+1} = \frac{1}{B_{i,i}} \left(b_i - \sum_{k=1,k\neq i}^{M} B_{i,k} x_k^n \right), i = 1, 2, ..., M.$$

It implies that $x_i^n \geqslant 0, i = 1, 2, ..., M, n = 1, 2, ...$ The limit vector will also be non-negative. Thus, for any non-negative vector b, vector $x = B^{-1}b$ will be non-negative. So, the matrix B^{-1} consists of non-negative numbers. Accordingly, the matrix $(E + \tau A)^{-1} = (B^{-1})^T$ exists and consists of the non-negative numbers.

The matrix A has the following property.

Lemma 1. *In one-dimensional case for any given grid function u, in two-dimensional case for any grid functions u_1, u_2 and for $\forall \tau > 0$:*

$$\left\| \left(\frac{1}{\tau} E + A \right)^{-1} \right\|_1 = \tau.$$

Moreover, the sum of the matrix elements $\left(\frac{1}{\tau} E + A \right)^{-1}$ in every column is exactly equal to τ, and the maximum element in every row is reached on the diagonal element, and this maximum is strict.

In order to prove the equality $\sum_{i=1}^{M} \left(\frac{1}{\tau} E + A \right)^{-1}_{i,j} = \frac{1}{\tau}$ it is enough to consider the auxiliary problem $\frac{1}{\tau}\rho^1 + A\rho^1 = \frac{1}{\tau}\rho^0$ with a given vector $\rho^0 = \{0, ...0, \underbrace{1}_{j}, 0, ...0\}$, $j \in [1, M]$, and then take the sum over the grid $\frac{1}{\tau} = \sum_{i=1}^{M} \rho_j^1 = \sum_{i=1}^{M} \left(\frac{1}{\tau} E + A \right)^{-1}_{i,j}$.

The mentioned properties of matrix elements can be proved by a direct calculation of inverse matrix elements, for example, using Cramer's rule. Such calculations are very simple, but rather cumbersome, so a detailed proof of the Lemma 1 is not given here, especially because the Lemma is not used for any further considerations.

Theorem 1. *A difference scheme for the equation of continuity $\rho_t + A(u^n)\rho^{n+1} = 0$ with any given grid function u^n has a unique solution, and if $\rho^0 \geqslant 0$, then $\rho^n \geqslant 0$ for $\forall n$. Difference scheme is conservative, i.e.*

$$\sum_{i=0}^{N-1} h\rho_i^n = \sum_{i=0}^{N-1} h\rho_i^0$$

in one-dimensional case and

$$\sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1} h_1h_2\rho_{ij}^n = \sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1} h_1h_2\rho_{ij}^0,$$

in case of two spatial dimensions.

From the non-negativity and conservatism we get $\|\rho^n\|_{L_{1,h}} = \text{const}$ for $\forall n$. If u^n is a given function and $\rho^0 \geqslant 0$, then the obtained equality is the condition of the weak stability of the difference scheme. However, $L_{1,h} \not\to L_1$ as $h \to 0$ because L_1 is not reflexive. Even in the one-dimensional case with a given function $u = \sin(3\pi x)$ the solution of equation $\dfrac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{u}) = 0$ will eventually be collected in two $\delta-$functions at the points $x = \pi, x = 3\pi$ with different coefficients depending on the initial conditions $\rho^0 \geqslant 0$, and in other points of the interval $[0,1]\,\rho(t,x) \to 0, t \to \infty$. The calculations with the specified difference schemes provide these results. Thus, the continuity equation should be understood in the sense of theory of distributions and generalized function: (the density of the continuous medium) $\rho(t,x) \geqslant 0$ everywhere (a.e. inappropriate) and belongs to the space of non-negative measures [1].

For the approximation of the equations of motion, we use the principle "difference against the flow", the same as in the approximation of continuity equation. Let's consider a fully implicit nonlinear difference scheme (Scheme I)

$$\begin{gathered}\rho_t + (\{\rho^{n+1}\}u^{n+1})_x = 0,\\ (\rho u)_t + (\{\rho^{n+1}u^{n+1}\}u^{n+1})_x + \frac{\gamma}{\gamma-1}\rho^{n+1}((\rho^{n+1})^{\gamma-1})_{\bar{x}} = 0,\end{gathered}$$

in one-dimensional case and

$$\begin{gathered}\rho_t + (\{\rho^{n+1}\}_1u_1^{n+1})_{x_1} + (\{\rho^{n+1}\}_2u_2^{n+1})_{x_2} = 0,\\ (\rho u_1)_t + (\{\rho^{n+1}u_1^{n+1}\}_1u_1^{n+1})_{x_1} + (\{\rho^{n+1}u_1^{n+1}\}_2u_2^{n+1})_{x_2} + \frac{\gamma}{\gamma-1}\rho^{n+1}((\rho^{n+1})^{\gamma-1})_{\bar{x}_1} = 0,\\ (\rho u_2)_t + (\{\rho^{n+1}u_2^{n+1}\}_1u_1^{n+1})_{x_1} + (\{\rho^{n+1}u_2^{n+1}\}_2u_2^{n+1})_{x_2} + \frac{\gamma}{\gamma-1}\rho^{n+1}((\rho^{n+1})^{\gamma-1})_{\bar{x}_2} = 0,\end{gathered}$$

in two-dimensional case. Operator braces $\{.\}_*$ are determined by the function u^{n+1} in one-dimensional case and by the functions u_1^{n+1}, u_2^{n+1} in two-dimensional case.

Further reasoning is not essentially different in one- and two-dimensional cases, so only one-dimensional finite-difference scheme is considered in detail.

Since the scheme is non-linear and implicit, it is required to ensure the solvability of the step problem

$$\rho^{n+1} + \tau(\{\rho^{n+1}\}u^{n+1})_x = \rho^n,$$
$$\rho^{n+1}u^{n+1} + \tau(\{\rho^{n+1}u^{n+1}\}u^{n+1})_x + \tau\frac{\gamma}{\gamma-1}\rho^{n+1}((\rho^{n+1})^{\gamma-1})_{\bar{x}} = \rho^n u^n.$$

Let's consider an iterative process, in which the operator braces $\{.\}$ are defined by the function v^k

$$\varrho^{k+1} + \tau(\{\varrho^{k+1}\}v^k)_x = \rho,$$
$$\varrho^{k+1}v^{k+1} + \tau(\{\varrho^{k+1}v^{k+1}\}v^k)_x + \tau\frac{\gamma}{\gamma-1}\varrho^{k+1}((\varrho^{k+1})^{\gamma-1})_{\bar{x}} = \rho u,$$

when $k = 0, 1, ...$ with initial condition $v^0 = u$. We assume that the initial velocity is compatible with the points of vanishing of ρ, i.e. $v^0 = u = 0$ if $\rho = 0$ at the points of the grid Ω_h.

From the first equation of the iterative process we get ϱ^1 for any already known v^0, and according to the above theorem $\varrho^1 \geqslant 0$ and $\|\varrho^1\|_{L_{1,h}} = \|\rho\|_{L_{1,h}}$.

The obtained grid function ϱ^1 has no more points of zeroing than ρ. Some points of zeroing ρ can remain points of zeroing of ϱ^1, but not necessarily.

From the second equation of the iterative process we exclude the points of zeroing of ϱ^1. Let $v^1 = 0$ at this point.

After exclusion of points of zeroing of $\varrho^1 = 0$ from the second equation of iterative process, we obtain the system of linear algebraic equations for finding v^1 with the matrix that has the following properties: on diagonal there are strictly positive numbers, off the diagonal there are non-positive numbers; there are strict diagonal dominance by the column. For such a matrix there exists an inverse, hence a grid function v^1 is uniquely determined. Then we assume that the velocity v^1 is compatible with the points of vanishing of ρ, i.e. $v^1 = 0$ if $\rho = 0$ at the points of the grid Ω_h.

So, two grid functions uniquely determined on a given grid function $v^0 = u$: first ϱ^1 and then v^1. The transition from the k-th iteration step to the $k+1$-th is carried out by the same rules used in the transition from 0 to 1.

The convergence of this iterative process is not considered. We only need to get uniform on k estimates of ϱ^k and v^k in some grid norms. Estimates that we need may depend on τ, h as the spaces of grid functions are finite-dimensional. Grid functions ϱ^k and v^k of the iterative process exist, so we can estimate them.

We multiply the first equation of the iterative process by $-\dfrac{(v^{k+1})^2}{2}$, and the second equation is multiplied by v^{k+1}. Then we add the resulting equations together and sum them up all over the grid Ω_h. We use the following equalities

$$\frac{1}{\tau}(\varrho^{k+1}v^{k+1}-\rho u)v^{k+1}=v^{k+1}\frac{1}{\tau}(\varrho^{k+1}-\rho)v^{k+1}+\rho\frac{1}{\tau}(v^{k+1}-u)v^{k+1}=$$
$$=\frac{1}{2\tau}(\varrho^{k+1}(v^{k+1})^2-\rho u^2)+\rho\frac{1}{2\tau}(v^{k+1}-u)^2+\frac{1}{2\tau}(\varrho^{k+1}-\rho)(v^{k+1})^2,$$

$$\left((\{\varrho^{k+1}v^{k+1}\}v^k)_x,v^{k+1}\right)-\left(\frac{1}{2}(\{\varrho^{k+1}\}v^k)_x,(v^{k+1})^2\right)=$$
$$=\left(\frac{1}{2}h(\{\varrho^{k+1}_{i+1}\}|v^k_{i+1}|),((v^{k+1})_x)^2\right)\geq 0.$$

The result is the summation identity

$$\sum_{i=1}^{N-1}h\frac{1}{2\tau}(\varrho^{k+1}_i(v^{k+1}_i)^2-\rho_i u_i^2)+\sum_{i=1}^{N-1}h\rho\frac{1}{2\tau}(v^{k+1}-u)^2+$$
$$+\sum_{i=1}^{N-1}h\frac{1}{2}h(\{\varrho^{k+1}\}|v^k|)((v^{k+1})_{\bar{x}})^2=-\sum_{i=1}^{N-1}h\frac{\gamma}{\gamma-1}\varrho^{k+1}_i((\varrho^{k+1}_i)^{\gamma-1})_{\bar{x}}v^{k+1}_i.$$

Since $\|\varrho^{k+1}\|_{L_{1,h}}=\|\rho\|_{L_{1,h}}$ for any k, this identity gives a uniform on k estimate $\|\varrho^{k+1}(v^{k+1})^2\|_{L_{1,h}}\leqslant C(h,\rho,u)$. Such an estimate is not sufficient to obtain an estimate $\|v^{k+1}\|\leqslant C$. To obtain this estimate, it is possible to use a term in the summation identity $\sum_{i=1}^{N-1}h\rho\frac{1}{2\tau}(v^{k+1}-u)^2$, but let there be a vanishing density $\rho_i=0, v^k_i=0$ at the point i, and in the next step of the iterative process this point becomes "massive", i.e. $\varrho^{k+1}_i>0$. It is impossible to estimate this value from below through the first equation of the iterative process

$$\frac{\varrho^{k+1}_i-\rho_i}{\tau}+\frac{-\varrho^{k+1}_{i+1}(-v^k_{i+1}+|v^k_{i+1}|)+\varrho^{k+1}_i(v^k_{i+1}+|v^k_{i+1}|-v^k_i+|v^k_i|)-\varrho^{k+1}_{i-1}(v^k_i+|v^k_i|)}{2h}=0,$$

thus $|v^{k+1}_i|$ can not be estimated from above. An iterative process was considered only to obtain summation identity.

Let's consider the case where there exist no points of zeroing ρ, i.e. at all points of grid $\rho>0$.

State the step problem as follows

$$\varrho+\tau(\{\varrho\}v)_x=\rho,$$
$$\varrho v+\tau(\{\varrho v\}v)_x+\tau\frac{\gamma}{\gamma-1}\varrho((\varrho)^{\gamma-1})_{\bar{x}}=\rho u.$$

Braces operators $\{.\}$ are defined by the unknown function v. Grid functions $\rho>0,u$ are given.

To prove the solvability of a nonlinear system of equations, we apply the following version of the principle of Leray-Schauder [2].

Let A be a compact operator in a separable normed space F, and let any possible solution $X \in F$ of the equation $X + \alpha A(X) = 0$ be uniformly bounded for $\alpha \in (0, 1]$. Then there exists at least one solution of the equation $X + A(X) = 0$.

In our case, F is the space of vectors

$$X = \begin{pmatrix} \varrho \\ v \end{pmatrix} = (\varrho_0, \varrho_1, \ldots, \varrho_{N-1}, v_0, v_1, \ldots, v_N)^T, \text{ and } v_0 = v_N = 0.$$

We define the operator $A_\varepsilon(X)$ with a parameter $\varepsilon > 0$:

$$A_\varepsilon(X) = \frac{1}{\varepsilon} \begin{pmatrix} \varrho + \tau(\{\varrho\}v)_x - \rho \\ \varrho v + \tau(\{\varrho v\}v)_x + \tau \dfrac{\gamma}{\gamma - 1} \varrho((\varrho)^{\gamma-1})_{\bar{x}} - \rho u \end{pmatrix}.$$

Then the equation $X + \alpha A_\varepsilon(X) = 0$ takes the form of a system of nonlinear difference equations

$$\frac{\varepsilon}{\alpha}\varrho + \varrho + \tau(\{\varrho\}v)_x - \rho = 0,$$
$$\frac{\varepsilon}{\alpha}v + \varrho v + \tau(\{\varrho v\}v)_x + \tau\frac{\gamma}{\gamma-1}\varrho((\varrho)^{\gamma-1})_{\bar{x}} - \rho u = 0.$$

In the definition of the grid operator $(\{\varrho\}v)_x$ we used the operation sign, which can give a discontinuous function, but $(\{\varrho\}v)_x = A(v)\varrho$, where the matrix $A(v)$ does not contain operation sign and contains but only operation $|.|$, which is a continuous function of its argument. Every convergent sequence X^k, which, due to finitedimensionality of vector space F, converges pointwise, is converted into pointwise convergent sequence $A_\varepsilon(X^k)$ by the operator of the problem. Therefore the conditions of the Leray-Schauder principle for the operator A_ε are satisfied.

Repeating the arguments of derivation of summation identity, but assuming the existence of a solution of $X + \alpha A_\varepsilon(X) = 0$, we get the inequalities

$$\left(\frac{\varepsilon}{\alpha} + 1\right)^{-1} \|\varrho\|_1 \leqslant \|\rho\|_1,$$
$$\sum_{i=1}^{N-1} h \left(\frac{\varepsilon}{\alpha}v^2 + \rho\frac{1}{2\tau}(v-u)^2\right) \leqslant C(h, \rho, u),$$

which are uniform on α. Thus, all the conditions of the Leray-Schauder principle are satisfied, and there exists at least one solution of the problem

$$\varepsilon\varrho_\varepsilon + \varrho_\varepsilon + \tau(\{\varrho_\varepsilon\}v_\varepsilon)_x - \rho = 0,$$
$$\varepsilon v_\varepsilon + \varrho_\varepsilon v_\varepsilon + \tau(\{\varrho_\varepsilon v_\varepsilon\}v_\varepsilon)_x + \tau\frac{\gamma}{\gamma-1}\varrho_\varepsilon((\varrho_\varepsilon)^{\gamma-1})_{\bar{x}} - \rho u = 0.$$

Next, we use the uniform on ε inequalities

$$(\varepsilon + 1)^{-1} \|\varrho_\varepsilon\|_1 \leqslant \|\rho\|_1,$$
$$\sum_{i=1}^{N-1} h \left(\varepsilon v_\varepsilon^2 + \rho\frac{1}{2\tau}(v_\varepsilon - u)^2\right) \leqslant C(h, \rho, u)$$

to pass to the limit as $\varepsilon \to 0$. We take convergent at all points of the grid subsequence of the uniformly bounded at all points of the grid functions $\varrho_\varepsilon, v_\varepsilon$, using a diagonal process. Passing to the limit, we obtain the solution of the step problem.

Theorem 2. *Suppose that the function ρ^0 is strictly greater than zero at the initial time. Then there exists a solution of the difference scheme*

$$\rho_t + (\{\rho^{n+1}\}u^{n+1})_x = 0,$$
$$(\rho u)_t + (\{\rho^{n+1}u^{n+1}\}u^{n+1})_x + \frac{\gamma}{\gamma-1}\rho^{n+1}((\rho^{n+1})^{\gamma-1})_{\bar{x}} = 0,$$

which satisfies the energy inequality

$$\sum_{i=0}^{N-1} h\left(\rho^{n+1}\frac{(u^{n+1})^2}{2} + \frac{1}{\gamma-1}(\rho^{n+1})^\gamma\right) \leqslant \sum_{i=0}^{N-1} h\left(\rho^n\frac{(u^n)^2}{2} + \frac{1}{\gamma-1}(\rho^n)^\gamma\right).$$

To prove this theorem, it suffices to verify the validity of the energy inequality. Once again, repeating the arguments used in the derivation of summation identity, we obtain a new summation identity

$$\sum_{i=1}^{N-1} h\frac{1}{2\tau}(\rho^{n+1}(u^{n+1})^2 - \rho(u^n)^2) + \sum_{i=1}^{N-1} h\rho\frac{1}{2\tau}(u^{n+1}-u^n)^2 +$$
$$+ \sum_{i=1}^{N-1} h\frac{1}{2}h(\{\rho^{n+1}\}|u^n|)((u^{n+1})_{\bar{x}})^2 = -\sum_{i=1}^{N-1} h\frac{\gamma}{\gamma-1}\rho^{n+1}((\rho^{n+1})^{\gamma-1})_{\bar{x}}u^{n+1}.$$

Now we take scalar product of the first equation of difference scheme by grid function $\frac{\gamma}{\gamma-1}(\rho^{n+1})^{\gamma-1}$ and add it to the new summation identity. Using equalities

$$(\{\rho^{n+1}\}u^{n+1})_x(\rho^{n+1})^{\gamma-1} = \underbrace{(\{\rho^{n+1}\}u^{n+1}\rho_i^{n+1})^{\gamma-1})_x}_{\sum_{\Omega_h}(.)=0} - \{\rho_{i+1}^{n+1}\}u_{i+1}^{n+1}((\rho_i^{n+1})^{\gamma-1})_x =$$
$$= \underbrace{(.)}_{\sum_{\Omega_h}(\cdot)=0} - (\rho_{i+1}^{n+1} - h\rho_x^{n+1}\max(0,\operatorname{sign}(u_{i+1}^{n+1})))u_{i+1}^{n+1}((\rho_i^{n+1})^{\gamma-1})_x =$$
$$= \underbrace{(.)}_{\sum_{\Omega_h}(\cdot)=0} + h\rho_x^{n+1}\max(0,|u_{i+1}^{n+1}|)((\rho^{n+1})^{\gamma-1})_x \underbrace{-(\rho_{i+1}^{n+1})u_{i+1}^{n+1}((\rho_i^{n+1})^{\gamma-1})_x}_{\text{reduced at }\sum_{\Omega_h}(\cdot)},$$

we obtain

$$\sum_{i=1}^{N-1} h\frac{1}{2\tau}(\rho^{n+1}(u^{n+1})^2 - \rho(u^n)^2) + \sum_{i=1}^{N-1} h\rho\frac{1}{2\tau}(u^{n+1}-u^n)^2 +$$
$$+ \sum_{i=0}^{N-1} h\frac{1}{\tau}(\rho^{n+1}-\rho^n)\frac{\gamma}{\gamma-1}(\rho^{n+1})^{\gamma-1} +$$
$$+ \sum_{i=1}^{N-1} h\frac{1}{2}h(\{\rho^{n+1}\}|u^n|)((u^{n+1})_{\bar{x}})^2 + \frac{\gamma}{\gamma-1}\sum_{i=1}^{N-1} h^2\rho_{\bar{x}}^{n+1}\max(0,|u^{n+1}|)))((\rho^{n+1})^{\gamma-1})_{\bar{x}} = 0.$$

The last summand is non-negative, because the function $f(\rho) = \rho^{\gamma-1}$ is monotone in ρ. In the third summand, in all points of grid we have

$$(\rho^{n+1} - \rho^n)\frac{\gamma}{\gamma-1}(\rho^{n+1})^{\gamma-1} \geqslant ((\rho^{n+1})^\gamma - (\rho^n)^\gamma)\frac{1}{\gamma-1},$$

by virtue of the Young inequality.

Using Young's inequality and removing non-negative terms in the last summation identity, we obtain energy inequality. Theorem 2 is completely proved for one-dimensional case.

For two-dimensional case Theorem 2 is also true.

A major shortcoming of Scheme I in the embodiment of non-linear implicit scheme is the restriction that ρ^0 is strictly greater than zero.

We introduce for Scheme I the following rule.

3. Rule of internal boundary conditions for Scheme I

At the grid point where $\rho_i^n = 0$ *the approximation of the equation of motion is replaced by an inner boundary condition* $u_i^{n+1} = 0$ *in one-dimensional case. In the case of two spatial dimensions, at the points* $\rho_{i,j}^n = 0$ *the approximation of the equations of motion is replaced by the internal boundary conditions* $u_{1,i,j}^{n+1} = u_{2,i,j}^{n+1} = 0$.

When deriving the summation identity in one dimensional case, the equation of motion was multiplied scalarly by u^{n+1}, the continuity equation by $-\frac{(u^{n+1})^2}{2}$, and the obtained results were summarized. In the case of using the Rule of internal boundary conditions, summation identity will be exactly the same. And the operator $A_\varepsilon(X)$, involved in the proof of the solvability of the step difference scheme with the application of the principle of Leray-Schauder will be defined on a smaller number of points $v_i, i \in \{0, 1, ..., N : \rho_i^n \neq 0\}$, but it will be continuous, and the solution $X + \alpha A_\varepsilon(X) = 0$ will be bounded uniformly in ε.

Conclusion. *Let* $\rho^n \geqslant 0$. *Then there exists a solution of the difference scheme with the internal boundary conditions*

$$\begin{aligned}
&\rho_t + (\{\rho^{n+1}\}u^{n+1})_x = 0,\\
&(\rho u)_t + (\{\rho^{n+1}u^{n+1}\}u^{n+1})_x + \frac{\gamma}{\gamma-1}\rho^{n+1}((\rho^{n+1})^{\gamma-1})_{\bar{x}} = 0, \text{ at } \rho^n \neq 0,\\
&u_i^{n+1} = 0 \text{ at } \rho_i^n = 0,
\end{aligned}$$

which satisfies the energy inequality

$$\sum_{i=0}^{N-1} h\left(\rho^{n+1}\frac{(u^{n+1})^2}{2} + \frac{1}{\gamma-1}(\rho^{n+1})^\gamma\right) \leqslant \sum_{i=0}^{N-1} h\left(\rho^n\frac{(u^n)^2}{2} - \frac{1}{\gamma-1}(\rho^n)^\gamma\right).$$

In two-dimensional case a similar result is true.

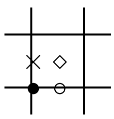
Rule of internal boundary conditions specifies how to carry out practical calculations for such schemes. If the density is too small (positive, e.g. due to the extremity of bit numbers in the computer), it is necessary to zero out the velocity. The density is calculated for all grid points. If the density is greater than zero (takes i.e. the minimum value is greater than a threshold) at some point of the grid, then at the next time step this point should be included in the set of points, where equation of motion is approximated.

For practical calculations the Scheme I in the embodiment of the linear implicit scheme

$$\rho_t + (\{\rho^{n+1}\}u^n)_x = 0,$$
$$(\rho u)_t + (\{\rho^{n+1}u^{n+1}\}u^n)_x + \frac{\gamma}{\gamma-1}\rho^{n+1}((\rho^{n+1})^{\gamma-1})_{\bar{x}} = 0$$
$$\text{(the braces operators are determined by the function } u^n).$$

has a disadvantage because it does not preserve possible symmetry of the currents available in the original problem.

Next, we consider another difference scheme, which we call the scheme II. In this scheme, the symmetry of the difference derivatives is used and grid functions ρ, u is considered on the shifted grids.



The numbering of the nodes of shifted grids in two-dimensional case:
- $\bullet$ the indices i, j of the orthogonal grid,
- $\diamond$ the indices i, j for the grid function ρ,
- $\times$ The indices i, j for the grid function u_1,
- $\circ$ the indices i, j for the grid function u_2.

For this scheme the energy inequality is true in the case of zeroing of density in the previous time step at some points. We use the following rule.

4. Rule of internal boundary conditions for Scheme II

At the grid point, where $[\rho_i^n] = 0$, the approximation of the equations of motion is replaced by an inner boundary condition $u_i^{n+1} = 0$ in one-dimensional case. In two-dimensional case, at point $[\rho_{i,j}^n]_1 = 0$ the approximation of the first equation of motion is replaced by an inner boundary condition $u_{1,i,j}^{n+1} = 0$, and at points $[\rho_{i,j}^n]_2 = 0$ the approximation of the second motion equation is replaced by an inner boundary condition $u_{2,i,j}^{n+1} = 0$.

In one-dimensional case

$$\rho_t + (\{\rho^{n+1}\}u^{n+1})_x = 0,$$
$$([\rho]u)_t + \frac{1}{2}(\{\rho^{n+1}u^{n+1}\}u^{n+1})_x + \frac{1}{2}(\{\rho^{n+1}\} < u^{n+1} > u^{n+1})_{\bar{x}} +$$
$$+ \frac{\gamma}{\gamma-1}[\rho^{n+1}]((\rho^{n+1})^{\gamma-1})_{\bar{x}} = 0, \text{ at } [\rho^n] \neq 0,$$
$$u^{n+1} = 0 \text{ if } [\rho_i^n] = 0.$$

Here we used the notation

$$< u^{n+1} > = \frac{u_{i+1}^{n+1} + u_i^{n+1}}{2} - \frac{h}{2}(u^{n+1})_x \operatorname{sign}(u_i^{n+1}) =$$
$$= u^{n+1} - h(u^{n+1})_x \max(0, \operatorname{sign}(u_i^{n+1})) = \begin{cases} u_i^{n+1}, & u_i^{n+1} \geqslant 0; \\ u_{i+1}^{n+1}, & u_i^{n+1} < 0. \end{cases}$$

Operators $\{.\}$, $< . >$ - depend on the unknown function u^{n+1}. The continuity equation is approximated at all grid points, the equation of motion is approximated only at those points where $[\rho] > 0$. At the points where $[\rho] = 0$ there is no approximation of the equations of motion, and $u^{n+1} = 0$ at such points, i.e. function u^{n+1} at these points is considered as the given one. And if such points exist, then the problem is divided into subsegments with the given boundary conditions. But the calculation of grid density function is not divided into subsegments, i.e. the calculation is carried out at all points.

In two-dimensional case

$$\rho_t + (\{\rho^{n+1}\}_1 u_1^{n+1})_{x_1} + (\{\rho^{n+1}\}_2 u_2^{n+1})_{x_2} = 0,$$
$$([\rho]_1 u_1)_t + \frac{1}{2}(\{\rho^{n+1} u_1^{n+1}\}_1 u_1^{n+1})_{x_1} + \frac{1}{2}(\{\rho^{n+1}\}_1 < u_1^{n+1} >_1 u_1^{n+1})_{\bar{x}_1} +$$
$$+([\{\rho^{n+1}\}_2 u_2^{n+1}]_1 [u_1^{n+1}]_2)_{x_2} + \frac{\gamma}{\gamma-1}[\rho^{n+1}]_1((\rho^{n+1})^{\gamma-1})_{\bar{x}_1} = 0,$$
$$([\rho]_2 u_2)_t + ([\{\rho^{n+1}\}_1 u_1^{n+1}]_2 [u_2^{n+1}]_1)_{x_1} +$$
$$\frac{1}{2}(\{\rho^{n+1} u_2^{n+1}\}_2 u_2^{n+1})_{x_2} + \frac{1}{2}(\{\rho^{n+1}\}_2 < u_2^{n+1} >_2 u_2^{n+1})_{\bar{x}_2} + \frac{\gamma}{\gamma-1}[\rho^{n+1}]_2((\rho^{n+1})^{\gamma-1})_{\bar{x}_2} = 0.$$

The first equation of motion is approximated at points where $[\rho]_1 > 0$. At the points where $[\rho]_1 = 0$, the function $u_1^{n+1} = 0$ is considered as given. The second equation of motion is approximated at points where $[\rho]_2 > 0$. At the points where $[\rho]_2 = 0$, the function $u_2^{n+1} = 0$ is considered as given. And if there is such a point, then the problem has internal boundary conditions, i.e. impermeability conditions on one of directions.

The choice of such a strange approximation in the equations of motion in divergent terms by perpendicular direction is explained below. In practical calculations, we use the corrected approximation of these terms.

In term of indices, in one-dimension case the approximation of the equation of motion has the following form

$$\frac{1}{2}\frac{(\rho_i^{n+1} + \rho_{i-1}^{n+1})u_i^{n+1} - (\rho_i^n + \rho_{i-1}^n)u_i^n}{\tau} +$$
$$\frac{1}{2}\frac{(-|v_{i+1}|\rho_{i+1} - |v_i|\rho_i + \rho_i v_i + \rho_{i+1}v_{i+1})\,u_{i+1}}{h} +$$
$$+\frac{1}{2}\frac{(|v_{i+1}|\rho_i + |v_i|\rho_i + |v_i|\rho_{i-1} + |v_{i-1}|\rho_{i-1} - v_i\rho_i + \rho_i v_{i+1} + \rho_{i-1}v_i - \rho_{i-1}v_{i-1})\,u_i}{h} +$$
$$+\frac{1}{2}\frac{(-|v_i|\rho_{i-1} - |v_{i-1}|\rho_{i-2} - \rho_{i-1}v_i - \rho_{i-2}v_{i-1})\,u_{i-1}}{h} +$$
$$+\frac{\gamma}{\gamma-1}\frac{\rho_i^{n+1} + \rho_{i-1}^{n+1}}{2}\frac{(\rho_i^{n+1})^{\gamma-1} - (\rho_{i-1}^{n+1})^{\gamma-1}}{h} = 0,$$

Here, middle line terms are grouped so that one can see the non-negativity of diagonal elements and non-positivity of other elements in the three-diagonal matrix, multiplied by u^{n+1}, upper indices omitted, which are $n+1$, the grid function $v = u^{n+1}$ introduced for clarity.

The rest of reasoning is the same as for the difference scheme I.

Since the scheme is non-linear and implicit, it is required to ensure the solvability of step problem

$$\rho^{n+1} + \tau(\{\rho^{n+1}\}u^{n+1})_x = \rho^n,$$
$$\left[\rho^{n+1}\right] u^{n+1} + \frac{\tau}{2}(\{\rho^{n+1}u^{n+1}\}u^{n+1})_x + \frac{\tau}{2}(\{\rho^{n+1}\} < u^{n+1} > u^{n+1})_{\bar{x}} +$$
$$+\tau\frac{\gamma}{\gamma-1}[\rho^{n+1}]((\rho^{n+1})^{\gamma-1})_{\bar{x}} = [\rho^n]u^n.$$

The second equation is defined only at the points where $[\rho^n] > 0$.

To get estimates for the norms of the solutions of the difference scheme we consider the iterative process

$$\varrho^{k+1} + \tau(\{\varrho^{k+1}\}v^k)_x = \rho,$$
$$\left[\varrho^{k+1}\right] v^{k+1} + \frac{\tau}{2}(\{\rho^{k+1}v^{k+1}\}v^k)_x + \frac{\tau}{2}(\{\varrho^{k+1}\} < v^{k+1} > v^k)_{\bar{x}} +$$
$$+\tau\frac{\gamma}{\gamma-1}[\varrho^{k+1}]((\varrho^{k+1})^{\gamma-1})_{\bar{x}} = [\rho]u.$$

when $k = 0, 1, ...$ with initial condition $v^0 = u$. The braces operators here are linear and depend on the known grid function v^k. Let us assume that the initial velocity is compatible with the points of zeroing $[\rho]$, i.e. $v^0 = u = 0$ if $[\rho] = 0$ at the points of the grid Ω_h, where the grid function v is defined.

In Scheme II the internal boundary conditions are known in advance, i.e. from the previous time step. At the grid point where $[\rho] = 0$ we have internal boundary condition $v^{k+1} = 0$.

We need uniform on k estimates of some grid norms of ϱ^k and v^k. These estimates may arbitrarily depend on τ, h.

We multiply the second equation in the above iterative process by v^{k+1}, apply the operation $[\cdot]$ to the first equation for all $[\rho^n] \geqslant 0$ and multiply it by $-\dfrac{(v^{k+1})^2}{2}$, and then summarize the resulting equalities over the grid Ω_h. If at some point of grid $[\rho^n] = 0$, then $v^{k+1} = 0$ is a known boundary condition. We use the following equalities

$$\frac{1}{\tau}([\varrho^{k+1}]v^{k+1} - [\rho]u)v^{k+1} = v^{k+1}\frac{1}{\tau}([\varrho^{k+1}] - [\rho])v^{k+1} + [\rho]\frac{1}{\tau}(v^{k+1} - u)v^{k+1} =$$
$$= \frac{1}{2\tau}([\varrho^{k+1}](v^{k+1})^2 - [\rho u^2]) + \underbrace{[\rho]\frac{1}{2}\tau(v^{k+1} - u)^2}_{\geqslant 0} + \frac{1}{2}\tau([\varrho^{k+1} - \rho])(v^{k+1})^2.$$

By definition of directed differences

$$\{\varrho^{k+1}v^{k+1}\} = \{\varrho^{k+1}\}\{v^{k+1}\}, \{v^{k+1}\} = \frac{v_i^{k+1} + v_{i-1}^{k+1}}{2} - \frac{h}{2}v_{\bar{x}}^{k+1}\mathrm{sign}(v^k),$$
$$< v^{k+1} >= \frac{v_{i+1}^{k+1} + v_i^{k+1}}{2} - \frac{h}{2}v_x^{k+1}\mathrm{sign}(v^k),$$
$$\{v_{i+1}^{k+1}\}v_x^{k+1} = \frac{1}{2}((v^{k+1})^2)_x - \frac{h}{2}\mathrm{sign}(v_{i+1}^k)(v_x^{k+1})^2,$$
$$< v_{i-1}^{k+1} > v_{\bar{x}}^{k+1} = \frac{1}{2}((v^{k+1})^2)_{\bar{x}} - \frac{h}{2}\mathrm{sign}(v_{i-1}^k)(v_{\bar{x}}^{k+1})^2.$$

Using the rules of difference differentiation we get

$$\underbrace{\left(\frac{1}{2}(\{\varrho^{k+1}v^{k+1}\}v^k)_x+\frac{1}{2}(\{\varrho^{k+1}\}<v^{k+1}>v^k)_{\bar{x}}\right)}_{\text{from the equation of motion}}v^{k+1}-$$

$$-\underbrace{\frac{1}{2}[(\{\varrho^{k+1}\}v^k)_x]}_{\text{from the equation of continuity}}(v^{k+1})^2=$$

$$=\underbrace{\frac{1}{2}((\{\varrho^{k+1}\}\{v^{k+1}\}v^{k+1}v^k+\{\varrho^{k+1}\}<v^{k+1}>v^{k+1}v^k-[\{\varrho^{k+1}\}v^k](v^{k+1})^2))_x}_{\sum_{\Omega_h}(\cdot)=0}+$$

$$+\underbrace{\frac{1}{4}h(\{\varrho^{k+1}_{i+1}\}|v^k_{i+1}|)((v^{k+1})_x)^2+\frac{1}{4}h(\{\varrho^{k+1}_{i-1}\}|v^k_{i-1}|)((v^{k+1})_{\bar{x}})^2}_{\geqslant 0}-$$

$$\underbrace{-\frac{1}{4}(\{\varrho^{k+1}_{i-1}\}v^k_{i-1})((v^{k+1})^2)_{\bar{x}}+\frac{1}{4}(\{\varrho^{k+1}_i\}v^k_i)((v^{k+1})^2)_x}_{\sum_{\Omega_h}(\cdot)=0}.$$

So, multiplying the difference equation of continuity by $-\dfrac{(v^{k+1})^2}{2}$, the difference equation of motion by v^{k+1} and summing over the grid, we obtain the summation identity

$$\sum_{i=1}^{N-1}h[\rho]\frac{1}{2\tau}(v^{k+1}-u)^2+\sum_{i=1}^{N-1}h\frac{1}{2\tau}([\varrho^{k+1}_i](v^{k+1}_i)^2-[\rho_i]u_i^2)+$$
$$+\sum_{i=1}^{N-1}h\frac{1}{4}h(\{\varrho^{k+1}_{i+1}\}|v^k_{i+1}|)((v^{k+1})_x)^2+\frac{1}{4}h(\{\varrho^{k+1}_{i-1}\}|v^k_{i-1}|)(v^{k+1})_{\bar{x}}=$$
$$=-\sum_{i=1}^{N-1}h\frac{\gamma}{\gamma-1}[\varrho^{k+1}_i]((\varrho^{k+1}_i)^{\gamma-1})_{\bar{x}}v^{k+1}_i.$$

Since $\|\varrho^{k+1}\|_{L_{1,h}}=\|\rho\|_{L_{1,h}}$ for any k, from the summation identities we get uniform on k estimate

$$\|[\varrho^{k+1}](v^{k+1})^2\|_{L_{1,h}}\leqslant C(h,\rho,u).$$

Now let's explain the choice of approximation of divergence in two-dimensional case

$$\frac{\partial(\rho v_1 u_2)}{\partial x_2}\sim([\{\rho^{n+1}\}_2u_2^{n+1}]_1[v_1^{n+1}]_2)_{x_2}.$$

The equation in the iterative process corresponding to the first component of the velocity is multiplied by v_1^{k+1}, and the operation $[\cdot]_1$ is applied to the first equation for all $[\rho^n]_1\geqslant 0$ which is then is multiplied by $-\dfrac{(v_1^{k+1})^2}{2}$. Then the resulting equalities are summed over the grid Ω_h. Let $u_2 = v_2^k, v_1 = v_1^{k+1}$.

$$\overbrace{-[(\{\varrho\}_2 u_2)_{x_2}]_1}^{\text{from the equation of continuity}} \cdot \frac{v_1^2}{2} = \underbrace{-([(\{\varrho\}_2 u_2)]_1 \frac{v_1^2}{2})_{x_2}}_{\sum_{\Omega_h}(\cdot)=0} + ([\{\varrho_{j+1}\}_2 u_{2,j+1}]_1)\left(\frac{v_1^2}{2}\right)_{x_2} =$$

$$= \underbrace{(.)}_{\sum_{\Omega_h}(\cdot)=0} + ([\{\varrho_{j+1}\}_2 u_{2,j+1}] - 1)[v_{1,j+1}]_2 v_{1x_2} = \underbrace{(.)}_{\sum_{\Omega_h}(\cdot)=0} + \underbrace{([\{\varrho\}_2 u_2]_1 [v_1]_2 v_1)_{x_2}}_{\sum_{\Omega_h}(\cdot)=0} -$$

$$- \underbrace{([\{\varrho\}_2 u_2]_1 [v_1]_2)_{x_2}}_{\text{from the 1st movement equation}} \cdot v_1.$$

Thus, in the summation identity the corresponding scalar products mutually vanish. Note that in this approximation there is no sign function , since $\{\varrho\}_2 u_2 = [\varrho]_2 u_2 - \frac{h_2}{2}\varrho_{\bar{x}_2} \cdot |u_2|$, and this approximation is a continuous operator of grid functions. However, for practical calculations this approximation is unsuitable, because tridiagonal matrix, which is multiplied by an unknown function $(v_{1,i,j-1}, v_{1,i,j}, v_{1,i,j+1})$, not necessarily has non-negative diagonal element, and non-positive other elements. This approximation can be corrected by the same rule, which was used in the approximation of continuity equation, i.e. we use the difference against the flow: $([\{\varrho\}_2 u_2]_1([v_1]_2 - \frac{h_2}{2}(v_1)_{\bar{x}_2}\text{sign}([\{\varrho\}_2 u_2]_1)))_{x_2} = (\{v_1\}_2[\{\varrho\}_2 u_2]_1)_{x_2} = (\{v_1\}_2 w_2)_{x_2}$. This approximation may be a two-point or single-point and does not contain the function sign. In terms of indices, this approximation has the following form

$$\frac{1}{2h_2}\left(-|w_{i,j+1}| + w_{i,j+1}\right) v_{1i,j+1} + \frac{1}{2h_2}\left(w_{i,j+1} + |w_{i,j+1}| - w_{i,j} - |w_{i,j}|\right) v_{1i,j} +$$

$$+ \frac{1}{2h_2}\left(-w_{i,j} + |w_{i,j}|\right) v_{1i,j-1},$$

where $w_{i,j} = [\{\varrho\}_2 u_2]_1$ is a known function, which determines the matrix of the problem for v_1.

Back to one-dimensional problem, from the first term of summation identity we get the estimate $\|(v^{k+1})^2\|_{L_{1,h}} \leqslant C$, since for the points where $[\rho] > 0$ there is a lower estimate for $[\rho] \geqslant \min\limits_{[\rho]>0}[\rho] = \text{const} > 0$, and at the points where $[\rho] = 0$ the function $v^{k+1} = 0$ is a known internal boundary condition. Next, we consider the problem in the operator form $A\begin{pmatrix} \varrho \\ v \end{pmatrix} = 0$:

$$A\begin{pmatrix} \varrho \\ v \end{pmatrix} = \begin{pmatrix} \varrho + \tau(\{\varrho\}v)_x - \rho, \\ [\varrho]\, v + \frac{\tau}{2}(\{\rho v\}v)_x + \frac{\tau}{2}(\{\varrho\} < v > v)_{\bar{x}} + \tau\frac{\gamma}{\gamma - 1}[\varrho]((\varrho)^{\gamma-1})_{\bar{x}} - [\rho]u \end{pmatrix}.$$

We pass from this problem to the one with $\varepsilon > 0$ in the following form: $X + A_\varepsilon(X) = 0, A_\varepsilon = \frac{1}{\varepsilon}A$. We apply to this problem the principle of Leray-Schauder. The operator A_ε

is continuous, as there is no sign function in the definitions of approximations for the first and second equations and the function $\mathrm{abs}(a) = |a|$ is continuous. Then we repeat the same arguments as in the derivation of estimates for the norms in the iterative process, and it turns out that the solution of the problem $X + \alpha A_\varepsilon(X) = 0$ with a completely continuous (due to the finite dimensionality of the grid) operator A_ε and any $\alpha \in (0, 1]$ has bounded norms irrespective of α. Therefore the solution of the problem with $\alpha = 1$ exists for $\forall\varepsilon$. Then we pass to the limit as $\varepsilon \to 0$ in the problem of $\varepsilon X + A(X) = 0$. Since the estimates $\|\varrho\|_{L_{1,h}} \leqslant \|\rho\|_{L_{1,h}}$, $\|(v)^2\|_{L_{1,h}} \leqslant C$ are independent of ε, the limit functions exist and are denoted by ρ^{n+1}, u^{n+1}.

Theorem 3. *Let $\rho^0 \geqslant 0$. The solution of the difference scheme II with internal boundary conditions exists and satisfies the energy inequality. In one-dimensional case the energy inequality takes the form*

$$\sum_{i=0}^{N-1} h\frac{1}{\gamma-1}((\rho^{n+1})^\gamma - (\rho^n)^\gamma) + \sum_{i=1}^{N-1} h\frac{1}{2}([\rho^{n+1}](u^{n+1})^2 - [\rho^n](u^n)^2) \leqslant 0.$$

To prove the theorem, it suffices to show the validity of energy inequality. We scalarly multiply the first equation in the difference scheme by $\dfrac{\gamma}{\gamma-1}(\rho^{n+1})^{\gamma-1}$ and add it to the previously obtained summation identity. In one-dimensional case we obtain the inequality

$$\sum_{i=0}^{N-1} h\rho_t^{n+1}\frac{\gamma}{\gamma-1}(\rho^{n+1})^{\gamma-1} + \sum_{i=1}^{N-1} h\frac{1}{2\tau}([\rho_i^{n+1}](u^{n+1})^2 - [\rho_i](u^n)^2) +$$
$$+ \sum_{i=0}^{N-1} h(\{\rho^{n+1}\}u^{n+1})_x\frac{\gamma}{\gamma-1}(\rho^{n+1})^{\gamma-1} + \sum_{i=1}^{N-1} h\frac{\gamma}{\gamma-1}[\rho^{n+1}]((\rho^{n+1})^{\gamma-1})_{\bar{x}}u^{n+1} \leqslant 0.$$

We apply Young's inequality to the first term as in the case of Scheme I, and for the last two terms we have

$$\sum_{i=0}^{N-1} h(\{\rho^{n+1}\}u^{n+1})_x\frac{\gamma}{\gamma-1}(\rho^{n+1})^{\gamma-1} + \sum_{i=1}^{N-1} h\frac{\gamma}{\gamma-1}[\rho^{n+1}]((\rho^{n+1})^{\gamma-1})_{\bar{x}}u^{n+1} =$$
$$= \frac{1}{2}\sum_{i=1}^{N-1} h(|u^{n+1}|(\rho^{n+1})_x\frac{\gamma}{\gamma-1}((\rho^{n+1})^{\gamma-1})_x \geqslant 0,$$

due to the monotonicity of the function $f(\rho) = \rho^{\gamma-1}$.

For the case of two spatial variables Theorem 3 is generalized in an obvious way. Energy inequality takes on the form

$$\sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1} h_1h_2\frac{1}{\gamma-1}((\rho^{n+1})^\gamma - (\rho^n)^\gamma) + \sum_{i=1}^{N_1-1}\sum_{j=0}^{N_2-1} h_1h_2\frac{1}{2}([\rho^{n+1}]_1(u_1^{n+1})^2 - [\rho^n]_1(u_1^n)^2) +$$
$$+ \sum_{i=0}^{N_1-1}\sum_{j=1}^{N_2-1} h_1h_2\frac{1}{2}([\rho^{n+1}]_2(u_2^{n+1})^2 - [\rho^n]_2(u_2^n)^2) \leqslant 0.$$

All results obtained for the difference Scheme II can be extended to non-orthogonal curvilinear grids, as well as to the problems on the grids in spherical and other coordinates.

By results we mean: conservative approximation of continuity equation, non-negativity of density, the presence of energy inequality, the ability to solve problems with vanishing density by zeroing the velocity in such cases.

In practical calculations, it is impossible to use non-linear implicit scheme. Therefore, linear implicit scheme is used in most cases. Availability of energy inequality allows to control stability of calculations by proper selection of time step τ in difference scheme. In addition, as it was seen in the derivation of energy inequality, it is obvious that due to the choice of approximation of the divergence operator in the equations of motion we can increase or decrease the influence of the grid viscosity. Thus, our theoretical do have practical importance.

Remark 1. *In this work, we do not consider a difference schemes for the equation of motion with viscous members and other skew-symmetric operators and nonzero external forces*

$$\frac{\partial \rho \boldsymbol{u}}{\partial t} + \operatorname{div}(\rho \boldsymbol{u} \times \boldsymbol{u}) + S(\mathbf{u}) + N(\mathbf{u}) + \operatorname{grad} p = \rho \mathbf{f}.$$

The construction of such schemes seems obvious enough.

Remark 2. *A more interesting problem is the formulation of boundary conditions for problems with areas of inflow and leakage. Matrix form of finite difference schemes gives advice on how to supply the boundary conditions to fulfil the conditions of conservatism, the nonnegativity of the density, and the energy inequality. We plan to discuss these questions in a separate article.*

5. The problems of calculation of large-scale sea currents using shallow water model

The system of equations of the shallow water theory [3] is very similar to barotropic gas equations with $\rho = h$, $p = \dfrac{g\,h^2}{2}$:

$$\frac{\partial}{\partial t} h + \frac{\partial}{\partial x}(h\,u) + \frac{\partial}{\partial y}(h\,v) = 0,$$

$$\frac{\partial}{\partial t}(h\,u) + \frac{\partial}{\partial x}(h\,u^2 + \frac{g\,h^2}{2}) + \frac{\partial}{\partial y}(h\,u\,v) = f_1 - g\,h\,\frac{\partial}{\partial x} b,$$

$$\frac{\partial}{\partial t}(h\,v) + \frac{\partial}{\partial x}(h\,u\,v) + \frac{\partial}{\partial y}(h\,v^2 + \frac{g\,h^2}{2}) = f_2 - g\,h\,\frac{\partial}{\partial y} b,$$

where h is a water depth (non-negative), $h + b$ is a level of the free surface, b is a bottom relief function, which, in this system of equations, may be defined up to an arbitrary additive constant;

u, v are the components of the velocity vector;

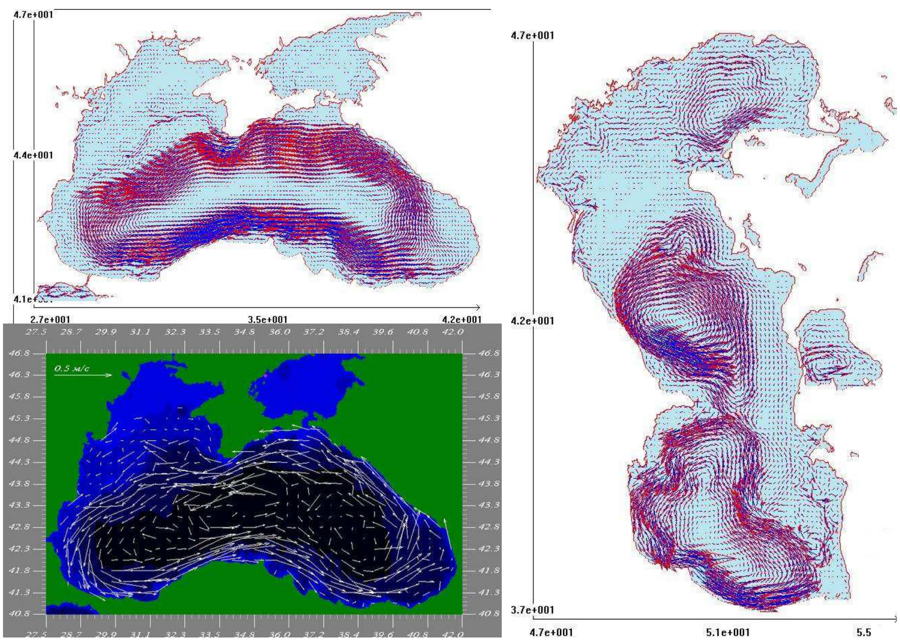
g is an acceleration due to gravity;

f_1, f_2 are the functions of external forces (Coriolis force, friction, wind). Quadratic friction is given by $\mathbf{f} = -\dfrac{\lambda\,\mathbf{v}\,|\mathbf{v}|}{2}$ (here $\mathbf{f} = (f_1, f_2)$, $\mathbf{v} = (u, v)$ vectors), and λ is the coefficient of hydraulic friction.

The strength and direction of the wind vector are set as dependent on atmospheric pressure and other parameters. The Coriolis force is given as the function of components of velocity and determines a skew-symmetric operator.

In the problem of shallow water, the vanishing of the depth of water is possible, i.e. the function h may be equal to zero. First, it is possible at the boundaries of the reservoir where wind and surge currents are calculated, and second, it is possible within the computational domain. For example, suppose that the profile of the bottom has some camber close to the water surface. Then as a result of level fluctuations of shallow water over the surface there may appear an island, and it may disappear and reappear. In this case, in accordance with the rule of the Scheme II, there are internal boundary conditions "no flow through the island". Note also that the equations of motion in the Scheme II can be divided into vanishing functions h, because the equations of motion are replaced by the internal boundary conditions at the points where $h = 0$. Therefore, the coefficient of hydraulic friction, for example, $\lambda = 2\,g\,n^2\,h^{\left(-\frac{1}{3}\right)}$, n is a bottom roughness coefficient by Manning ($n = .7e\text{-}2$), is defined for all points where the grid velocity is computed. But this ratio is acceptable in the case of very shallow water. In our calculations the coefficient of hydraulic friction was chosen in different form, i.e. it was defined experimentally.

The pictures below show the results of calculations for two tasks by Scheme II. The calculation areas are located on the ellipsoid (more precisely, on the geoid), the system of equations was written in the corresponding coordinate system, the vertical component of the Coriolis force is not taken into account. Other parameters were chosen from Earth's data: the acceleration due to gravity, the speed of rotation of the Earth, shape of Earth's surface and the areas of location in latitude and longitude of the Earth.



Calculations were performed on a linear implicit scheme similar to the iterative process for Scheme II. To calculate the grid function h on the next time step we used the factorization of the transition operator, which ensures the equivalence of directions. For the equations of motion we also built a scheme that allow the efficient parallelization of computations. The water depth in both seas was selected according to ETOPO1 grid with increments of $1'$.

The rule of internal boundary conditions was applied with a threshold of depth 1 mm. For the values of the grid function h (averaged for the corresponding direction) less than 1 mm, the velocity was taken equal to zero.

We chose the initial conditions $u = v = 0$, $h + b = \text{const}$, which satisfy the equations of the stationary problem. However, the corresponding solution is unstable. In this initial state we introduced a very few distortion and then we tracked the entire path of transition from one stationary solution (unstable) to another (steady) through a non stationary path that is described by a system of shallow water equations.

In the first problem, the obtained stationary solution was compared to the experimental data of the currents in the Black Sea [4]. In the second problem, related to the Caspian Sea, the experimental data are not available or insufficient. However, the calculation results of first problem are close to the experimental data, and this gives hope that the obtained pictures of the currents in the Caspian Sea are proper.

Calculations were performed on computers with OS Windows7 using our own programs on C-language. We used the gcc compiler in the shell Dev-Cpp. Parallelization of calculations was performed on multithread technology using pthreads.2 package. The computational time on the grids with step $1'$ on a computer with a Quad-core i7-2630 2.00 GHz was 8-10 hours, and the optimal number of threads based on the configuration of areas was found to be 20. On the sparse grid in increments of $2'$ calculation time does not exceed 1 hour. For even more sparse grid in increments of $3'$ computation time is only a few minutes. The results of calculations on grids with different steps are almost identical.

References

[1] V.S. Vladimirov, *Generalized functions in mathematical physics*, Second edition, revised and enlarged, Moscow, Nauka, 1979.

[2] O.A. Ladyzhenskaya, *Mathematical questions of dynamics of viscous incompressible fluids*, Second edition, revised and enlarged, Moscow, Nauka, 1970.

[3] A.G. Kulikovskii, N.In. Pogorelov, A.Yu. Semenov, *Mathematical problems of numerical solution of hyperbolic systems of equations*, Moscow, Nauka, 2001.

[4] O.P. Nikitin, S.Yu. Kas'yanov, *Atlas of surface currents of the black sea from drifter data*, Moscow, State Oceanographic Institute of Roshydromet, 2008.

[5] P.L. Lions, *Mathematical Topics in Fluid Mechanics*, v. 2, Compressible Models. Oxford Lecture Series in Mathematics and its Applications 10, Oxford, Clarendon Press, 1998.

[6] A.A. Samarskii, *The Theory of Difference Schemes*, Marcel Dekker, Inc., New York Basel, 2001.

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