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On Bessel Property and Unconditional Basicity of the Systems of Root Vector-functions of a Dirac type Operator

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Abstract. In this paper, we consider Dirac type one-dimensional operator $Dy = B\frac{dy}{dx} + P(x)\,y$, $y(x) = (y_1(x), y_2(x))^T$, where $B = \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix}$; $b_1 > 0$, $b_2 < 0$, $P(x) = \begin{pmatrix} p_1(x) & 0 \\ 0 & p_2(x) \end{pmatrix}$, $p_1(x)$ and $p_2(x)$ are complex-valued functions defined on arbitrary finite interval $G(a,b)$ of a real straightline, and establish the criterion of Bessel property and unconditional basicity of the system of root vector-functions of this operator.

Key Words and Phrases: Bessel inequality, unconditional basicity, root vector-functions.

2010 Mathematics Subject Classifications: 34L10, 42A20

1. Introduction

In [1, 2], necessary and sufficient conditions of Bessel property and unconditional basicity in $L_2(G)$ of the system of root functions of second order ordinary differential equations are established. Later, these and other issues for higher order ordinary differential operators and one-dimensional Dirac operator were studied in the papers [3, 4, 5, 6, 7, 8].

In the present paper, we establish the validity of the Bessel inequality and unconditional basicity in $L_2^2(G)$ of the system of root functions of a Dirac type one-dimensional operator.

Let $L_p^2(G)$, $p \geq 1$ be a space of two-component vector-functions

$$f(x) = (f_1(x),\, f_2(x))^T$$

with the norm

$$\|f\|_{p,2} = \left[\int_G \left(|f_1(x)|^2 + |f_2(x)|^2\right)^{p/2} dx\right]^{1/p}.$$

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In the case $p = \infty$ the norm is determined by the equality

$$\|f\|_{\infty,2} = \sup_{x\in G} vrai\ |f(x)|\,.$$

It is clear that for arbitrary functions $f(x) \in L_p^2(G)$, $g(x) \in L_q^2(G)$, where $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p \le \infty$, the "scalar" derivative $(f,g) = \int_G \sum_{j=1}^{2} f_j(x)\, \overline{g_j(x)}\, dx$ is defined.

Consider the Dirac type one-dimensional operator

$$Dy = B\frac{dy}{dx} + P(x)\, y, \quad y(x) = (y_1(x)\,,\, y_2(x))^T\,,$$

where $B = \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix}$; $b_1 > 0$, $b_2 < 0$; $P(x) = \begin{pmatrix} p_1(x) & 0 \\ 0 & p_2(x) \end{pmatrix}$, $p_1(x)$ and $p_2(x)$ are complex-valued functions defined on an arbitrary interval $G = (a,b)$ of a real straight line.

Following the paper [1], by the eigen function of the operator D, corresponding to the complex eigenvalue λ, we will mean any identically nonzero complex-valued function $\overset{0}{u}(x)$ that is continuous on any closed subinterval of the interval G and satisfies the equation $D\overset{0}{u} = \lambda\overset{0}{u}$ almost everywhere in G.

In the similar way, by the associated function of order l, $l \ge 1$, corresponding to the same λ and the eigen function $\overset{0}{u}(x)$ we will mean any complex-valued vector-function $\overset{l}{u}(x)$ that is absolutely continuous on any closed subinterval of G and satisfies the equation $D\overset{l}{u} = \lambda\overset{l}{u} + \overset{l-1}{u}$ almost everywhere in G.

Let $\{u_k(x)\}_{k=1}^{\infty}$ be an arbitrary system composed of eigen and associated functions of the operator D, $\{\lambda_k\}_{k=1}^{\infty}$ be the corresponding system of eigen values. Also, let the function $u_k(x)$ be included in the system $\{u_k(x)\}_{k=1}^{\infty}$ together with appropriate associated functions of less order.

We say that for the given system of functions $\varphi_k(x) \in L_2^2(G)$ the Bessel inequality is fulfilled if there exists a constant M such that for an arbitrary vector-function $f(x) \in L_2^2(G)$, the following inequality is valid:

$$\sum_{k=1}^{\infty} |(\varphi_k, f)|^2 \le M\ \|f\|_{2,2}^2\,, \tag{1}$$

where the constant M is independent of $f(x)$.

2. Main value formula

Lemma 1. *(The mean value formula). If $p_1(x)$ and $p_2(x)$ belong to the class $L_1^{loc}(G)$ and the points $x-t$, x, $x+t$ lie in the domain G, then the following formulas are valid:*

$$\overset{l}{u}(x+t)=\left[\cos\frac{\lambda}{\sqrt{|b_1b_2|}}t\,I-\sin\frac{\lambda}{\sqrt{|b_1b_2|}}t\,\frac{B}{\sqrt{|b_1b_2|}}\right]\overset{l}{u}(x)+$$

$$+B^{-1}\int_x^{x+t}\left(\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-\xi+x)\,\frac{B}{\sqrt{|b_1b_2|}}-\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-\xi+x)\,I\right)\times$$

$$\times\left[P(\xi)\,\overset{l}{u}(\xi)-\overset{l-1}{u}(\xi)\right]d\xi, \tag{2}$$

$$\overset{l}{u}(x-t)=\left[\cos\frac{\lambda}{\sqrt{|b_1b_2|}}t\,I+\sin\frac{\lambda}{\sqrt{|b_1b_2|}}t\,\frac{B}{\sqrt{|b_1b_2|}}\right]\overset{l}{u}(x)+$$

$$+B^{-1}\int_{x-t}^{x}\left(\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t+\xi-x)\,\frac{B}{\sqrt{|b_1b_2|}}+\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t+\xi-x)\,I\right)\times$$

$$\times\left[P(\xi)\,\overset{l}{u}(\xi)-\overset{l-1}{u}(\xi)\right]d\xi, \tag{3}$$

$$\overset{l}{u}(x+t)+\overset{l}{u}(x-t)=2\,\overset{l}{u}(x)\cos\frac{\lambda}{\sqrt{|b_1b_2|}}t+$$

$$+B^{-1}\int_{x-t}^{x+t}\left(\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)\,\frac{B}{\sqrt{|b_1b_2|}}+sgn(\xi-x)\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)\,I\right)\times$$

$$\times\left[P(\xi)\,\overset{l}{u}(\xi)-\overset{l-1}{u}(\xi)\right]d\xi, \tag{4}$$

where I is a unit operator in E^2.

Proof. To derive formulas (2) and (3), it suffices to apply the operator

$$\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|\xi-x|)\,I+sgn(\xi-x)\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)\,\frac{B}{\sqrt{|b_1b_2|}}$$

to the equation $L\overset{l}{u}(\xi)=\lambda\overset{l}{u}(\xi)+\overset{l-1}{u}(\xi)$ and integrate with respect to the parameter ξ from x to $x+t$ (from $x-t$ to x), and then integrate by parts in the expression of the form

$$\int_x^{x+t}\left(\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-\xi+x)\,I-\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-\xi+x)\,\frac{B}{\sqrt{|b_1b_2|}}\right)Bd\overset{l}{u}(\xi)$$

$$\left(\int_{x-t}^{x}\left(\cos\frac{\lambda}{\sqrt{|b_1b_2|}}\left(t+\xi-x\right)I+\sin\frac{\lambda}{\sqrt{|b_1b_2|}}\left(t+\xi-x\right)\frac{B}{\sqrt{|b_1b_2|}}\right)Bd\overset{l}{u}\left(\xi\right)\right)$$

having grouped the similar terms. Formula (4) follows from formulas (2) and (3). The lemma is proved. ◀

In the present paper we prove the following criterion of Bessel property and unconditional basicity of the system $\{\varphi_k(x)\}_{k=1}^{\infty}$, where $\varphi_k(x) = u_k(x)\,\|u_k\|_{2,2}^{-1}$.

3. Criterion of Bessel property and unconditional basicity

Theorem 1. *Let G be a finite interval, the functions $p_1(x)$ and $p_2(x)$ belong to the class $L_2(G)$, the lengths of the chains of root functions be uniformly bounded and there exist a constant C_0 such that*

$$|Jm\,\lambda_k| \le C_0,\ k = 1, 2, ... \tag{5}$$

Then, for the system of functions $\{\varphi_k(x)\}$, where $\varphi_k(x) = u_k(x)\,\|u_k\|_{2,2}^{-1}$, to satisfy the Bessel inequality, it is necessary and sufficient that there exist a constant K such that

$$\sum_{|Re\,\lambda_k-\nu|\le 1} 1\ \le K, \tag{6}$$

where ν is an arbitrary real number.

Proof. Necessity. Denote by R the number $\left(\frac{n_0}{\sqrt{|b_1b_2|}}\left(1+C_0\right)\right)^{-1}$, where $\frac{n_0}{\sqrt{|b_1b_2|}} \ge 1$ is chosen in such a way that $R \le mesG/4$ and for any set $E \in \overline{G}$, $mes\,E \le 2R$, the inequality $\max\left\{\|p_1\|_{2,E}, \|p_2\|_{2,E}\right\} \le \frac{1}{L}$ is fulfilled, where L is a positive integer to be defined below. This is possible because of summability of the functions $p_1(x)$ and $p_2(x)$.

Let $0 \le t \le R$, $x \in \left[a, \frac{a+b}{2}\right]$. Write the mean value formula (4) for the points x, $x+t$, $x+2t \in \overline{G}$:

$$u_k(x) = 2u_k(x+t)\cos\left(\frac{\lambda_k}{\sqrt{|b_1b_2|}}t\right) - u_k(x+2t) +$$
$$+B^{-1}\int_{x}^{x+2t}\left\{\sin\left(\frac{\lambda_k}{\sqrt{|b_1b_2|}}\left(t-|x+t-\xi|\right)\right)\frac{B}{\sqrt{|b_1b_2|}}+\right.$$
$$\left.+sgn\left(\xi-x-t\right)\cos\left(\frac{\lambda_k}{\sqrt{|b_1b_2|}}\left(t-|x+t-\xi|\right)\right)I\right\}\times$$
$$\times\left[P\left(\xi\right)u_k\left(\xi\right)-\theta_k u_{k-1}\left(\xi\right)\right]d\xi \tag{7}$$

where $\theta_k = 0$, if $u_k(x)$ is an eigen vector-function; $\theta_k = 1$ if $u_k(x)$ is an associated vector-function, and in the last case we assume $\lambda_k = \lambda_{k-1}$.◄

Adding and subtracting $2u_k(x+t)\cos\left(\frac{\nu}{\sqrt{|b_1 b_2|}}t\right)$ to the right side of inequality (7), we represent this formula in the form

$$u_k(x) = 2u_k(x+t)\cos\left(\frac{\nu}{\sqrt{|b_1 b_2|}}t\right) - u_k(x+2t) +$$
$$+4u_k(x+t)\sin\left(\frac{\lambda_k+\nu}{2\sqrt{|b_1 b_2|}}t\right)\sin\left(\frac{\nu-\lambda_k}{2\sqrt{|b_1 b_2|}}t\right) +$$
$$+B^{-1}\int_x^{x+2t}\left\{\sin\left(\frac{\lambda_k}{\sqrt{|b_1 b_2|}}(t-|x+t-\xi|)\right)\frac{B}{\sqrt{|b_1 b_2|}}+\right.$$
$$\left.+sgn(\xi-x-t)\cos\left(\frac{\lambda_k}{\sqrt{|b_1 b_2|}}(t-|x+t-\xi|)\right)I\right\}\times$$
$$\times\left[P(\xi)\,u_k(\xi) - \theta_k u_{k-1}(\xi)\right]d\xi.$$

Integrate the last relation with respect to t from 0 to R to obtain:

$$u_k^i(x) = \frac{1}{R}\int_G u_k^i(t)\,\upsilon(t)\,dt + \frac{4}{R}\int_0^R u_k^i(x+t)\sin\left(\frac{\lambda_k+\nu}{2\sqrt{|b_1 b_2|}}\right)\sin\left(\frac{\nu-\lambda_k}{2\sqrt{|b_1 b_2|}}t\right)dt+$$
$$+\frac{1}{R}B^{-1}\int_0^R\int_x^{x+2t}\left\{\sin\left(\frac{\lambda_k}{\sqrt{|b_1 b_2|}}(t-|x+t-\xi|)\right)\frac{B}{\sqrt{|b_1 b_2|}}+\right.$$
$$\left.+sgn(\xi-x-t)\cos\left(\frac{\lambda_k}{\sqrt{|b_1 b_2|}}(t-|x+t-\xi|)\right)I\right\}\times$$
$$\times\left[(P(\xi)\,u_k(\xi) - \theta_k u_{k-1}(\xi))\right]_i d\xi\,dt, \quad i = 1, 2, ... \tag{8}$$

where $u_k^i(x)$ is the i -th component of the vector $u_k(x)$, $[\,]_i$ is the i -th component of the corresponding vector

$$\upsilon(t) = 2\cos\left(\frac{\nu}{\sqrt{|b_1 b_2|}}(x-t)\right) - \frac{1}{2} \quad \text{for } x \le t \le x+R,$$

$\upsilon(t) = -\frac{1}{2}$ for $x+R < t < x+2R$ and $\upsilon(t) = 0$ in the remaining cases.

Introduce the set of indices $J_\nu = \{k : |Re\lambda_k - \nu| \le 1, \; |Jm\lambda_k| \le C_0\}$.

Let $k \in J_\nu$. Using the inequalities $|\sin z|, |\cos z| \le 2$ and $|\sin z| \le 2|z|$ for $|Jm\, z| \le 1$, from relation (8) we find

$$\left|u_k^i(x)\right| \le \frac{1}{R}\left|\int_G u_k^i(t)\, \upsilon(t)\, dt\right| + \frac{8}{R\sqrt{|b_1 b_2|}}\, |\lambda_k - \nu| \int_0^R \left|u_k^i(x+t)\right| t\, dt +$$

$$+\frac{2}{R\sqrt{|b_1 b_2|}} \int_0^R \int_x^{x+2t} \left\{|p_1(\xi)| \left|u_k^1(\xi)\right| + |p_2(\xi)| \left|u_k^2(\xi)\right|\right\} d\xi\, dt +$$

$$+\frac{2}{R\,|b_1 b_2|} \int_0^R \int_x^{x+2t} \left\{|p_1(\xi)| \left|b_2\, u_k^1(\xi)\right| + |p_2(\xi)| \left|b_1\, u_k^2(\xi)\right|\right\} d\xi\, dt +$$

$$+\frac{2}{R\sqrt{|b_1 b_2|}} \int_0^R \int_x^{x+2t} \left\{\left| u_{k-1}^1(\xi)\right| + \left| u_{k-1}^2(\xi)\right|\right\} d\xi\, dt +$$

$$+\frac{2}{R\,|b_1 b_2|} \int_0^R \int_x^{x+2t} \left\{ \left|b_2\, u_{k-1}^1(\xi)\right| + \left|b_1\, u_{k-1}^2(\xi)\right|\right\} d\xi\, dt \le$$

$$\le \frac{1}{R}\left|\int_G u_k^i(t)\, \upsilon(t)\, dt\right| + \frac{8\sqrt{R}}{\sqrt{3}} \frac{1}{\sqrt{|b_1 b_2|}} (1 + C_0) \left\|u_k^i\right\|_{2,G} +$$

$$+\frac{2}{\sqrt{|b_1 b_2|}} \left(\|p_1\|_{2,E} \left\|u_k^1\right\|_{2,G} + \|p_2\|_{2,E} \left\|u_k^2\right\|_{2,G}\right) +$$

$$+\frac{2}{|b_1 b_2|} \left(\|p_1\|_{2,E} \left\|b_2\, u_k^1\right\|_{2,G} + \|p_2\|_{2,E} \left\|b_1\, u_k^2\right\|_{2,G}\right) +$$

$$+\frac{4R}{\sqrt{|b_1 b_2|}} \left(\left\|u_{k-1}^1\right\|_{\infty,G} + \left\|u_{k-1}^2\right\|_{\infty,G}\right) + \frac{4R}{|b_1 b_2|} \left(\left\|b_2\, u_{k-1}^1\right\|_{\infty,G} + \left\|b_1\, u_{k-1}^2\right\|_{\infty,G}\right) \le$$

$$\le \frac{1}{R}\left|\int_G u_k^i(t)\, \upsilon(t)\, dt\right| + \frac{8\sqrt{R}}{\sqrt{3}\,|b_1 b_2|} (1 + C_0) \left\|u_k^i\right\|_{2,G} + 4\frac{\|u_k\|_{2,G}}{L\sqrt{|b_1 b_2|}} + 4\frac{(|b_1| + |b_2|)}{L\,|b_1 b_2|} \|u_k\|_{2,G} +$$

$$+\frac{4R}{\sqrt{|b_1 b_2|}} \|u_{k-1}\|_{\infty,G} + \frac{4R\,(|b_1| + |b_2|)}{|b_1 b_2|} \|u_{k-1}\|_{\infty,G} = \frac{1}{R}\left|\int_G u_k^i(t)\, \upsilon(t)\, dt\right| +$$

$$+\frac{8\sqrt{R}}{\sqrt{3}\,|b_1 b_2|} (1 + C_0) \left\|u_k^i\right\|_{2,G} + \left(\frac{4}{L\sqrt{|b_1 b_2|}} + \frac{4\,(|b_1| + |b_2|)}{L\,|b_1 b_2|}\right) \|u_k\|_{2,G} +$$

$$+\left(\frac{4R}{\sqrt{|b_1 b_2|}} + \frac{4R\,(|b_1| + |b_2|)}{|b_1 b_2|}\right) \|u_{k-1}\|_{\infty,G}\,.$$

Consequently

$$\left|u_k^i(x)\right| \le \frac{1}{R}\left|\int_G u_k^i(t)\, \upsilon(t)\, dt\right| +$$

$$+\left(\frac{8\sqrt{R}}{\sqrt{3\,|b_1 b_2|}}(1+C_0)+\frac{4}{L\,\sqrt{|b_1 b_2|}}+\frac{4\,(|b_1|+|b_2|)}{L\,|b_1 b_2|}\right)\|u_k\|_{2,G}+$$

$$+\left(\frac{4R}{\sqrt{|b_1 b_2|}}+\frac{4R\,(|b_1|+|b_2|)}{|b_1 b_2|}\right)\|u_{k-1}\|_{\infty,2};\ i=1,2.$$

In [7], the estimations

$$\overset{l-1}{\|u\|}_{\infty,2}\le C^1(l,G,b_1,b_2)\,(1+|Jm\,\lambda|)\,\overset{l}{\|u\|}_{\infty,2}, \tag{9}$$

$$\overset{l}{\|u\|}_{\infty,2}\le C^2(l,G,b_1,b_2)\,(1+|Jm\,\lambda|)^{\frac{1}{r}}\,\overset{l}{\|u\|}_{r,2},\ \ 1\le r<\infty, \tag{10}$$

were obtained under the conditions $b_1=1$ and $b_2=-1$. For arbitrary $b_1>0$ and $b_2<0$, the estimations (9) and (10) are proved similarly.

Applying estimations (9) and (10), for $r=2$ we get the inequality

$$\left|u_k^i(x)\right|\le\frac{1}{R}\left|\int_G u_k^i(t)\,\upsilon(t)\,dt\right|+\left\{8(1+C_0)\frac{\sqrt{R}}{\sqrt{3\,|b_1 b_2|}}+\frac{4}{L}\left(\frac{1}{\sqrt{|b_1 b_2|}}+\frac{|b_1|+|b_2|}{|b_1 b_2|}\right)+\right.$$

$$+\left(\frac{4R}{\sqrt{|b_1 b_2|}}+\frac{4R\,(|b_1|+|b_2|)}{|b_1 b_2|}\right)C^1(n_k,G,b_1,b_2)\times$$

$$\left.\times C^2(n_k,G,b_1,b_2)\,(1+C_0)^{3/2}\right\}\|u_k\|_{2,2}\,. \tag{11}$$

Because of uniform boundedness of the lengths of the chains of associated functions, the inequality $C^1(n_k,G,b_1,b_2)\,C^2(n_k,G,b_1,b_2)\le j=const$ is true.
From (11), by virtue of the inequality $\left|\sum_{i=1}^m a_i\right|^q\le m^{q-1}\sum_{i=1}^m|a_i|^q$, for $q=2$ we have

$$\frac{|u_k(x)|^2}{\|u_k\|_{2,2}^2}\le\frac{3}{R^2}\left\{\left|\int_G\frac{u_k^1(t)}{\|u_k\|_{2,2}}\upsilon(t)\,dt\right|^2+\left|\int_G\frac{u_k^2(t)}{\|u_k\|_{2,2}}\upsilon(t)\,dt\right|^2\right\}+$$

$$+3\cdot 2^2\left\{8(1+C_0)\frac{\sqrt{R}}{\sqrt{3\,|b_1 b_2|}}+\frac{4}{L}\left(\frac{1}{\sqrt{|b_1 b_2|}}+\frac{|b_1|+|b_2|}{|b_1 b_2|}\right)+\right.$$

$$\left.+\left(\frac{4R}{\sqrt{|b_1 b_2|}}+\frac{4R\,(|b_1|+|b_2|)}{|b_1 b_2|}\right)C^1(n_k,G,b_1,b_2)\,C^2(n_k,G,b_1,b_2)\,(1+C_0)^{3/2}\right\}^2. \tag{12}$$

The number n_0 is chosen sufficiently large (the numbers R and L respectively as well) to satisfy

$$3\cdot 2^2\left\{8(1+C_0)\frac{\sqrt{R}}{\sqrt{3\,|b_1 b_2|}}+\frac{4}{L}\left(\frac{1}{\sqrt{|b_1 b_2|}}+\frac{|b_1|+|b_2|}{|b_1 b_2|}\right)+\right.$$

$$+\left(\frac{4R}{\sqrt{|b_1 b_2|}}+\frac{4R\,(|b_1|+|b_2|)}{|b_1 b_2|}\right) C^1\,(n_k,G,b_1,b_2)\times$$

$$\times C^2\,(n_k,G,b_1,b_2)\;(1+C_0)^{3/2}\Big\}^2\le\frac{1}{2\,mes\,G}.$$

Then, for any finite set of indices $J\subset J_\nu$ and for $x\in\left[a,\frac{a+b}{2}\right]$ from (12) we get the inequality

$$\sum_{k\in J}|u_k\,(x)|^2\,\|u_k\|_{2,2}^{-2}\le\frac{3}{R^2}\sum_{k\in J}\left\{\left|\int_G\frac{u_k^1\,(t)}{\|u_k\|_{2,2}}\,\upsilon\,(t)\;dt\right|^2+\left|\int_G\frac{u_k^2\,(t)}{\|u_k\|_{2,2}}\,\upsilon\,(t)\;dt\right|^2\right\}+$$

$$+\frac{1}{2\,mesG}\sum_{k\in J}1. \tag{13}$$

We establish this inequality for $x\in\left[\frac{a+b}{2},\,b\right]$ in the same way.
Applying the Bessel inequality, from (13) we have

$$\sum_{k\in J}|u_k\,(x)|\;\|u_k\|_{2,2}^{-2}\le 2\,M\,3R^{-2}\,\|\upsilon\,\|_{2,G}^2+\frac{1}{2\,mes\,G}\sum_{k\in J}1.$$

Taking into account the estimate $\|\upsilon\|_{2,G}=O\left(R^{\frac{1}{2}}\right)$, and then integrating with respect to $x\in G$, from the last inequality we get the validity of the following estimation for any finite set $J\subset J_\nu$:

$$\sum_{k\in J}1\le const\;R^{2\left(\frac{1}{2}-1\right)}+\frac{1}{2}\sum_{k\in J}1.$$

Consequently

$$\sum_{k\in J}1\le const\;R^{-1}.$$

Hence, by virtue of arbitrariness of the finite set $J\subset J_\nu$, we get it follows the necessity of the inequality (6).

Sufficiency. For simplicity we take $G=(0,2\pi)$. Writing out the formula (2) for $u_k\,(x+t)$ with $x=0$ and then multiplying it scalarly by the vector-function $f\,(t)=(f_1\,(t)\,,f_2\,(t))^T\in L_2^2(G)$ we arrive at the conclusion that to prove the validity of the Bessel inequality for the system $\varphi_k\,(x)=u_k\,(x)\;\|u_k\|_{2,2}^{-1}$, it suffices to establish the validity of the following inequalities:

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_i\,(t)}\;\cos\left(\frac{\lambda_k}{\sqrt{|b_1 b_2|}}t\right)\,dt\right|^2\left|\varphi_k^i\,(0)\right|^2\le C\;\|f\|_{2,2}^2\,,\quad i=1,2, \tag{14}$$

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_i(t)}\ \sin\left(\frac{\lambda_k}{\sqrt{|b_1b_2|}}t\right)\ dt\right|^2\left|\varphi_k^{3-i}(0)\right|^2\leq C\ \|f\|_{2,2}^2\,,\quad i=1,2, \tag{15}$$

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_1(t)}\ \int_0^t p_1(\xi)\ \varphi_k^1(\xi)\sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-\xi)\ d\xi\,dt\right|^2\leq C\ \|f\|_{2,2}^2\,, \tag{16}$$

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_1(t)}\ \int_0^t p_2(\xi)\ \varphi_k^2(\xi)\cos\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-\xi)\ d\xi\,dt\right|^2\leq C\ \|f\|_{2,2}^2\,, \tag{17}$$

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_2(t)}\ \int_0^t p_1(\xi)\ \varphi_k^1(\xi)\cos\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-\xi)\ d\xi\,dt\right|^2\leq C\ \|f\|_{2,2}^2\,, \tag{18}$$

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_2(t)}\ \int_0^t p_2(\xi)\ \varphi_k^2(\xi)\sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-\xi)\ d\xi\,dt\right|^2\leq C\ \|f\|_{2,2}^2\,, \tag{19}$$

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_i(t)}\ \int_0^t \frac{u_{k-1}^i(\xi)}{\|u_k\|_{2,2}}\sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-\xi)\ d\xi\,dt\right|^2\leq C\ \|f\|_{2,2}^2\,,\quad i=1,2, \tag{20}$$

$$\sum_{k=1}^{\infty}\left|\int_0^{2\pi}\overline{f_i(t)}\ \int_0^t \frac{u_{k-1}^{3-i}(\xi)}{\|u_k\|_{2,2}}\cos\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-\xi)\ d\xi\,dt\right|^2\leq C\ \|f\|_{2,2}^2\,,\quad i=1,2, \tag{21}$$

The proof of inequalities (14), (15), (20) and (21) is similar to the one of corresponding inequalities in [2]. Here, the validity of estimations (9) and (10) should be taken into account.

Prove the inequality (16) by the method of [7]. Denote by J_k the expressions standing under the summation sign. Then

$$J_k=\left|\int_0^{2\pi}p_1(\xi)\ \varphi_k^1(\xi)\left(\int_\xi^{2\pi}\overline{f_1(t)}\ \sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-\xi)\ dt\right)\ d\xi\right|^2\leq$$

$$\leq 2\pi\int_0^{2\pi}|p_1(\xi)|^2\left|\varphi_k^1(\xi)\right|\left|\int_\xi^{2\pi}\overline{f_1(t)}\ \sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}(t-\xi)\,dt\right|^2 d\xi,$$

whence, taking into account the estimation (10), we have

$$J_k\leq C\int_0^{2\pi}|p_1(\xi)|^2\left|\int_0^{2\pi}\overline{g(t,\xi)}\ \sin\frac{\lambda_k}{\sqrt{|b_1b_2|}}t\,dt\right|^2 d\xi=C\,J_k, \tag{22}$$

where $g(t,\xi) = f_1(\xi + t)$ for $\xi \le t + \xi \le 2\pi$ and $g(t,\xi) = 0$ for $2\pi < t + \xi \le 2\pi + \xi$. It is clear that for any fixed $\xi \in [0, 2\pi]$ we have $\int_0^{2\pi} |g(t,\xi)|^2 dt \le \|f_1\|^2_{L_2(0,2\pi)}$. Furthermore, under the conditions (5) and (6), for any $\varphi(t) \in L_2(0, 2\pi)$ the validity of the inequality

$$\sum_{k=1}^{\infty} \left| \int_0^{2\pi} \overline{\varphi(t)} \, \sin \frac{\lambda_k}{\sqrt{|b_1 b_2|}} t \right|^2 \le C \|\varphi\|^2_{L_2(0,2\pi)}$$

was established in [2]. Therefore, for any positive integer N we have

$$\sum_{k=1}^{N} J_k = \int_0^{2\pi} |p_1(\xi)|^2 \sum_{k=1}^{N} \left| \int_0^{2\pi} \overline{g(t,\xi)} \, \sin \frac{\lambda_k}{\sqrt{|b_1 b_2|}} dt \right|^2 \le C \|p_1\|^2_{L_2(0,2\pi)} \|f_1\|^2_{L_2(0,2\pi)} .$$

Because of the arbitrariness of N, we find $\sum_{k=1}^{\infty} J_k \le C \|f\|^2_{2,2}$. Hence, from (22) it follows that the inequality (16) is true. Inequalities (17)-(19) are proved in the same way. Theorem 1 is proved.

Denote by D^* an operator formally associated to the operator D : $D^* \upsilon = B \frac{d\upsilon}{dx} + \overline{P(x)} \, \upsilon(x)$, where $\overline{P(x)} = \begin{pmatrix} \overline{p_1(x)} & 0 \\ 0 & \overline{p_2(x)} \end{pmatrix}$.

Based on Theorem 1, we can also prove the following one using the known method:

Theorem 2. *Let G be a finite interval, $\{u_k(x)\}$ be an arbitrary, closed in $L_2^2(G)$ and minimal system consisting of eigen and associated functions of the operator D, the system $\{\upsilon_k\}$ be biorthogonally associated with $\{u_k(x)\}$ in $L_2^2(G)$, consisting of eigen and associated functions of the operator D^* and closed in $L_2^2(G)$, the length of any chain of the root vectors be uniformly bounded and condition* (5) *be fulfilled.*

Then the necessary and sufficient condition for unconditional basicity of the system $\{u_k(x)\}$ in $L_2^2(G)$ is the existence of constants M and M_1 such that the inequalities (6) *and*

$$\|u_k\|_{2,2} \, \|\upsilon_k\|_{2,2} \le M_1, \tag{23}$$

are true for any $k = 1, 2$.

Note that under the conditions of Theorem 2, the validity of inequalities (6) *and* (23) *is a necessary and sufficient condition for Riesz basicity of the system $\left\{u_k(x) \, \|u_k\|^{-1}_{2,2}\right\}$ in $L_2^2(G)$.*

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