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Boundedness of Pseudo-differential Operators on Homogeneous Herz-type Hardy Space with Variable Exponent

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Abstract. In this paper, we obtain some boundedness of the pseudo-differential operator T on homogeneous Herz-type Hardy space with variable exponent.

Key Words and Phrases: pseudo-differential operator, Herz-type Hardy space, variable exponent.

2010 Mathematics Subject Classifications: 42B30, 42B35, 46E30

1. Introduction

The theory of function spaces with variable exponent has been extensively studied by researchers since the work of Kováčik and Rákosník [5] appeared in 1991. In [1] and [7], the boundedness of some integral operators on variable L^p spaces was studied. In [8], the Herz-type Hardy spaces with variable exponent were defined and their atomic characterizations were given.

The boundedness of pseudo-differential operators on Herz-type Hardy spaces was studied by many authors (see [2, 6]). Inspired by [3, 8], we will prove the boundedness of pseudo-differential operators of order zero on the space $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, which generalizes some known results.

We first briefly recall some standard notations. Given an open set $\Omega \subset \mathbb{R}^n$, and a measurable function $p(\cdot) : \Omega \longrightarrow [1, \infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

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These spaces are referred to as variable L^p spaces, since they generalized the standard L^p spaces: if $p(x) = p$ is constant, then $L^{p(\cdot)}(\Omega)$ is isometrically isomorphic to $L^p(\Omega)$.

The space $L^{p(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(\Omega) := \{f : f \in L^{p(\cdot)}(E) \text{ for all compact subsets } E \subset \Omega\}.$$

Define $\mathcal{P}(\Omega)$ to be the set of $p(\cdot) : \Omega \longrightarrow [1, \infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in \Omega\} > 1, \quad p^+ = \operatorname{ess\,sup}\{p(x) : x \in \Omega\} < \infty.$$

Denote $p'(x) = p(x)/(p(x) - 1)$. Let $\mathcal{B}(\Omega)$ be the set of $p(\cdot) \in \mathcal{P}(\Omega)$ such that the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\Omega)$.

In variable L^p spaces there are some important lemmas.

Lemma 1 ([5]). *Let $p(\cdot) \in \mathcal{P}(\Omega)$. If $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, then fg is integrable on Ω and*

$$\int_\Omega |f(x)g(x)|dx \le r_p \|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{p'(\cdot)}(\Omega)},$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

The last inequality is called the generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 2 ([4]). *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1} \text{ and } \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where δ_1, δ_2 are constants with $0 < \delta_1, \delta_2 < 1$ and χ_S, χ_B are the characteristic functions of S, B, respectively.

Throughout this paper δ_2 is the same as in Lemma 2.

Lemma 3 ([4]). *Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C.$$

Next we recall the definition of the homogeneous Herz-type spaces with variable exponent. Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the sets of all positive and non-negative integers, respectively. Also denote $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\tilde{\chi}_0 = \chi_{B_0}$.

Definition 1 ([4]). *Let $\alpha \in \mathbb{R}, 0 < p \le \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by*

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p}.$$

In [8], the authors gave the definition of homogeneous Herz-type Hardy space with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and the atomic decomposition characterization. $\mathcal{S}(\mathbb{R}^n)$ denotes the space of Schwartz functions, and $\mathcal{S}'(\mathbb{R}^n)$ denotes the dual space of $\mathcal{S}(\mathbb{R}^n)$. Let $G_N(f)(x)$ be the grand maximal function of $f(x)$ defined by

$$G_N(f)(x) = \sup_{\phi \in \mathcal{A}_N} |\phi^*_\nabla(f)(x)|,$$

where $\mathcal{A}_N = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta| \le N} |x^\alpha D^\beta \phi(x)| \le 1\}$ and $N > n+1$, ϕ^*_∇ is the nontangential maximal operator defined by

$$\phi^*_\nabla(f)(x) = \sup_{|y-x|<t} |\phi_t * f(y)|$$

with $\phi_t(x) = t^{-n}\phi(x/t)$.

Definition 2 ([8])**.** *Let $\alpha \in \mathbb{R}, 0 < p < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $N > n+1$. The homogeneous Herz-type Hardy space with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by*

$$H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N(f)(x) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \right\}$$

and $\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

For $x \in \mathbb{R}$ we denote by $[x]$ the largest integer less than or equal to x.

Definition 3 ([8])**.** *Let $n\delta_2 \le \alpha < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and non-negative integer $s \ge [\alpha - n\delta_2]$. A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-atom, if it satisfies*

(i) supp $a \subset B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$.
(ii) $\|a\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le |B(0,r)|^{-\alpha/n}$.
(iii) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0, |\beta| \le s$.

If $r = 2^k$ for some $k \in \mathbb{Z}$ in Definition 3, then the corresponding central $(\alpha, q(\cdot))$-atom is called a dyadic central $(\alpha, q(\cdot))$-atom.

Lemma 4 ([8])**.** *Let $n\delta_2 \le \alpha < \infty,\ 0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ if and only if*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k\,, \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each a_k is a central $(\alpha, q(\cdot))$-atom with support contained in B_k and $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$. *Moreover*

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p},$$

where the infimum is taken over all above decompositions of f.

In [9], we gave some real-variable characterizations for $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ with integral 1. For $t > 0$, set $\phi_t(x) = t^{-n}\phi(x/t)$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, define the maximal operator ϕ^*_+, $\phi^*_{\nabla,N}$ (with $N > 1$) and ϕ^{**}_M (with $M \in \mathbb{N}$) by

$$\phi^*_+(f)(x) = \sup_{t>0} |(f * \phi_t)(x)|, \;\; \phi^*_{\nabla,N}(f)(x) = \sup_{t>0} \sup_{|x-y|<Nt} |(f * \phi_t)(y)|$$

and

$$\phi^{**}_M(f)(x) = \sup_{(y,t)\in\mathbb{R}^{n+1}_+} |(f * \phi_t)(y)| \left(\frac{t}{|x-y| + t} \right)^M.$$

Lemma 5 ([9]). *Let $0 < \alpha < \infty$, $0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, the following statements are equivalent:*

(i) $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

(ii) For some $N > 1$, $\phi^*_{\nabla,N}(f) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

(iii) $\phi^*_{\nabla}(f) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

(iv) $\phi^*_+(f) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Moreover

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \|\phi^*_{\nabla,N}(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \|\phi^*_{\nabla}(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \|\phi^*_+(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

2. Main result and its proof

In this section, we will prove the boundedness of pseudo-differential operators on homogeneous Herz-type Hardy space with variable exponent.

Theorem 1. *Let $n\delta_2 \le \alpha < \infty, 0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. If $Tf(x) = \int_{\mathbb{R}^n} \hat{f}(x)\sigma(x,\xi)$ $e^{2\pi i x\cdot\xi}d\xi$ with $\sigma \in \mathbb{S}^0$, that is, $\sigma \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $|D^\alpha_x D^\beta_\xi \sigma(x,\xi)| \le C_{\alpha,\beta}(1 + |\xi|)^{-|\beta|}$, then $\|Tf\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.*

Proof. Let $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. By Lemma 4, we have $f(x) = \sum_{k=-\infty}^{\infty} \lambda_k a_k(x)$ in distributional sense. Then we consider two cases with $0 < p \le 1$ and $1 < p < \infty$.

The case $0 < p \le 1$. In this case, we only need to show that $\|Ta_k\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le C$ and C is independent of a_k. If $k \le 0$, then

$$\begin{aligned}
\|Ta_k\|^p_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|G_N(Ta_k)\chi_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&= \sum_{j=-\infty}^{k+2} 2^{j\alpha p}\|G_N(Ta_k)\chi_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} + \sum_{j=k+3}^{\infty} 2^{j\alpha p}\|G_N(Ta_k)\chi_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&= I_1 + I_2.
\end{aligned}$$

For I_1, using the $L^{q(\cdot)}(\mathbb{R}^n)$-boundedness of $\mathcal{M}$, we have

$$\begin{aligned}
I_1 &= \sum_{j=-\infty}^{k+2} 2^{j\alpha p}\|G_N(Ta_k)\chi_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C\sum_{j=-\infty}^{k+2} 2^{j\alpha p}\|\mathcal{M}(Ta_k)\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C\sum_{j=-\infty}^{k+2} 2^{j\alpha p}\|a_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C\sum_{j=-\infty}^{k+2} 2^{(j-k)\alpha p} \le C.
\end{aligned}$$

To estimate I_2, by Theorem 4 in [3], we can write

$$\phi_t * (Ta_k)(x) = \int_{\mathbb{R}^n} K_t(x, x-z)a_k(z)dz.$$

Then we expand $K_t(x, x-z)$ in a Taylor series about $z = 0$. By the vanishing moments of a_k, we get that

$$\phi_t * (Ta_k)(x) = \sum_{|\alpha|=N+1} \int_{\mathbb{R}^n} D_z^\alpha K_t(x, x-\theta z)z^\alpha a_k(z)dz,$$

where $\theta \in (0,1)$ and $N \in \mathbb{Z}_+$ satisfying $\alpha - n\delta_2 < N+1$. Noting that $x \in A_j$ with $j \ge k+3$, by Theorem 4 in [3], we can obtain that

$$\begin{aligned}
|\phi_t * (Ta_k)(x)| &\le \frac{C}{|x|^{n+N+1}} \int_{\mathbb{R}^n} |z|^{N+1}a_k(z)dz \\
&\le \frac{C2^{k(N+1)}}{|x|^{n+N+1}} \int_{\mathbb{R}^n} a_k(z)dz \\
&\le \frac{C2^{k(N+1)}}{|x|^{n+N+1}} \|a_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

$$\leq \frac{C2^{k(N+1)}}{2^{j(n+N+1)}}|B_k|^{-\alpha/n}\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.$$

So by Lemma 2 and Lemma 3 we have

$$\begin{aligned}
I_2 &\leq C\sum_{j=k+3}^{\infty} 2^{p[k(N+1)-j(n+N+1)]}\left(\frac{|B_j|}{|B_k|}\right)^{p\alpha/n}\|\chi_{B_k}\|^p_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C\sum_{j=k+3}^{\infty} 2^{p[(k-j)(N+1)-jn]}\left(\frac{|B_j|}{|B_k|}\right)^{p\alpha/n}\|\chi_{B_k}\|^p_{L^{q'(\cdot)}(\mathbb{R}^n)}\left(|B_j|\|\chi_{B_j}\|^{-1}_{L^{q'(\cdot)}(\mathbb{R}^n)}\right)^p\\
&= C\sum_{j=k+3}^{\infty} 2^{p(k-j)(N+1-\alpha)}\left(\frac{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\right)^p\\
&\leq C\sum_{j=k+3}^{\infty} 2^{p(k-j)(N+1-\alpha+n\delta_2)} \leq C.
\end{aligned}$$

If $k > 0$, we choose a radial smooth function η such that $\operatorname{supp}\eta \subset B(0,1)$ and η equals 1 near the origin. We split $T = T_1 + T_2$ by decomposing $K(x,z) = K_1(x,z) + K_2(x,z) = \eta(z)K(x,z)+(1-\eta(z))K(x,z)$. Then T_1 and T_2 are of order zero. Noting $\operatorname{supp}\tilde{\phi}^*_+(T_1a_k) \subset B_{k+1}$ and $L^{q(\cdot)}(\mathbb{R}^n)$-boundedness of $\mathcal{M}$, we get

$$\begin{aligned}
\|T_1a_k\|^p_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \sum_{j=-\infty}^{k+1} 2^{j\alpha p}\|\phi^*_+(T_1a_k)\chi_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C\sum_{j=-\infty}^{k+1} 2^{j\alpha p}\|\mathcal{M}(T_1a_k)\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C\sum_{j=-\infty}^{k+1} 2^{j\alpha p}\|T_1a_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C\sum_{j=-\infty}^{k+1} 2^{j\alpha p}\|a_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C\sum_{j=-\infty}^{k+1} 2^{(j-k)\alpha p} \leq C.
\end{aligned}$$

Now let's estimate $T_2a_k(x)$. We have

$$|(K_2)_t(x,z)| \leq C_M(1+|z|)^{-M} \tag{1}$$

for any $M \geq n$ (see [3, Theorem 4]). Then we write

$$\|T_2a_k\|^p_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|\phi^*_+(T_2a_k)\chi_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}$$

$$
\begin{aligned}
&= \sum_{j=-\infty}^{k+2} 2^{j\alpha p} \|\phi_+^*(T_2 a_k)\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p + \sum_{j=k+3}^{\infty} 2^{j\alpha p} \|\phi_+^*(T_2 a_k)\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \\
&= J_1 + J_2.
\end{aligned}
$$

For J_1, we can obtain the desirable estimate by a similar method used for I_1. For J_2, noting that $x \in A_j$ and $l \geq k+3$, by (1) we can obtain that

$$
\begin{aligned}
|\phi_t * (T_2 a_k)(x)| &= \left| \int_{\mathbb{R}^n} (K_2)_t(x, x-z) a_k(z) dz \right| \\
&\leq C_M \int_{\mathbb{R}^n} \frac{1}{(1+|x-z|)^M} |a_k(z)| dz \\
&\leq \frac{C}{|x|^{n+N+1}} \int_{\mathbb{R}^n} |a_k(z)| dz \\
&\leq \frac{C 2^{k(N+1)}}{|x|^{n+N+1}} |B_k|^{-\alpha/n} \|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)},
\end{aligned}
$$

where we take $N \in \mathbb{Z}_+$ satisfying $\alpha - n\delta_2 < N+1$. So it easily follows that $J_2 \leq C$.

The case $1 < p < \infty$. In this case, we write

$$
\begin{aligned}
\|Tf\|_{H\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} &= \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|\phi_+^*(Tf)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p} \\
&\leq \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{\infty} |\lambda_j| \|\phi_+^*(Ta_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
&\leq \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \|\phi_+^*(Ta_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
&\quad + \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k}^{\infty} |\lambda_j| \|\phi_+^*(Ta_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
&= U_1 + U_2.
\end{aligned}
$$

Similar to I_1, we can get the estimates of U_2. For U_1, we continue to decompose it as follows

$$
\begin{aligned}
U_1 &= \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \|\phi_+^*(Ta_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
&\leq \left\{ \sum_{k=-\infty}^{0} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \|\phi_+^*(Ta_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p}
\end{aligned}
$$

$$
\begin{aligned}
&+\left\{\sum_{k=1}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-1} |\lambda_j| \|\phi_+^*(T_1 a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&+\left\{\sum_{k=1}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-1} |\lambda_j| \|\phi_+^*(T_2 a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
= \quad & V_1 + V_2 + V_3.
\end{aligned}
$$

For V_2, it is easy to get the estimate by a similar method used for I_1. Using the vanishing moments for V_1 and (1) for V_3, we easily obtain that if $x \in A_k$ and $k \geq j+1$, then

$$
|\phi_+^*(Ta_j)(x)|, |\phi_+^*(T_2 a_j)(x)| \leq \frac{C2^{j(N+1)}}{|x|^{n+N+1}} |B_j|^{-\alpha/n} \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)},
$$

where we choose $N \in \mathbb{Z}_+$ such that $\alpha - n\delta_2 < N+1$. From this, it easily follows that $V_1 + V_3 \leq C$.

This completes the proof of Theorem 1. ◄

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