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Fixed Point Theorems and an Application in Parametric Metric Spaces

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Abstract. In this paper, we give concepts of coupled fixed and coupled coincidence point in parametric metric spaces. We also prove a coupled fixed point theorem in this space and give a corollary and an example about the main result. Finally, we give an application to homotopy with proof.

Key Words and Phrases: coupled fixed point, parametric metric space, coupled coincidence.
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1. Introduction

Fixed point theory with all its applications have been a popular topic in the science world. It is used in some areas such as nonlinear analysis, engineering, mathematical economics, mathematical biology, functional analysis, etc. The concept of a metric space was introduced by Frechet [10]. Then many mathematicians have studied fixed points of contractive mappings. After the introduction of Banach contraction principle, the study of existence and uniqueness of fixed points and common fixed points have been a major area of interest.

The notion of coupled fixed point was first given in [4]. Bhaskar and Lakshmikantham [4] proved the existence of a coupled fixed point for F under a weak contractivity condition, established the uniqueness under an additional assumption on the metric space and showed that the components of the coupled fixed point are equal. In [12], coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings were proved in partially ordered complete metric spaces. Mustafa et al. [13] proved some coupled coincidence fixed point theorems for nonlinear (ψ, φ)-weakly contractive mappings in partially ordered G_b-metric spaces. For other works related to coupled fixed point theorems in metric spaces, see [1, 2, 3, 6, 7, 8, 9, 14, 15, 16, 18, 19, 20, 21, 22].

The concept of parametric metric space was first given in [11], where some fixed point theorems in complete parametric metric spaces were proved. Rao et al. [17] introduced parametric s-metric space and proved two unique common fixed point theorems in this space.

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In this study, our aim is to prove a coupled fixed point theorem in parametric metric spaces. For this purpose, we give new definitions and a lemma with the proof. Then, we present a corollary and an example related to the main result. Moreover, we prove a theorem and a homotopy result.

2. Preliminaries

In this section, we give some definitions which are useful for main result in this paper.

Definition 1. *[11]. Let X be a nonempty set and $p : X \times X \times (0, \infty) \to [0, \infty)$ be a mapping. We say that p is a parametric metric on X if,*

(1) $p(x, y, t) = 0$ for all $t > 0$ if and only if $x = y$;

(2) $p(x, y, t) = p(y, x, t)$ for all $t > 0$;

(3) $p(x, y, t) \leq p(x, z, t) + p(z, y, t)$ for all $x, y, z \in X$ and all $t > 0$,

and the pair (X, p) is called a parametric metric space.

Example 1. *[11]. Let X denote the set of all functions $f : (0, \infty) \to \mathbb{R}$. Define $p : X \times X \times (0, \infty) \to [0, \infty)$ by $p(f, g, t) = |f(t) - g(t)|$ for all $f, g \in X$ and all $t > 0$. Then (X, p) is a parametric metric space.*

Definition 2. *[11]. Let (X, p) be a parametric metric space, and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is called the limit of the sequence $\{x_n\}$, if $\lim_{n\to\infty} p(x, x_n, t) = 0$ for all $t > 0$, and the sequence $\{x_n\}$ is said to be convergent to x, denoted by $x_n \to x$ as $n \to \infty$.*

We remark that if (X, p) is a parametric metric space and $\{x_n\}, \{y_n\}$ are two sequences in X such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $\lim_{m,n\to\infty} p(x_n, y_m, t) = p(x, y, t)$ for all $t > 0$, that is, p is continuous in its two variables.

Definition 3. *[11]. Let (X, p) be a parametric metric space.*
- *A sequence $\{x_n\}$ is said to be a Cauchy if and only if $\lim_{m,n\to\infty} p(x_n, y_m, t) = 0$ for all $t > 0$.*
- *A parametric metric space (X, p) is called complete if and only if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$.*
- *Let $x \in X$ and $r > 0$. The set*

$$B(x, r) = \{y \in X : p(x, y, t) < r, \ \forall t > 0\}$$

is called an open ball with center at x and radius $r > 0$.

Definition 4. *[11]. Let (X, p) be a parametric metric space and let $T : X \to X$ be a mapping. If for any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, $Tx_n \to Tx$ as $n \to \infty$, then T is a continuous mapping at $x \in X$.*

3. Main Results

In this section, we first give some definitions on coupled and coincidence fixed point in parametric metric spaces.

Definition 5. *An element* $(x, y) \in X \times X$ *is called a coupled fixed point of a mapping* $F : X \times X \to X$ *if* $F(x, y) = x$ *and* $F(y, x) = y$.

Definition 6. *An element* $(x, y) \in X \times X$ *is called a coupled coincidence point of the mappings* $F : X \times X \to X$ *and* $g : X \to X$ *if* $F(x, y) = gx$ *and* $F(y, x) = gy$.

Definition 7. *Let* X *be a nonempty set. The maps* F *and* g *are said to be commutative if* $gF(x, y) = F(gx, gy)$.

Lemma 1. *Let* (X, p) *be a parametric metric space. Assume that* $F : X \times X \to X$ *and* $g : X \to X$ *satisfy the following condition:*

$$p(F(x, y), F(a, b), t) \leq \alpha[p(gx, ga, t) + p(gy, gb, t)] \tag{1}$$

for all $x, y, a, b \in X$ *and* $t > 0$. *Let* (x, y) *be a coupled coincidence point of the mappings* F *and* g. *If* $\alpha \in [0, \frac{1}{2})$, *then there are following equalities:*

$$F(x, y) = gx = gy = F(y, x).$$

Proof. From the definition of a coupled coincidence point, we have $F(x, y) = gx$ and $F(y, x) = gy$ for F and g. Assume that $gx \neq gy$. From (1) and the axiom (2) of the definition of parametric metric space, we get

$$\begin{aligned} p(gx, gy, t) = p(F(x, y), F(y, x), t) &\leq \alpha[p(gx, gy, t) + p(gy, gx, t)] \\ &= \alpha[p(gx, gy, t) + p(gx, gy, t)] \\ &= 2\alpha p(gx, gy, t) \\ &< p(gx, gy, t), \end{aligned}$$

which is a contradiction. Thus $gx = gy$ and so we obtain

$$F(x, y) = gx = gy = F(y, x).$$

◄

Theorem 1. *Let* (X, p) *be a parametric metric space. Let* $F : X \times X \to X$ *and* $g : X \to X$ *be two maps such that*

$$p(F(x, y), F(a, b), t) \leq \alpha[p(gx, ga, t) + p(gy, gb, t)] \tag{2}$$

for all $x, y, a, b \in X$ *and* $t > 0$. *Suppose that* F *and* g *satisfy the four conditions:*

(i) $F(X \times X) \subseteq g(X)$,

(ii) $g(X)$ *is a complete parametric metric space,*

(iii) g *is continuous,*

(iv) F *and* g *are commutative.*

If $\alpha \in (0, \frac{1}{2})$, *then there is a unique* x *in* X *such that* $gx = F(x,x) = x$.

Proof. Let $x_0, y_0 \in X$. From (i), we could choose two points $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we again can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. If we continue like this, we construct two sequences (x_n) and (y_n) in X such that

$$gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n).$$

By (2), we obtain

$$p(gx_n, gx_{n+1}, t) = p(F(x_{n-1}, y_{n-1}), F(x_n, y_n), t) \leq \alpha[p(gx_{n-1}, gx_n, t) + p(gy_{n-1}, gy_n, t)]$$

for $n \in \mathbb{N}$ and all $t > 0$. The inequalities

$$\begin{aligned} p(gx_{n-1}, gx_n, t) &= p(F(x_{n-2}, y_{n-2}), F(x_{n-1}, y_{n-1}), t) \\ &\leq \alpha[p(gx_{n-2}, gx_{n-1}, t) + p(gy_{n-2}, gy_{n-1}, t)] \end{aligned}$$

and

$$\begin{aligned} p(gy_{n-1}, gy_n, t) &= p(F(y_{n-2}, x_{n-2}), F(y_{n-1}, x_{n-1}), t) \\ &\leq \alpha[p(gy_{n-2}, gy_{n-1}, t) + p(gx_{n-2}, gx_{n-1}, t)] \end{aligned}$$

give that

$$p(gx_{n-1}, gx_n, t) + p(gy_{n-1}, gy_n, t) \leq 2\alpha[p(gx_{n-2}, gx_{n-1}, t) + p(gy_{n-2}, gy_{n-1}, t)]$$

holds for all $n \in \mathbb{N}$ and $t > 0$. Thus we get

$$\begin{aligned} p(gx_n, gx_{n+1}, t) &\leq \alpha[p(gx_{n-1}, gx_n, t) + p(gy_{n-1}, gy_n, t)] \\ &\leq 2\alpha^2[p(gx_{n-2}, gx_{n-1}, t) + p(gy_{n-2}, gy_{n-1}, t)] \\ &\vdots \\ &\leq \frac{1}{2}(2\alpha)^n[p(gx_0, gx_1, t) + p(gy_0, gy_1, t)]. \end{aligned}$$

As a result, we have

$$p(gx_n, gx_{n+1}, t) \leq \frac{1}{2}(2\alpha)^n[p(gx_0, gx_1, t) + p(gy_0, gy_1, t)] \tag{3}$$

for every $n \in \mathbb{N}$ and $t > 0$. Let $m, n \in \mathbb{N}$ with $m > n$. From the axiom (3) of the definition of parametric metric space, we have

$$p(gx_n, gx_m, t) \leq p(gx_n, gx_{n+1}, t) + p(gx_{n+1}, gx_{n+2}, t) + \ldots + p(gx_{m-1}, gx_m, t).$$

From the inequality (3), we get

$$\begin{aligned} p(gx_n, gx_m, t) &\leq \frac{1}{2}\left(\sum_{i=n}^{m-1}(2\alpha)^i\right)[p(gx_0, gx_1, t) + p(gy_0, gy_1, t)] \\ &\leq \frac{(2\alpha)^n}{2(1-2\alpha)}[p(gx_0, gx_1, t) + p(gy_0, gy_1, t)], \end{aligned}$$

because $1 - 2\alpha > 0$. If we take limits as $m, n \to \infty$, we get

$$\lim_{m,n\to\infty} p(gx_n, gx_m, t) = 0,$$

so (gx_n) is a Cauchy sequence in $g(X)$. Similarly, it can be shown that (gy_n) is a Cauchy sequence in $g(X)$. Since $g(X)$ is a complete parametric metric space, (gx_n) and (gy_n) are convergent to $x \in X$ and $y \in X$, respectively. By condition (iii), we get (ggx_n) is convergent to gx and (ggy_n) is convergent to gy. Moreover, from (iv), there are

$$ggx_{n+1} = g(F(x_n, y_n)) = F(gx_n, gy_n) \quad \text{and} \quad ggy_{n+1} = g(F(y_n, x_n)) = F(gy_n, gx_n).$$

Therefore from (iii), we have

$$\begin{aligned} gx = g(\lim_{n\to\infty} gx_n) &= \lim_{n\to\infty} g(F(x_{n-1}, y_{n-1})) \\ &= \lim_{n\to\infty} F(gx_{n-1}, gy_{n-1}) \\ &= F(\lim_{n\to\infty} gx_{n-1}, \lim_{n\to\infty} gy_{n-1}) \\ &= F(x, y) \end{aligned}$$

and

$$\begin{aligned} gy = g(\lim_{n\to\infty} gy_n) &= \lim_{n\to\infty} g(F(y_{n-1}, x_{n-1})) \\ &= \lim_{n\to\infty} F(gy_{n-1}, gx_{n-1}) \\ &= F(\lim_{n\to\infty} gy_{n-1}, \lim_{n\to\infty} gx_{n-1}) \\ &= F(y, x). \end{aligned}$$

Thus

$$\begin{aligned} p(ggx_{n+1}, F(x, y), t) &= p(F(gx_n, gy_n), F(x, y), t) \\ &\leq \alpha[p(ggx_n, gx, t) + p(ggy_n, gy, t)]. \end{aligned}$$

From Lemma 1, (x, y) is a coupled fixed point of the mappings F and g, and so

$$gx = F(x, y) = F(y, x) = gy.$$

From the fact that (gx_{n+1}) is a subsequence of (gx_n), (gx_{n+1}) is convergent to x and thus

$$\begin{aligned} p(gx_{n+1}, gx, t) &= p(F(x_n, y_n), F(x, y), t) \\ &\leq \alpha[p(gx_n, gx, t) + p(gy_n, gy, t)]. \end{aligned}$$

Letting $n \to \infty$ and using the fact that p is continuous, we have

$$p(x, gx, t) \leq \alpha[p(x, gx, t) + p(y, gy, t)]. \tag{4}$$

In a similar way, it can be shown that

$$p(y, gy, t) \leq \alpha[p(x, gx, t) + p(y, gy, t)]. \tag{5}$$

From (4) and (5), we obtain

$$p(x, gx, t) + p(y, gy, t) \leq 2\alpha[p(x, gx, t) + p(y, gy, t)]. \tag{6}$$

Inequality (6) happens only if $p(x, gx, t) = 0$ and $p(y, gy, t) = 0$ as $2\alpha < 1$.

On the other hand, we get

$$\begin{aligned} p(gx, gx_{n+1}, t) &= p(F(x, y), F(x_n, y_n), t) \\ &\leq \alpha[p(gx, gx_n, t) + p(gy, gy_n, t)]. \end{aligned}$$

If we take limit as $n \to \infty$ and consider the continuity of p, we have

$$p(gx, x, t) \leq \alpha[p(gx, x, t) + p(gy, y, t)]. \tag{7}$$

Similarly, we can show the validity of the following inequality:

$$p(gy, y, t) \leq \alpha[p(gx, x, t) + p(gy, y, t)]. \tag{8}$$

By (10) and (11), we get

$$p(gx, x, t) + p(gy, y, t) \leq 2\alpha[p(gx, x, t) + p(gy, y, t)]. \tag{9}$$

Since $2\alpha < 1$, we conclude that (12) happens only when $p(x, gx, t) = 0$ and $p(y, gy, t) = 0$. Thus we have $x = gx$ and $y = gy$, that is,

$$gx = F(x, x) = x.$$

Let's prove the uniqueness. We take $z \in X$ with $z \neq x$ such that

$$z = gz = F(z, z).$$

So

$$\begin{aligned} p(x, z, t) = p(F(x, x), F(z, z), t) &\leq \alpha[p(gx, gz, t) + p(gx, gz, t)] \\ &= 2\alpha p(gx, gz, t) \\ &= 2\alpha p(x, z, t) \\ &< p(x, z, t), \end{aligned}$$

which is a contradiction because $2\alpha < 1$. Thus there is a unique common fixed point of the maps F and g. ◄

Corollary 1. *Let (X,p) be a complete parametric metric space and $F : X \times X \to X$ be a continuous map such that*

$$p(F(x,y), F(a,b), t) \leq \alpha[p(x,a,t) + p(y,b,t)]$$

for all $x, y, a, b \in X$ and $t > 0$. If $\alpha \in [0, \frac{1}{2})$, then there is a unique x in X such that $F(x,x) = x$.

Proof. Consider a mapping $g : X \to X$ defined by $gx = x$. It is clear that F and g satisfy all hypotheses of Theorem 1. As a result, we get the validity of corollary. ◄

Example 2. *Consider $X = [0,\infty)$. Let $p : X \times X \times (0,\infty) \to X$ be defined by*

$$p(x,y,t) = \frac{|x-y|}{t}$$

for all $x, y \in X$ and all $t > 0$. It can be easily shown that (X,p) is a complete parametric metric space.

Suppose that the mappings $F : X \times X \to X$ and $g : X \to X$ are defined as

$$F(x,y) = \frac{x+y}{5}, \qquad g(x) = 3x.$$

Then g is continuous, $g(X) = [0,\infty) = X$ is a complete parametric metric space and

$$F(X \times X) = [0,\infty) \subseteq g(X).$$

Since

$$gF(x,y) = g\left(\frac{x+y}{5}\right) = \frac{3(x+y)}{5} = F(gx, gy),$$

we conclude that F and g are commutative. We have

$$\begin{aligned}
p(F(x,y), F(a,b), t) = p\left(\frac{x+y}{5}, \frac{a+b}{5}, t\right) &= \left|\frac{x+y-a-b}{5t}\right| \\
&= \frac{1}{5t}|x+y-a-b| \\
&\leq \frac{1}{5t}(|x-a| + |y-b|) \\
&\leq \frac{1}{4t}3(|x-a| + |y-b|) \\
&= \frac{1}{4t}[|3x-3a| + |3y-3b|] \\
&= \frac{1}{4}[\frac{|3x-3a|}{t} + \frac{|3y-3b|}{t}] \\
&= \frac{1}{4}[p(3x,3a,t) + p(3y,3b,t)] \\
&= \frac{1}{4}[p(gx,ga,t) + p(gy,gb,t)]
\end{aligned}$$

for all $x, y, a, b \in X$ and all $t > 0$. Since all conditions in Theorem 1 are satisfied by F and g, we obtain that there is a unique $0 \in X$ such that $F(0,0) = g(0) = 0$.

We now prove a theorem which is an analogue of Boyd-Wong fixed point theorem [5].

Theorem 2. *Let (X,p) be a complete parametric metric space and $T : X \to X$ be a function such that*

$$p(Tx,Ty,t) \leq \phi[p(x,y,t)], \tag{10}$$

where $t > 0$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a real function, upper semicontinuous from the right and satisfying

$$\phi(t) < t \qquad for \quad t > 0. \tag{11}$$

Assume that there exists an element $x \in X$ such that $p(x,Tx,t) < \infty$. Then T has a unique fixed point $y \in X$ and $T^n x \to y$ as $n \to \infty$ for each $x \in X$.

Proof. If we define $\alpha_n = p(T^{n-1}x, T^n x, t)$ for an element $x \in X$, then we have

$$\begin{aligned}\alpha_{n+1} =& p(T^n x, T^{n+1}x, t)\\ =& p(TT^{n-1}x, TT^n x, t)\\ \leq& \phi[p(T^{n-1}x, T^n x, t)]\\ <& p(T^{n-1}x, T^n x, t)\\ =& \alpha_n.\end{aligned}$$

Therefore $\{\alpha_n\}$ is a decreasing sequence and so it has a limit a. Suppose that $a > 0$. $\alpha_{n+1} \leq \phi(\alpha_n)$ and upper semicontinuity from the right of ϕ give the following result:

$$a \leq \lim_{\alpha_n \to a+} \sup \phi(\alpha_n) \leq \phi(a).$$

But the last statement is in contradiction with (11). Thus, we get

$$\lim_{n\to\infty} p(T^{n-1}x, T^n x, t) = 0.$$

We now show that $\{T^n x\}$ is a Cauchy sequence. Suppose that $\{T^n x\}$ is not Cauchy sequence. Then there exists an $\epsilon > 0$ such that for each $n \in \mathbb{N}$ there is $m = m(n) > n$ such that

$$p(T^n x, T^m x, t) \geq \epsilon. \tag{12}$$

We can assume that $m(n)$ is the smallest integer for which (12) holds. It means

$$p(T^n x, T^{m-1}x, t) < \epsilon.$$

From the triangle inequality, we get

$$\begin{aligned}\epsilon \leq p(T^n x, T^m x, t) \leq& p(T^n x, T^{m-1}x, t) + p(T^{m-1}x, T^m x, t)\\ \leq& \epsilon + p(T^{m-1}x, T^m x, t).\end{aligned}$$

Since $\lim_{m\to\infty} p(T^{m-1}x, T^m x, t) = 0$, we have

$$\gamma_n = p(T^n, T^m x, t) \to \epsilon + \quad \text{as} \quad m \to \infty.$$

It is clear that $m > n$ implies $p(T^m x, T^{m+1}x, t) \leq p(T^n x, T^{n+1}x, t)$. Thus we obtain

$$\begin{aligned}\epsilon \leq \gamma_n = p(T^n x, T^m x, t) &\leq p(T^n x, T^{n+1}x, t) + p(TT^n x, TT^m x, t) + p(T^m x, T^{m+1}x, t)\\ &\leq 2p(T^n x, T^{n+1}x, t) + \phi[p(T^n x, T^m x, t)]\\ &= 2p(T^n x, T^{n+1}x, t) + \phi(\gamma_n).\end{aligned}$$

Using the continuity of ϕ, we get

$$\epsilon \leq \lim_{n\to\infty} 2p(T^n x, T^{n+1}x, t) + \lim_{n\to\infty} \sup \phi(\gamma_n) < \phi(\epsilon),$$

which contradicts (11). As a result, $\{T^n x\}$ is a Cauchy sequence and as X is complete, $\{T^n x\}$ converges to x_0 in X. From (10) and (11), as T is continuous, we get

$$Tx_0 = T(\lim_{n\to\infty} T^n x) = \lim_{n\to\infty} T(T^n x) = \lim_{n\to\infty} T^{n+1}x = x_0.$$

Thus, the limit point x_0 of $\{T^n x\}$ is a fixed point of T.

We now prove the uniqueness. For this purpose, let u be another fixed point of T. Then

$$\begin{aligned}p(u, x_0, t) &= p(Tu, Tx_0, t)\\ &\leq \phi[p(u, x_0, t)]\\ &< p(u, x_0, t),\end{aligned}$$

but this is a contradiction, so $u = x_0$ and T has a unique fixed point. ◀

4. An Application to Homotopy

In this section, we give a homotopy result using Theorem 2.

Theorem 3. *Let (X, p) be a complete parametric metric space. Consider an open subset U and a closed subset V of X with $U \subset V$. Let the function $H : V \times [0, 1] \to X$ satisfy the following conditions:*

(1) For each $x \in V \setminus U$ and each $t \in [0, 1]$, $x \neq H(x, t)$,

(2) there exists $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$, a continuous non-decreasing function satisfying $\gamma(t) < t$ such that for each $t \in [0, 1]$ and each $x, y \in V$ we have

$$p(H(x, t), H(y, t), t) \leq \gamma[p(x, y, t)],$$

(3) there exists a continuous function $\alpha : [0, 1] \to \mathbb{R}$ such that

$$p(H(x, t), H(x, s), t) \leq |\alpha(t) - \alpha(s)|$$

for all $t, s \in [0, 1]$ and each $x \in V$,

(4) $\psi : [0, +\infty) \to [0, +\infty)$ is strictly non-decreasing where $\psi(x) = x - \gamma(x)$.

Then $H(.,0)$ has a fixed point if and only if $H(.,1)$ has a fixed point.

Proof. Consider the set

$$G = \{t \in [0,1] \mid x = H(x,t) \text{ for some } x \in U\}.$$

($\Rightarrow$) Assume that $H(.,0)$ has a fixed point. Since (1) holds, G is a non-empty set because there is an element $0 \in G$. We will show that G is both closed and open in $[0,1]$. By the connectedness of $[0,1]$, we have the required result because $G = [0,1]$.

We prove that G is open in $[0,1]$. Let $t_0 \in G$ and $x_0 \in U$ with $x_0 = H(x_0,t_0)$. Since U is open in X, there exists $r > 0$ such that $B(x_0,r) \subseteq U$. Consider $\varepsilon = \psi(r+p(x,x_0,t)) > 0$. There exists $\beta(\varepsilon) > 0$ such that $|\alpha(t) - \alpha(t_0)| < \varepsilon$ for all $t \in (t_0 - \beta(\varepsilon), t_0 + \beta(\varepsilon))$ because α is continuous in t_0.

Let $t \in (t_0 - \beta(\varepsilon), t_0 + \beta(\varepsilon))$. Then for $x \in \overline{B(x_0,r)} = \{x \in X \mid p(x,x_0,t) \leq r\}$, we have

$$\begin{aligned} p(H(x,t),x_0,t) =& p(H(x,t),H(x_0,t_0),t) \\ \leq& p(H(x,t),H(x,t_0),t) + p(H(x,t_0),H(x_0,t_0),t) \\ \leq& |\alpha(t) - \alpha(t_0)| + \gamma[p(x,x_0,t)] \\ \leq& \varepsilon + p(x,x_0,t) \\ =& \psi(r + p(x,x_0,t)) + p(x,x_0,t) \\ =& r + p(x,x_0,t) - \gamma(r + p(x,x_0,t)) + p(x,x_0,t) \\ \leq& r + 2p(x,x_0,t) - r - p(x,x_0,t) \\ =& p(x,x_0,t) \\ \leq& r \end{aligned}$$

and $H(x,t) \in \overline{B(x_0,r)}$. Therefore,

$$H(.,t) : \overline{B(x_0,r)} \to \overline{B(x_0,r)}$$

for every fixed $t \in (t_0 - \beta(\varepsilon), t_0 + \beta(\varepsilon))$. $H(.,t)$ has a fixed point in V because all hypotheses of Theorem 2 are satisfied. However, this fixed point must be in U as (1) holds. Thus $(t_0 - \beta(\varepsilon), t_0 + \beta(\varepsilon)) \subseteq G$ and so G is open in $[0,1]$.

In the next step, we show that G is closed in $[0,1]$. Consider a sequence $\{t_n\}_{n \in \mathbb{N}^*}$ in G with $t_n \to t^* \in [0,1]$ as $n \to +\infty$. Our aim is to show $t^* \in G$. From the definition of G, there exists $x_n \in U$ with $x_n = H(x_n,t_n)$ for all $n \in \mathbb{N}^*$. For $m,n \in \mathbb{N}^*$, we have

$$\begin{aligned} p(x_n,x_m,t) =& p(H(x_n,t_n),H(x_m,t_m),t) \\ \leq& p(H(x_n,t_n),H(x_n,t_m),t) + p(H(x_n,t_m),H(x_m,t_m),t) \\ \leq& |\alpha(t_n) - \alpha(t_m)| + \gamma[p(x_n,x_m,t)]. \end{aligned}$$

This implies that

$$\psi(p(x_n,x_m,t)) \leq |\alpha(t_n) - \alpha(t_m)|,$$

and from (4) we conclude

$$p(x_n, x_m, t) \leq \psi^{-1}(|\alpha(t_n) - \alpha(t_m)|).$$

Since ψ^{-1} and α are continuous and $\{t_n\}_{n\in\mathbb{N}^*}$ is convergent, we obtain

$$\lim_{n,m\to+\infty} p(x_n, x_m, t) = 0,$$

that is, $\{x_n\}_{n\in\mathbb{N}^*}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in V$ with

$$\lim_{n\to+\infty} \omega_\lambda(x^*, x_n) = 0.$$

On the other hand, since

$$\begin{aligned} p(x_n, H(x^*, t^*), t) =& p(H(x_n, t_n), H(x^*, t^*), t) \\ \leq& p(H(x_n, t_n), H(x_n, t^*), t) + p(H(x_n, t^*), H(x^*, t^*), t) \\ \leq& |\alpha(t_n) - \alpha(t^*)| + \gamma[p(x_n, x^*, t)], \end{aligned}$$

we obtain $\lim_{n\to+\infty} p(x_n, H(x^*, t^*), t) = 0$ and thus

$$p(x^*, H(x^*, t^*), t) = \lim_{n\to+\infty} p(x_n, H(x^*, t^*), t) = 0.$$

The last statement implies that $x^* = H(x^*, t^*)$. From (1), we get $x^* \in U$. Thus $t^* \in G$ and G is closed in $[0, 1]$.
($\Leftarrow$) If the same argument is applied as above, the required result is obtained. ◀

5. Conclusion

Searching for coupled fixed point theorems in various metric spaces has become of great interest in recent years. Especially, researchers in this area are currently focusing on interesting and useful applications of fixed point theorems. In this sense, we give some coupled fixed point theorems for parametric metric spaces and a homotopy result as an application. We believe that all results in this paper will develop the fixed point theory.

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