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Determining the Coefficients in the Right Side of the System of Elliptic Equations

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Abstract. The goal of this paper is to study the well-posedness of an inverse problem of determining the unknown coefficients on the right side of the system of elliptic equations of reaction-diffusion type. The inverse problem is considered in a bounded domain in the case of the Dirichlet boundary condition with additional integral information. The sought-for coefficients don't depend on the space variable. The uniqueness theorem for the solution of the considered inverse problem is proved, and the "conditional" stability estimate is obtained.

Key Words and Phrases: inverse problem, system of elliptic equations, uniqueness, "conditional" stability.

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1. Introduction

Let be $D \subset R^n$ be a convex, bounded domain with rather smooth boundary ∂D, $D' \subset R^{n-1}$ be a projection of D on a hyperplane $y = x_n = 0$, $x = (x_1, ..., x_{n-1})$ and (x, y) be arbitrary points in the domains D' and D, respectively, $D = D' \times (\gamma_1(x), \gamma_2(x))$, $\gamma_1(x)$ and $\gamma_2(x)$ be the given rather smooth functions. The functional spaces $C^l(\cdot)$, $C^{l+\alpha}(\cdot)$, $l = 0, 1, 2$, $0 < \alpha < 1$ and the norms on these spaces were determined, for example, in [1, p. 29]. $u = (u_1, ..., u_m)$, $\|u\|_l = \sum_{k=1}^m \|u_k\|_{C^l} = \sum_{k=1}^m \sum_{i=0}^l \sup\limits_D \left|D_x^i u_k\right|$, $u_{kx_j} = \frac{\partial u_k}{\partial x_j}$, $\Delta u_k = \sum_{i=1}^u \frac{\partial^2 u_k}{\partial x_i}$, $D_x^i u_k(x)$ are all possible derivatives of the function $u_k(x)$ with respect to x_j.

2. Problem statement and main result

We consider an inverse problem of determining a pair of functions

$$\left\{f_k(x), u_k(x, y), k = \overline{1, m}\right\}$$

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from the conditions

$$\Delta u_k = f_k(x)\, g_k(u), \quad (x,y) \in D, \tag{1}$$

$$u_k(x,y) = \varphi_k(x,y), \quad (x,y) \in \partial D, \tag{2}$$

$$\int_{\gamma_1(x)}^{\gamma_2(x)} u_k(x,y)\, dy = h_k(x), \quad x \in \bar{D}' \tag{3}$$

by the known functions $g_k(p)$, $\varphi_k(x,y)$, $h_k(x)$.

Similar inverse problems are ill-posed in Hadamard sense, and for scalar equations they have been studied in [2, 3, 4, 5].

For the data of the problem (1)-(3) we make the following assumptions:
1^0. The functions $g_k(p) = g_k(p_1, ..., p_m)$ are defined and continuous for any $p \in R^m$; there exists a constant $c_0 > 0$ such that for all $p, q \in R^m$

$$|g_k(p) - g_k(q)| \le c_0 \sum_{k=1}^{m} |p_k - q_k|;$$

2^0. $\varphi_k(x,y) \in C^{2+\alpha}(\partial D)$;
3^0. $h_k(x) \in C^{2+\alpha}(\bar{D}')$;
4^0. $\gamma_1(x), \ \gamma_2(x) \in C^{1+\alpha}(\bar{D}')$.

Definition 1. *The pair of functions* $\left\{f_k(x), u_k(x,y), k = \overline{1,m}\right\}$ *is said to be the solution of the problem* (1)-(3) *if*
1) $f_k(x) \in C(\bar{D}')$,
2) $u_k(x,y) \in C^2(D) \bigcap C(\bar{D})$,
3) *the relations* (1)-(3) *are satisfied for these functions in the usual sense.*

The uniqueness theorem and the "conditional" stability estimate for the solution of inverse problems are most important issues when treating their well-posedness.

Let $\left\{f_k^i(x), u_k^i(x,y), k = \overline{1,m}\right\}$ be a solution of the problem (1)-(3) corresponding to the data $g_k^i(u^i)$, $\varphi_k^i(x,y)$, $h_k^i(x)$, $i = 1, 2$.

Definition 2. *We say that the solution of the problem* (1)-(3) *is stable if for any* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that for*

$$\left\|g^1 - g^2\right\|_0 \le \delta, \left\|\varphi^1 - \varphi^2\right\|_2 \le \delta, \left\|h^1 - h^2\right\|_2 \le \delta$$

the inequality $\left\|u^1 - u^2\right\|_0 + \left\|f^1 - f^2\right\|_0 \le \varepsilon$ *is true.*

Theorem 1. *Let*

1. *the functions* $g_k^i(u)$, $\varphi_k^i(x,y)$, $h_k^i(x)$, $k=\overline{1,m}$, $i=1,2$ *satisfy the conditions* 1^0–3^0, *respectively;*

2. *there exist the solution* $\left\{f_k^i(x), u_k^i(x,y), k=\overline{1,m}, i=1,2\right\}$ *of the problem* (1)-(3) *in the sense of Definition 1 which belongs to*

$K_\alpha=\left\{(f,u)\left|f(x)\in C^\alpha\left(\bar{D}'\right),\quad u(x,y)\in C^{2+\alpha}\left(\bar{D}\right),\ |f(x)|\le c_1,\ x\in\bar{D}',\ \left|D^l_{x,y}u(x,y)\right|\le\right.\right.$ $c_2,\ l=0,1,2,\quad(x,y)\in\bar{D},\ c_1,c_2$ *are some constant numbers*$\}$;

Then the solution of the problem (1)-(3) *for* $(x,y)\in\bar{D}$ *is unique and the "conditional" stability estimate is valid:*

$$\left\|f^1-f^2\right\|_0+\left\|u^1-u^2\right\|_0\le c_3\left[\left\|g^1-g^2\right\|_0+\left\|\varphi^1-\varphi^2\right\|_2+\left\|h^1-h^2\right\|_2\right], \tag{4}$$

where $c_3>0$ *is independent of the data of the problem* (1)-(3) *and the set* K_α.

Proof.

Integrating the equation (1) with respect to y over the interval $(\gamma_1(x),\ \gamma_2(x))$ and taking into account the conditions 1^0-3^0, we get:

$$f_k(x)=\left[\Delta h_k(x)+u_{ky}(x,\ \gamma_2(x))-u_{ky}(x,\gamma_1(x))\right]\left(\int_{\gamma_1(x)}^{\gamma_2(x)}g_k(u)\,dx\right)^{-1}. \tag{5}$$

Writing out the system (1), (2), (5) for the functions $g_k^1\left(u^1\right)$, $\varphi_k^1(x,y)$, $h_k^1(x)$ and $g_k^2\left(u^2\right)$, $\varphi_k^2(x,y)$, $h_k^2(x)$, respectively, and then subtracting the corresponding relations, we get that the functions $z_k(x,y)=u_k^1(x,y)-u_k^2(x,y)$, $\lambda_k(x)=f_k^1(x)-f_k^2(x)$, $\delta_{1k}(u)=g_k^1(u)-g_k^2(u)$, $\delta_{2k}(x,y)=\varphi_k^1(x,y)-\varphi_k^2(x,y)$, $\delta_{3k}(x)=h_k^1(x)-h_k^2(x)$ satisfy the conditions of the following system:

$$\Delta z_k(x,y)=\lambda_k(x)\,g_k^1\left(u^1\right)+F_k\left(x,u^1,u^2\right),\ (x,y)\in D, \tag{6}$$

$$z_k(x,y)=\delta_{2k}(x,y),\ (x,y)\in\partial D, \tag{7}$$

$$\lambda_k(x)=\left[z_{kx}(x,\gamma_2(x))-z_{kx}(x,\gamma_1(x))\right]\left(\int_{\gamma_1(x)}^{\gamma_2(x)}g_k^1\left(u^1\right)dy\right)^{-1}+H_k(x),\ x\in\bar{D}', \tag{8}$$

where

$$F_k\left(x,u^1,u^2\right)=f_k^2(x)\left[\delta_{1k}\left(u^1\right)+g_k^2\left(u^1\right)-g_k^2\left(u^2\right)\right],$$

$$H_k(x) = \Delta\delta_{3k}(x)\left(\int_{\gamma_1(x)}^{\gamma_2(x)} g_k^1(u')\,dy\right)^{-1} +$$

$$+\left\{\left[u_{ky}^2(x,\gamma_2(x)) - u_{ky}^2(x,\gamma_1(x)) + \Delta h_k^2(x)\right]\int_{\gamma_1(x)}^{\gamma_2(x)}\left[g_k^1(u^2) - g_k^1(u^1) - \delta_{3k}(u^2)\right]dy\right\}\times$$

$$\times\left\{\int_{\gamma_1(x)}^{\gamma_2(x)} g_k^1(u^1)\,dy\int_{\gamma_1(x)}^{\gamma_2(x)} g(u^2)\,dy\right\}^{-1}.$$

Let the functions $\Phi_k^i(x,y) \in C^{2+\alpha}(\bar{D})$, $k=\overline{1,m}$, $i=1,2$ be such that $\Phi_k^i(x,y) \equiv \varphi_k^i(x,y)$, $(x,y)\in\partial D$, $\|\Phi\|_2 \le \|\varphi\|_2$. Denote $\delta_{4k}(x,y) = \Phi_k^1(x,y) - \Phi_k^2(x,y)$, $k=\overline{1,m}$.

It is easy to show that the function $w_k(x,y) = z_k(x,y) - \delta_{4k}(x,y)$ satisfies the following system:

$$\Delta w_k(x,y) = \lambda_k(x)\,g_k^1(u^1) + F_k(x,u^1,u^2) - \Delta\delta_{4k}(x,y),\ \ (x,y)\in D, \tag{9}$$

$$w_k(x,y) = 0, (x,y)\in\partial D. \tag{10}$$

Under the assumptions of the theorem, the right side of the equation (9) is a Holderian function. In other words, the data of the problem (9), (10) of determining $w_k(x,y)$ satisfy the conditions of a theorem stated in [1, p. 157]. So, there exists a classical solution of the problem (9), (10), and it can be represented in the following form [1, p. 151]:

$$w_k(x,y) = \int_D G(x,y;\xi,\eta)\left[\lambda(\xi)\,g_k^1(u^1) + F_k(\xi,u^1,u^2) - \Delta\delta_{4k}(\xi,\eta)\,d\xi d\eta\right]d\xi d\eta,$$

where $G(x,y;\xi,\eta)$ is the Green function (9), (10). Hence for $z_k(x,y)$ we get:

$$z_k(x,y) = \delta_{4k}(x,y) + \int_D G(x,y;\xi,\eta)\left[\lambda_k(\xi)\,g_k^1(u^1) + F_k(y,u^1,u^2) - \Delta\delta_{4k}(\xi,\eta)\right]d\xi d\eta. \tag{11}$$

Estimate the function $z_k(x,y)$, $k=\overline{1,m}$. From (11) we get:

$$|z_k(x,y)| \le |\delta_{4k}(x,y)| + \int_D |G(x,y;\xi,\eta)|\times$$

$$\times\left[|\lambda_k(\xi)|\left|g_k^1(u^1)\right| + \left|F_k(\xi,u^1,u^2)\right| + |\Delta\delta_{4k}(\xi,\eta)|\right]d\xi d\eta, \tag{12}$$

By definition, the first summand on the right side of (12) is estimated as follows:

$$\left|\delta_{4k}\left(x,y\right)\right| = \left|\Phi_k^1\left(x,y\right) - \Phi_k^2\left(x,y\right)\right| \leq \left\|\Phi^1 - \Phi^2\right\|_0, (x,y) \in \bar{D}. \tag{13}$$

The integrand expression in (12) is estimated with the consideration of theorem conditions as follows:

$$\left|\lambda_k\left(x\right)\right|\left|g_k^1\left(u^1\right)\right| + \left|F_k\left(x,u^1,u^2\right)\right| + \left|\Delta\delta_{4k}\left(x,y\right)\right| \leq \left|\lambda_k\left(x\right)\right|\left|g_k^1\left(u^1\right)\right| + \left|f_k^2\left(x\right)\right|\left|\delta_{1k}\left(u^1\right)\right| +$$

$$+\left|f_k^2\left(x\right)\right|\left|g_k^2\left(u^1\right) - g_k^2\left(u^2\right)\right| \leq c_4\left|\lambda_k\left(x\right)\right| + c_5\left|z_k\left(x,y\right)\right| + c_6\left\|g^1 - g^2\right\|_0, (x,y) \in \bar{D}, \tag{14}$$

where $c_4, c_5, c_6 > 0$ depend on the data of the problem (1)-(3) and the set K_α.

Taking into account the inequalities (13), (14) and the estimate for the Green function [1, p. 153]

$$\left|D_x^l G\left(x,\xi\right)\right| \leq c_7\left|x - \xi\right|^{-n+2-l},\ l = 0,1,2 \tag{15}$$

in (12), we get

$$\left|z_k\left(x,y\right)\right| \leq c_8\left[\left\|\delta_1\right\|_0 + \left\|\delta_4\right\|_2\right] + c_9 mesD \cdot \theta,\ \ (x,y) \in \bar{D}, \tag{16}$$

where $c_8, c_9 > 0$ depend on the data of the problem (1)-(3) and the set K_α, $\theta = \left\|u^1 - u^2\right\|_0 + \left\|f^1 - f^2\right\|_0$.

By virtue of theorem conditions, differentiating (11) with respect to y we get:

$$z_{ky}\left(x,y\right) = \delta_{4ky}\left(x,y\right) + \int_D G_y\left(x,y;\xi,\eta\right) \times$$

$$\times\left[\lambda_k\left(\xi\right) g_k^1\left(u^1\right) + F_k\left(\xi,u^1,u^2\right) - \Delta\delta_{4k}\left(\xi,\eta\right)\right] d\xi d\eta.$$

Taking into account the derivatives of the Green functions (15), inequalities (3), (4), and proceeding in the same way as in deriving the inequality (16), we get:

$$\left|z_k\left(x,y\right)\right| \leq c_{10}\left[\left\|\delta_1\right\|_0 + \left\|\delta_4\right\|_2\right] + c_{11} mesD \cdot \theta,\ \ (x,y) \in \bar{D}, \tag{17}$$

where $c_{10}, c_{11} > 0$ depend on the data of the problem (1)-(3) and the set K_α.

Now estimate $\lambda_k\left(x\right)$. From (8) we have:

$$\left|\lambda_k\left(x\right)\right| \leq \left[\left|z_{ky}\left(x,\gamma_2\left(x\right)\right)\right| + \left|z_{ky}\left(x,\gamma_1\left(x\right)\right)\right|\right]\left|\int_{\gamma_1(x)}^{\gamma_2(x)} g_k^1\left(u^1\right) dy\right|^{-1} +$$

$$+\left|\Delta\delta_{3k}(x)\right|\left|\int_{\gamma_1(x)}^{\gamma_2(x)} g_k^1\left(u^1\right)dy\right|^{-1}+$$

$$+\left\{\left[\left|u_{ky}^2\left(x,\gamma_2(x)\right)\right|+\left|u_{ky}^2\left(x,\gamma_1(x)\right)\right|+\left|\Delta h_k^2(x)\right|\right]\right.$$

$$\left.\int_{\gamma_1(x)}^{\gamma_2(x)}\left[\left|g_k^1\left(u^2\right)-g_k^1\left(u^1\right)\right|+\left|\delta_{3k}\left(u^2\right)\right|\right]dy\right\}\times$$

$$\times\left\{\int_{\gamma_1(x)}^{\gamma_2(x)} g_k^1\left(u^1\right)dy\cdot\int_{\gamma_1(x)}^{\gamma_2(x)} g_k^2\left(u^2\right)dy\right\}^{-1}.$$

Taking into account the inequality (17), using theorem conditions and proceeding in the same way as in deriving the inequality (16), we get:

$$\left|\lambda_k(x)\right|\le c_{12}\left[\left\|\delta_1\right\|_0+\left\|\delta_3\right\|_2\right]+c_{13} mesD\cdot\theta,\ \ (x,y)\in\bar{D}^1, \tag{18}$$

where $c_{12}, c_{13}>0$ depend on the data of the problem (1)-(3) and the set K_α.

Inequalities (16) and (18) are satisfied for any $(x,y)\in\bar{D}$. So they should be satisfied for maximum values of the left-hand sides as well:

$$\|z\|_0\le c_8\left[\left\|\delta_1\right\|_0+\left\|\delta_4\right\|_2\right]+c_9 mesD\cdot\theta,$$

$$\|\lambda\|_0\le c_{12}\left[\left\|\delta_1\right\|_0+\left\|\delta_4\right\|_2+\left\|\delta_3\right\|_2\right]+c_{13} mesD\cdot\theta$$

Inequalities (17), (18) are satisfied for every $(x,y)\in\bar{D}$. Therefore, they should be satisfied also for maximum values of the left-hand sides. Thus, we get

$$\theta\le c_{14}\left[\left\|\delta_1\right\|_0+\left\|\delta_3\right\|_2+\left\|\delta_4\right\|_2\right]+c_{15} mesD\cdot\theta,\ \ (x,y)\in\bar{D}. \tag{19}$$

If $c_{15} mesD<1$, then from inequality (19) we get the estimate (4). If otherwise, we proceed as follows: divide the domain D into the subspaces D_i $\left(i=\overline{1,n}\right)$ with smooth boundaries in such a way that $\bar{D}_i\subset\bar{D}$, $\bigcup_{i=1}^N\bar{D}_i=\bar{D}$ and $c_{15}\max\limits_{1\le i\le N}\{mesD_i\}<1$. Define the finite functions with supports D_i as follows:

$$\theta_i(x,y)=\begin{cases}\frac{1}{N}, & (x,y)\in D_i\\ 0, & (x,y)\in R^n\backslash\bar{D}_i,\quad i=\overline{1,N},\end{cases} \tag{20}$$

Using (20), we represent the functions $u_k(x,y)$ and $\lambda_k(x)$ in the following form:

$$\begin{array}{l}\sum_{q=1}^N u_{kq}(x,y)=u_k(x,y)\,,\ \ u_{kq}(x,y)=u_k(x,y)\,\theta_q(x,y)\,,\\ \sum_{q=1}^N\lambda_{kq}(x)=\lambda_k(x)\,,\ \ \lambda_k(x)=\lambda_{kq}(x)\,\frac{1}{N}.\end{array}$$

Further, multiplying the system (6), (7), (8) by $\theta_i(x,y)$, $i=\overline{1,N}$, and proceeding in the same way as in deriving the inequality (19), we get:

$$\|z_q\|_0 + \|\lambda_q\|_0 \le c_{14}\left[\|\delta_1\|_0 + \|\delta_3\|_2 + \|\delta_4\|_2\right] + c_{15} mesD \cdot \theta, \quad q=\overline{1,N}, \quad (x,y)\in \bar{D}.$$

Combining the last inequalities, we get the estimate (4). The uniqueness of the solution of (1)-(3) follows from (4) with

$$g_k^1(u) = g_k^2(u), \quad \varphi_k^1(x,y) = \varphi_k^2(x,y), \quad h_k^1(x) = h_k^2(x).$$

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