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Characterizations for the Nonsingular Integral Operator and its Commutators on Generalized Orlicz-Morrey Spaces

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Abstract. We show continuity in generalized Orlicz-Morrey spaces $M^{\Phi,\varphi}(\mathbb{R}^n_+)$ of nonsingular integral operators and its commutators with BMO functions. We shall give necessary and sufficient conditions for the boundedness of the nonsingular integral operator and its commutators on generalized Orlicz-Morrey spaces $M^{\Phi,\varphi}(\mathbb{R}^n_+)$.

Key Words and Phrases: generalized Orlicz-Morrey spaces, nonsingular integral, commutator, BMO.

2010 Mathematics Subject Classifications: 42B20, 42B35

1. Introduction

The classical Morrey spaces were introduced by Morrey [26] to study the local behavior of solutions to second-order elliptic partial differential equations. Although such spaces allow to describe local properties of functions better than Lebesgue spaces, they have some unpleasant issues. It is well known that Morrey spaces are non separable and that the usual classes of nice functions are not dense in such spaces. Moreover, various Morrey spaces are defined in the process of study. Mizuhara [25] and Nakai [27] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$. Later, Guliyev [10] defined the generalized Morrey spaces $M^{p,\varphi}$ with normalized norm

$$\|f\|_{M^{p,\varphi}} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-1/p} \|f\|_{L^p(B(x,r))},$$

where the function φ is a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. Here and everywhere in the sequel $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius r centered at x and $|B(x,r)| = v_n r^n$ is its Lebesgue measure, where v_n is the volume of the unit ball in $\mathbb{R}^n$.

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The Orlicz spaces were first introduced by Orlicz in [31, 32] as generalizations of Lebesgue spaces $L^p(\mathbb{R}^n)$. Since then, the theory of Orlicz spaces themselves has been well developed and the spaces have been widely used in probability, statistics, potential theory, partial differential equations, as well as harmonic analysis and some other fields of analysis.

In [6], the generalized Orlicz-Morrey space $M^{\Phi,\varphi}(\mathbb{R}^n)$ was introduced to unify Orlicz and generalized Morrey spaces. Other definitions of generalized Orlicz-Morrey spaces can be found in [28] and [34]. In words of [16], our generalized Orlicz-Morrey space is the third kind and the ones in [28] and [34] are the first kind and the second kind, respectively. According to the examples in [9], one can say that the generalized Orlicz-Morrey spaces of the first kind and the third kind are different and that second kind and third kind are different. However, we do not know the relation between the first and the second kind.

Note that, Orlicz-Morrey spaces unify Orlicz and generalized Morrey spaces. We extend some results on generalized Morrey space in the papers [1, 8, 10, 12, 13, 17, 18] to the case of Orlicz-Morrey space in [6, 14, 15, 16].

As based on the results of [10, 12], the following conditions were introduced in [6] (see, also [14]) for the boundedness of the singular integral operators on $M^{\Phi,\varphi}(\mathbb{R}^n)$:

$$\int_r^{\infty} \Big(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)} \Big) \, \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} \le C \, \varphi_2(x,r), \tag{1}$$

where C does not depend on x and r.

Consider the half-space $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0,\infty)$. For $x = (x', x_n) \in \mathbb{R}^n_+$, let $\tilde{x} = (x', -x_n)$ be the "reflected point". Let $x \in \mathbb{R}^n_+$. The nonsingular integral operator $\widetilde{T}$ is defined by

$$\widetilde{T} f(x) = \int_{\mathbb{R}^n_+} \frac{|f(y)|}{|\tilde{x} - y|^n} \, dy, \quad \tilde{x} = (x', -x_n). \tag{2}$$

The commutators generated by $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and the operator $\widetilde{T}$ are defined by

$$[b, \widetilde{T}] f(x) = \int_{\mathbb{R}^n_+} \frac{b(x) - b(y)}{|\tilde{x} - y|^n} f(y) dy.$$

The operator $|b, \widetilde{T}|$ is defined by

$$|b, \widetilde{T}| f(x) = \int_{\mathbb{R}^n_+} \frac{|b(x) - b(y)|}{|\tilde{x} - y|^n} f(y) dy.$$

The operator $\widetilde{T}$ and its commutator appear in [4] in connection with boundary estimates for solutions to elliptic equations.

Therefore, the purpose of this paper is mainly to study the boundedness of the nonsingular integral operator $\widetilde{T}$ and its commutators $[b, \widetilde{T}]$ on generalized Orlicz-Morrey spaces of the third kind $M^{\Phi,\varphi}(\mathbb{R}^n_+)$.

A function $\varphi : (0,\infty) \to (0,\infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C > 0$ such that

$$\varphi(r) \le C\varphi(s) \qquad (\text{resp. } \varphi(r) \ge C\varphi(s)) \quad \text{for } r \le s.$$

For a Young function Φ, we denote by $\mathcal{G}_\Phi$ the set of all decreasing functions $\varphi : (0,\infty) \to (0,\infty)$ such that $t \in (0,\infty) \mapsto \Phi^{-1}(t^{-n})\varphi(t)^{-1}$ is almost decreasing.

The following results are the fundamental theorems in this paper:

Theorem 1. *Let $\Phi \in \Delta'$ and $\varphi_1, \varphi_2 \in \Omega_\Phi$.*

1. *The condition* (1) *is sufficient for the boundedness of $\widetilde{T}$ from $M^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ to $WM^{\Phi,\varphi_2}(\mathbb{R}^n_+)$. If, in addition, $\Phi \in \nabla_2$, then the condition* (1) *is sufficient for the boundedness of $\widetilde{T}$ from $M^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ to $M^{\Phi,\varphi_2}(\mathbb{R}^n_+)$.*

2. *If $\varphi_1 \in \mathcal{G}_\Phi$, then the condition*

$$\varphi_1(x,r) \le C\varphi_2(x,r), \tag{3}$$

where C does not depend on x and r, is necessary for the boundedness of $\widetilde{T}$ from $M^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ to $WM^{\Phi,\varphi_2}(\mathbb{R}^n_+)$ and from $M^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ to $M^{\Phi,\varphi_2}(\mathbb{R}^n_+)$.

3. *If $\varphi_1 \in \mathcal{G}_\Phi$ satisfies the regularity type condition*

$$\int_t^\infty \varphi_1(r)\frac{dr}{r} \le C\varphi_1(t), \tag{4}$$

for all $t > 0$, where $C > 0$ does not depend on t, then the condition (3) *is necessary and sufficient for the boundedness of $\widetilde{T}$ from $M^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ to $WM^{\Phi,\varphi_2}(\mathbb{R}^n_+)$. If, in addition, $\Phi \in \nabla_2$, then the condition* (3) *is necessary and sufficient for the boundedness of $\widetilde{T}$ from $M^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ to $M^{\Phi,\varphi_2}(\mathbb{R}^n_+)$.*

If we take $\Phi(t) = t^p$, $p \in [1,\infty)$ in Theorem 1, we get the following new result for generalized Morrey spaces.

Corollary 2. *Let $p \in [1,\infty)$ and $\varphi_1, \varphi_2 \in \Omega_p \equiv \Omega_{t^p}$.*

1. *The condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt \le C\varphi_2(r), \tag{5}$$

for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of $\widetilde{T}$ from $M^{p,\varphi_1}(\mathbb{R}^n_+)$ to $WM^{p,\varphi_2}(\mathbb{R}^n_+)$. If $1 < p < \infty$, then the condition (5) *is sufficient for the boundedness of $\widetilde{T}$ from $M^{p,\varphi_1}(\mathbb{R}^n_+)$ to $M^{p,\varphi_2}(\mathbb{R}^n_+)$.*

2. *If* $\varphi_1 \in \mathcal{G}_p$, *then the condition* (3) *is necessary for the boundedness of* $\widetilde{T}$ *from* $M^{p,\varphi_1}(\mathbb{R}^n_+)$ *to* $WM^{p,\varphi_2}(\mathbb{R}^n_+)$ *and from* $M^{p,\varphi_1}(\mathbb{R}^n_+)$ *to* $M^{p,\varphi_2}(\mathbb{R}^n_+)$.

3. *If* $\varphi_1 \in \mathcal{G}_p$ *satisfies the regularity condition* (4), *then the condition* (3) *is necessary and sufficient for the boundedness of* $\widetilde{T}$ *from* $M^{p,\varphi_1}(\mathbb{R}^n_+)$ *to* $WM^{p,\varphi_2}(\mathbb{R}^n_+)$. *If, in addition,* $1 < p < \infty$, *then the condition* (3) *is necessary and sufficient for the boundedness of* $\widetilde{T}$ *from* $M^{p,\varphi_1}(\mathbb{R}^n_+)$ *to* $M^{p,\varphi_2}(\mathbb{R}^n_+)$.

Theorem 3. *Let* $b \in BMO(\mathbb{R}^n_+)$, $\Phi \in \Delta'$ *and* $\varphi_1, \varphi_2 \in \Omega_\Phi$.

1. *If* $\Phi \in \nabla_2$, *then the condition*

$$\int_r^\infty \Big(1+\ln\frac{t}{r}\Big)\Big(\operatorname*{ess\,inf}_{t<s<\infty}\frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)}\Big)\,\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t} \le C\,\varphi_2(x,r), \tag{6}$$

where C *does not depend on* x *and* r, *is sufficient for the boundedness of* $|b,\widetilde{T}|$ *from* $M^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ *to* $M^{\Phi,\varphi_2}(\mathbb{R}^n_+)$.

2. *If* $\varphi_1 \in \mathcal{G}_\Phi$, *then the condition* (3) *is necessary for the boundedness of* $|b,\widetilde{T}|$ *from* $M^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ *to* $M^{\Phi,\varphi_2}(\mathbb{R}^n_+)$.

3. *If* $\Phi \in \nabla_2$ *and* $\varphi_1 \in \mathcal{G}_\Phi$ *satisfies the regularity type condition*

$$\int_t^\infty \Big(1+\ln\frac{t}{r}\Big)\varphi_1(r)\frac{dr}{r} \le C\varphi_1(t), \tag{7}$$

for all $t > 0$, *where* $C > 0$ *does not depend on* t, *then the condition* (3) *is necessary and sufficient for the boundedness of* $|b,\widetilde{T}|$ *from* $M^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ *to* $M^{\Phi,\varphi_2}(\mathbb{R}^n_+)$.

If we take $\Phi(t) = t^p$, $p \in [1,\infty)$ in Theorem 3, we get the following new result for generalized Morrey spaces.

Corollary 4. *Let* $p \in [1,\infty)$, $\varphi_1, \varphi_2 \in \Omega_p$ *and* $b \in BMO(\mathbb{R}^n_+)$.

1. *If* $1 < p < \infty$, *then the condition*

$$\int_r^\infty \Big(1+\ln\frac{t}{r}\Big)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt \le C\varphi_2(r),$$

for all $r > 0$, *where* $C > 0$ *does not depend on* r, *is sufficient for the boundedness of* $|b,\widetilde{T}|$ *from* $M^{p,\varphi_1}(\mathbb{R}^n_+)$ *to* $M^{p,\varphi_2}(\mathbb{R}^n_+)$.

2. *If* $\varphi_1 \in \mathcal{G}_p$, *then the condition* (3) *is necessary for the boundedness of* $|b,\widetilde{T}|$ *from* $M^{p,\varphi_1}(\mathbb{R}^n_+)$ *to* $M^{p,\varphi_2}(\mathbb{R}^n_+)$.

3. *If* $1 < p < \infty$ *and* $\varphi_1 \in \mathcal{G}_p$ *satisfies the regularity type condition* (7), *then the condition* (3) *is necessary and sufficient for the boundedness of* $|b,\widetilde{T}|$ *from* $M^{p,\varphi_1}(\mathbb{R}^n_+)$ *to* $M^{p,\varphi_2}(\mathbb{R}^n_+)$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Definitions and Preliminary Results

2.1. On Young Functions and Orlicz Spaces

We recall the definition of Young functions.

Definition 5. *A function* $\Phi : [0,\infty) \to [0,\infty]$ *is called a Young function if* Φ *is convex, left-continuous,* $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ *and* $\lim\limits_{r\to\infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \ge s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \le r < \infty.$$

It is well known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0, \tag{8}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\} & , \quad r \in [0,\infty) \\ \infty & , \quad r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \le k\Phi(r) \text{ for } r > 0$$

for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \qquad r \ge 0,$$

for some $k > 1$.

Definition 6. *(Orlicz space). For a Young function Φ, the set*

$$L^{\Phi}(\mathbb{R}^n_+) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n_+) : \int_{\mathbb{R}^n_+} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L^{\Phi}(\mathbb{R}^n_+) = L^p(\mathbb{R}^n_+)$. If $\Phi(r) = 0$, $(0 \le r \le 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L^{\Phi}(\mathbb{R}^n_+) = L^{\infty}(\mathbb{R}^n_+)$. The space $L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^n_+)$ is defined as the set of all functions f such that $f\chi_B \in L^{\Phi}(\mathbb{R}^n_+)$ for all balls $B \subset \mathbb{R}^n_+$.

$L^{\Phi}(\mathbb{R}^n_+)$ is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(\mathbb{R}^n_+)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n_+} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \le 1 \right\}.$$

We note that

$$\int_{\mathbb{R}^n_+} \Phi\Big(\frac{|f(x)|}{\|f\|_{L^{\Phi}(\mathbb{R}^n_+)}}\Big)dx \le 1. \tag{9}$$

The weak Orlicz space

$$WL^{\Phi}(\mathbb{R}^n_+) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n_+) : \|f\|_{WL^{\Phi}(\mathbb{R}^n_+)} < +\infty\}$$

is defined by the norm

$$\|f\|_{WL^{\Phi}(\mathbb{R}^n_+)} = \inf \Big\{ \lambda > 0 \ : \ \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda}, \ t\Big) \ \le 1 \Big\}.$$

Lemma 7. *([22], Lemma 1.3.2) Let $\Phi \in \Delta_2$. Then there exist $p > 1$ and $b > 1$ such that*

$$\frac{\Phi(t_2)}{t_2^p} \le \frac{b\Phi(t_1)}{t_1^p}$$

for $0 < t_1 < t_2$.

Lemma 8. *([33], Proposition 62.20) Let Φ be a Young function with canonical representation*

$$\Phi(t) = \int_0^t \varphi(s)ds, \quad t \ge 0.$$

(1) Assume that $\Phi \in \Delta_2$. More precisely $\Phi(2t) \le A\Phi(t)$ for some $A \ge 2$. If $p > 1 + \log_2 A$, then

$$\int_t^{\infty} \frac{\varphi(s)}{s^p} ds \lesssim \frac{\Phi(t)}{t^p}, \quad t > 0.$$

(2) Assume that $\Phi \in \nabla_2$. Then

$$\int_0^t \frac{\varphi(s)}{s} ds \lesssim \frac{\Phi(t)}{t}, \quad t > 0.$$

The following lemmas are valid.

Lemma 9. *[2, 24] Let Φ be a Young function and B a set in $\mathbb{R}^n_+$ with finite Lebesgue measure. Then*

$$\|\chi_B\|_{WL^\Phi(\mathbb{R}^n_+)} = \|\chi_B\|_{L^\Phi(\mathbb{R}^n_+)} = \frac{1}{\Phi^{-1}\left(|B|^{-1}\right)}.$$

Lemma 10. *For a Young function Φ and all balls B in $\mathbb{R}^n_+$, the following inequality is valid*

$$\|f\|_{L^1(B)} \le 2|B|\Phi^{-1}\left(|B|^{-1}\right)\|f\|_{L^\Phi(B)}.$$

2.2. Generalized Orlicz-Morrey Space

Various versions of generalized Orlicz-Morrey spaces were introduced in [28], [34] and [6]. We used the definition of [6] which runs as follows.

We now define generalized Orlicz-Morrey spaces of the third kind. The *generalized Orlicz-Morrey space $M^{\Phi,\phi}(\mathbb{R}^n_+)$ of the third kind* is defined as the set of all measurable functions f for which the norm

$$\|f\|_{M^{\Phi,\phi}(\mathbb{R}^n_+)} \equiv \sup_{x\in\mathbb{R}^n_+,\, r>0} \frac{1}{\phi(x,r)}\Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right)\|f\|_{L^\Phi(\mathcal{B}^+(x,r))}$$

is finite, where $\mathcal{B}^+(x,r) = B(x,r)\cap\mathbb{R}^n_+$. Also by $WM^{\Phi,\varphi}(\mathbb{R}^n_+)$ we denote the *weak generalized Orlicz-Morrey space of the third kind* of all functions $f \in WL^\Phi_{\rm loc}(\mathbb{R}^n_+)$ for which

$$\|f\|_{WM^{\Phi,\varphi}(\mathbb{R}^n_+)} = \sup_{x\in\mathbb{R}^n_+, r>0} \varphi(x,r)^{-1}\Phi^{-1}(|\mathcal{B}^+(x,r)|^{-1})\,\|f\|_{WL^\Phi(\mathcal{B}^+(x,r))} < \infty,$$

where $WL^\Phi(\mathcal{B}^+(x,r))$ denotes the weak L^Φ-space of measurable functions f for which

$$\|f\|_{WL^\Phi(\mathcal{B}^+(x,r))} \equiv \|f\chi_{\mathcal{B}^+(x,r)}\|_{WL^\Phi(\mathbb{R}^n_+)}.$$

Note that $M^{\Phi,\phi}(\mathbb{R}^n_+)$ covers many classical function spaces.

Example 11. *Let $1 \le q \le p < \infty$ and $\Phi \in \Delta_2 \cap \nabla_2$. From the following special cases, we see that our results will cover the Lebesgue space $L^p(\mathbb{R}^n_+)$, the classical Morrey space $M^p_q(\mathbb{R}^n_+)$, the generalized Morrey space $M^{\phi,p}(\mathbb{R}^n_+)$ and the Orlicz space $L^\Phi(\mathbb{R}^n_+)$ with norm coincidence:*

1. *If $\Phi(t) = t^p$ and $\phi(t) = t^{-\frac{n}{p}}$, then $M^{\Phi,\phi}(\mathbb{R}^n_+) = L^p(\mathbb{R}^n_+)$ with norm equivalence.*
2. *If $\Phi(t) = t^q$ and $\phi(t) = t^{-\frac{n}{p}}$, then $M^{\Phi,\phi}(\mathbb{R}^n_+)$, which is denoted by $M^p_q(\mathbb{R}^n_+)$, is the classical Morrey space.*

3. *If* $\Phi(t) = t^p$*, then* $M^{\Phi,\phi}(\mathbb{R}^n_+) = M^{p,\phi}(\mathbb{R}^n_+)$ *is the generalized Morrey space which was discussed in [10, 25, 27].*

4. *If* $\phi(t) = \Phi^{-1}(t^{-n})$*, then* $M^{\Phi,\phi}(\mathbb{R}^n_+) = L^{\Phi}(\mathbb{R}^n_+)$*, which is beyond the reach of generalized Orlicz-Morrey spaces of the second kind defined in [9] according to an example constructed in [34].*

Other definitions of generalized Orlicz-Morrey spaces can be found in [28, 29, 30, 9]. Therefore, our definition of generalized Orlicz-Morrey spaces here is named "third kind".

In the case $\varphi(x,r) = \frac{\Phi^{-1}\left(|B(x,r)|^{-1}\right)}{\Phi^{-1}\left(|B(x,r)|^{-\lambda/n}\right)}$, we get the Orlicz-Morrey space $\mathcal{M}^{\Phi,\lambda}(\mathbb{R}^n)$ from generalized Orlicz-Morrey space $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$. We refer to [7, Lemmas 2.8 and 2.9] for more information about Orlicz-Morrey spaces.

Lemma 12. *[7, Lemma 2.12] Let* Φ *be a Young function and* φ *be a positive measurable function on* $\mathbb{R}^n \times (0,\infty)$*.*

(i) If

$$\sup_{t<r<\infty} \frac{\Phi^{-1}(|B(x,r)|^{-1})}{\varphi(x,r)} = \infty \quad \text{for some } t>0 \text{ and for all } x \in \mathbb{R}^n, \tag{10}$$

then $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) = \Theta$*.*

(ii) If $\Phi \in \Delta'$ *and*

$$\sup_{0<r<\tau} \varphi(x,r)^{-1} = \infty \quad \text{for some } \tau>0 \text{ and for all } x \in \mathbb{R}^n, \tag{11}$$

then $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) = \Theta$*.*

Remark 13. *Let* Φ *be a Young function. We denote by* Ω_Φ *the sets of all positive measurable functions* φ *on* $\mathbb{R}^n \times (0,\infty)$ *such that for all* $t>0$*,*

$$\sup_{x\in\mathbb{R}^n} \left\| \frac{\Phi^{-1}(|B(x,r)|^{-1})}{\varphi(x,r)} \right\|_{L^\infty(t,\infty)} < \infty,$$

and

$$\sup_{x\in\mathbb{R}^n} \left\| \varphi(x,r)^{-1} \right\|_{L^\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 12, we always assume that $\varphi \in \Omega_\Phi$ *and* $\Phi \in \Delta'$*.*

The following lemma plays a key role in our main results.

Lemma 14. *Let $\mathcal{B}_0^+ := \mathcal{B}^+(x_0, r_0)$ be a ball in $\mathbb{R}^n_+$. If $\varphi \in \mathcal{G}_\Phi$, then there exists $C > 0$ such that*

$$\frac{1}{\varphi(r_0)} \le \|\chi_{\mathcal{B}_0^+}\|_{M^{\Phi,\varphi}(\mathbb{R}^n_+)} \le \frac{C}{\varphi(r_0)}.$$

Proof. Let $\mathcal{B}^+ = \mathcal{B}^+(x, r)$ denote an arbitrary ball in $\mathbb{R}^n_+$. By the definition and Lemma 9, it is easy to see that

$$\|\chi_{\mathcal{B}_0^+}\|_{M^{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n_+, r>0} \varphi(r)^{-1}\Phi^{-1}(|\mathcal{B}^+|^{-1}) \frac{1}{\Phi^{-1}(|\mathcal{B}^+ \cap \mathcal{B}_0^+|^{-1})}$$

$$\ge \varphi(r_0)^{-1}\Phi^{-1}(|\mathcal{B}_0^+|^{-1}) \frac{1}{\Phi^{-1}(|\mathcal{B}_0^+ \cap \mathcal{B}_0^+|^{-1})} = \frac{1}{\varphi(r_0)}.$$

Now if $r \le r_0$, then $\varphi(r_0) \le C\varphi(r)$ and

$$\varphi(r)^{-1}\Phi^{-1}(|\mathcal{B}^+|^{-1})\|\chi_{\mathcal{B}_0^+}\|_{L^\Phi(\mathcal{B}^+)} \le \frac{1}{\varphi(r)} \le \frac{C}{\varphi(r_0)}.$$

On the other hand, if $r \ge r_0$, then $\frac{\varphi(r_0)}{\Phi^{-1}(|\mathcal{B}_0^+|^{-1})} \le C\frac{\varphi(r)}{\Phi^{-1}(|\mathcal{B}^+|^{-1})}$ and

$$\varphi(r)^{-1}\Phi^{-1}(|\mathcal{B}^+|^{-1})\|\chi_{\mathcal{B}_0^+}\|_{L^\Phi(\mathcal{B}^+)} \le \frac{C}{\varphi(r_0)}.$$

This completes the proof.◄

3. Nonsingular integral operators in the Orlicz space $L^\Phi(\mathbb{R}^n_+)$

The following theorem was proved in [5].

Theorem 15. *Let $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^n_+)$. Then there exists a constant C_p independent of f, such that*

$$\|\widetilde{T}f\|_{L^p(\mathbb{R}^n_+)} \le C_p\|f\|_{L^p(\mathbb{R}^n_+)}, \quad 1 < p < \infty$$

and

$$\|\widetilde{T}f\|_{WL^1(\mathbb{R}^n_+)} \le C_1\|f\|_{L^1(\mathbb{R}^n_+)}.$$

Theorem 16. *Let Φ be a Young function and $\widetilde{T}$ be a nonsingular integral operator, defined by (2). If $\Phi \in \Delta_2 \bigcap \nabla_2$, then the operator $\widetilde{T}$ is bounded on $L^\Phi(\mathbb{R}^n_+)$ and if $\Phi \in \Delta_2$, then the operator $\widetilde{T}$ is bounded from $L^\Phi(\mathbb{R}^n_+)$ to $WL^\Phi(\mathbb{R}^n_+)$.*

Proof. First, let's prove that for $\Phi \in \Delta_2$ the nonsingular integral operator $\widetilde{T}$ is bounded from $L^\Phi(\mathbb{R}^n_+)$ to $WL^\Phi(\mathbb{R}^n_+)$.

We take $f \in L^\Phi(\mathbb{R}^n_+)$ satisfying $\|f\|_{L^\Phi} = 1$. Fix $\lambda > 0$ and define $f_1 = \chi_{\{|f|>\lambda\}} \cdot f$ and $f_2 = \chi_{\{|f|\le\lambda\}} \cdot f$. Then $f = f_1 + f_2$. We have

$$\big|\{x \in \mathbb{R}^n_+ : \big|\widetilde{T}f(x)\big| > \lambda\}\big| \le \big|\{x \in \mathbb{R}^n_+ : \big|\widetilde{T}f_1(x)\big| > \frac{\lambda}{2}\}\big| + \big|\{x \in \mathbb{R}^n_+ : \big|\widetilde{T}f_2(x)\big| > \frac{\lambda}{2}\}\big|$$

and

$$\begin{aligned}&\Phi(\lambda)\big|\{x \in \mathbb{R}^n_+ : \big|\widetilde{T}f(x)\big| > \lambda\}\big| \\ &\le |\Phi(\lambda)\{x \in \mathbb{R}^n_+ : \big|\widetilde{T}f_1(x)\big| > \frac{\lambda}{2}\}| + \Phi(\lambda)|\{x \in \mathbb{R}^n_+ : \big|\widetilde{T}f_2(x)\big| > \frac{\lambda}{2}\}|.\end{aligned}$$

We know that from the weak (1,1) boundedness and L^p, $p \in (1,\infty)$ boundedness of $\widetilde{T}$

$$\big|\{x \in \mathbb{R}^n_+ : \big|\widetilde{T}(\chi_{\{|f|>\lambda\}} \cdot f)(x)\big| > \lambda\}\big| \lesssim \frac{1}{\lambda}\int_{\{x\in\mathbb{R}^n_+ : |f(x)|>\lambda\}} |f(x)|dx$$

and

$$\big|\{x \in \mathbb{R}^n_+ : \big|\widetilde{T}(\chi_{\{|f|\le\lambda\}} \cdot f)(x)\big| > \lambda\}\big| \lesssim \frac{1}{\lambda^p}\int_{\{x\in\mathbb{R}^n_+ : |f(x)|\le\lambda\}} |f(x)|^p dx.$$

Since $f_1 \in WL^1(\mathbb{R}^n_+)$ and $\frac{\Phi(\lambda)}{\lambda}$ is increasing, we have

$$\begin{aligned}&\Phi(\lambda)\big|\{x \in \mathbb{R}^n_+ : |\widetilde{T}f_1(x)| > \frac{\lambda}{2}\}\big| \lesssim \frac{\Phi(\lambda)}{\lambda}\int_{\mathbb{R}^n_+} |f_1(x)|dx \\ &= \frac{\Phi(\lambda)}{\lambda}\int_{\{x\in\mathbb{R}^n_+ : |f(x)|>\lambda\}} |f(x)|dx \lesssim \int_{\mathbb{R}^n_+} |f(x)|\frac{\Phi(|f(x)|)}{|f(x)|}dx = \int_{\mathbb{R}^n_+} \Phi(|f(x)|)dx.\end{aligned}$$

By Lemma 7 and $f_2 \in L^p(\mathbb{R}^n_+)$ we have

$$\begin{aligned}&\Phi(\lambda)\,\big|\{x \in \mathbb{R}^n_+ : |\widetilde{T}f_2(x)| > \frac{\lambda}{2}\}\big| \lesssim \frac{\Phi(\lambda)}{\lambda^p}\int_{\mathbb{R}^n_+} |f_2(x)|^p dx \\ &= \frac{\Phi(\lambda)}{\lambda^p}\int_{\{x\in\mathbb{R}^n_+ : |f(x)|\le\lambda\}} |f(x)|^p dx \lesssim \int_{\mathbb{R}^n_+} |f(x)|^p\frac{\Phi(|f(x)|)}{|f(x)|^p}dx = \int_{\mathbb{R}^n_+} \Phi(|f(x)|)dx.\end{aligned}$$

Thus we get

$$\big|\{x \in \mathbb{R}^n_+ : \big|\widetilde{T}f(x)\big| > \lambda\}\big| \le \frac{C}{\Phi(\lambda)}\int_{\mathbb{R}^n_+} \Phi(|f(x)|)dx \le \frac{1}{\Phi\left(\frac{\lambda}{C\|f\|_{L^\Phi}}\right)}.$$

Since $\|\cdot\|_{L^\Phi}$ norm is homogeneous, this inequality is true for every $f \in L^\Phi(\mathbb{R}^n_+)$.

Now let's prove that for $\Phi \in \Delta_2 \bigcap \nabla_2$ the nonsingular integral operator $\widetilde{T}$ is bounded in $L^\Phi(\mathbb{R}^n_+)$.

Using the distribution functions, we have

$$\begin{aligned}\int_{\mathbb{R}^n_+} \Phi\left(\frac{|\widetilde{T}f(x)|}{\Lambda}\right) dx &= \frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{\lambda}{\Lambda}\right) |\{x \in \mathbb{R}^n_+ : |\widetilde{T}f(x)| > \lambda\}| d\lambda \\ &= \frac{2}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right) |\{x \in \mathbb{R}^n_+ : |\widetilde{T}f(x)| > 2\lambda\}| d\lambda.\end{aligned}$$

The following inequality is valid:

$$\begin{aligned}\left|\{x \in \mathbb{R}^n_+ : \left|\widetilde{T}f(x)\right| > 2\lambda\}\right| \le \left|\{x \in \mathbb{R}^n_+ : \left|\widetilde{T}(\chi_{\{|f|>\lambda\}} \cdot f)\right| > \lambda\}\right| \\ + |\{x \in \mathbb{R}^n_+ : |\widetilde{T}(\chi_{\{|f|\le\lambda\}} \cdot f)(x)| > \lambda\}|.\end{aligned}$$

Let $p > 1$ be sufficiently large. By the weak $(1,1)$ boundedness and L^p-boundedness of $\widetilde{T}$ (see Theorem 15) we have

$$\left|\{x \in \mathbb{R}^n_+ : \left|\widetilde{T}(\chi_{\{|f|>\lambda\}} \cdot f)(x)\right| > \lambda\}\right| \lesssim \frac{1}{\lambda}\int_{\{x\in\mathbb{R}^n_+ : |f(x)|>\lambda\}} |f(x)| dx$$

and

$$\left|\{x \in \mathbb{R}^n_+ : \left|\widetilde{T}(\chi_{\{|f|\le\lambda\}} \cdot f)(x)\right| > \lambda\}\right| \lesssim \frac{1}{\lambda^p}\int_{\{x\in\mathbb{R}^n_+ : |f(x)|\le\lambda\}} |f(x)|^p dx.$$

The same calculation as we used for the maximal operator works for the first term to obtain

$$\frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right) |\{x \in \mathbb{R}^n_+ : |\widetilde{T}(\chi_{\{|f|>\lambda\}} \cdot f)(x)| > \lambda\}| d\lambda \le \int_{\mathbb{R}^n_+} \Phi\left(\frac{c|f(x)|}{\Lambda}\right) dx. \tag{12}$$

As for the second term, a similar computation still works, but we use the fact that $\Phi \in \Delta_2$.

$$\begin{aligned}&\frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right) \left|\{x \in \mathbb{R}^n_+ : \left|\widetilde{T}(\chi_{\{|f|\le\lambda\}} \cdot f)(x)\right| > \lambda\}\right| d\lambda \\ &\lesssim \frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right) \left(\int_{\{x\in\mathbb{R}^n_+ : |f(x)|\le\lambda\}} |f(x)|^p dx\right) \frac{d\lambda}{\lambda^p} \\ &\lesssim \frac{1}{\Lambda}\int_{\mathbb{R}^n_+} |f(x)|^p \left(\int_{|f(x)|}^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right) \frac{d\lambda}{\lambda^p}\right) dx.\end{aligned}$$

Using Lemma 8 (1), we have

$$\frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right)\left|\{x\in\mathbb{R}^n_+ : \left|\widetilde{T}(\chi_{\{|f|\le\lambda\}}\cdot f)(x)\right|>\lambda\}\right|d\lambda$$

$$\lesssim \int_{\mathbb{R}^n_+}\Phi\left(\frac{2|f(x)|}{\Lambda}\right)dx \le \int_{\mathbb{R}^n_+}\Phi\left(\frac{c|f(x)|}{\Lambda}\right)dx. \tag{13}$$

Thus, putting together (12) and (13), we obtain

$$\int_{\mathbb{R}^n_+}\Phi\left(\frac{|\widetilde{T}f(x)|}{\Lambda}\right)dx \le \int_{\mathbb{R}^n_+}\Phi\left(\frac{c_0|f(x)|}{\Lambda}\right)dx.$$

Again we shall label the constant we want to distinguish from other less important constants. As before, if we set $\Lambda = c_2\|f\|_{L^\Phi}(\mathbb{R}^n_+)$, then we obtain

$$\int_{\mathbb{R}^n_+}\Phi\left(\frac{|\widetilde{T}f(x)|}{\Lambda}\right)dx \le 1.$$

Hence the operator norm of $\widetilde{T}$ is less than c_2.◀

4. Nonsingular integral operators in the space $M^{\Phi,\varphi}(\mathbb{R}^n_+)$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H^*_w g(t) := \int_t^\infty g(s)w(s)ds, \quad 0<t<\infty,$$

where w is a weight.

The following theorem was proved in [11, 13] and in the case $w=1$ in [3].

Theorem 17. *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)H^*_w g(t) \le C\sup_{t>0} v_1(t)g(t) \tag{14}$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t)\int_t^\infty \frac{w(s)ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{15}$$

Moreover, the value $C=B$ is the best constant for (14).

Remark 18. *In* (14) *and* (15) *it is assumed that* $\frac{1}{\infty} = 0$ *and* $0 \cdot \infty = 0$.

For any $x \in \mathbb{R}^n_+$ define $\tilde{x} = (x', -x_n)$ and recall that $x^0 = (x', 0)$. Also define $\mathcal{B}^+_r \equiv \mathcal{B}^+(x^0, r) = B(x^0, r) \cap \mathbb{R}^n_+$, $2\mathcal{B}^+_r = \mathcal{B}^+(x^0, 2r)$.

Lemma 19. *Let* Φ *be any Young function and* $f \in L^\Phi_{\rm loc}(\mathbb{R}^n_+)$ *be such that*

$$\int_1^\infty \|f\|_{L^\Phi(\mathcal{B}^+(x^0,t))}\, \Phi^{-1}\big(t^{-n}\big)\, \frac{dt}{t} < \infty \tag{16}$$

i) If $\Phi \in \Delta_2 \bigcap \nabla_2$*, then*

$$\|\widetilde{T} f\|_{L^\Phi(\mathcal{B}^+(x^0,r))} \le \frac{C}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^\infty \|f\|_{L^\Phi(\mathcal{B}^+(x^0,t))}\, \Phi^{-1}\big(t^{-n}\big)\, \frac{dt}{t}. \tag{17}$$

ii) If $\Phi \in \Delta_2$*, then*

$$\|\widetilde{T} f\|_{WL^\Phi(\mathcal{B}^+(x^0,r))} \le \frac{C}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^\infty \|f\|_{L^\Phi(\mathcal{B}^+(x^0,t))}\, \Phi^{-1}\big(t^{-n}\big)\, \frac{dt}{t}, \tag{18}$$

where the constants are independent of x^0*,* r *and* f*.*

Proof. *i)* Denote $\mathcal{B}^+_r = \mathcal{B}^+(x^0, r)$, $\mathcal{B}^+_t = \mathcal{B}^+(x^0, t)$ and for any $f \in L^\Phi_{\rm loc}(\mathbb{R}^n_+)$ write $f = f_1 + f_2$ with $f_1 = f\chi_{2\mathcal{B}^+_r}$ and $f_2 = f\chi_{(2\mathcal{B}^+_r)^c}$. Because of the Φ-boundedness of the operator $\widetilde{T}$ (see Theorem 16) and $f_1 \in L^\Phi(\mathbb{R}^n_+)$, we have

$$\|\widetilde{T} f_1\|_{L^\Phi(\mathcal{B}^+_r)} \le \|\widetilde{T} f_1\|_{L^\Phi(\mathbb{R}^n_+)} \le C\|f_1\|_{L^\Phi(\mathbb{R}^n_+)} = C\|f\|_{L^\Phi(2\mathcal{B}^+_r)}.$$

It is easy to see that for arbitrary points $x \in \mathcal{B}^+_r$ and $y \in (2\mathcal{B}^+_r)^c$ it holds

$$\frac{1}{2}|x^0 - y| \le |\tilde{x} - y| \le \frac{3}{2}|x^0 - y|. \tag{19}$$

Applying the Fubini theorem to $\widetilde{T} f_2$, we get

$$\begin{aligned}
|\widetilde{T} f_2(x)| &\le C \int_{\mathbb{R}^n_+} \frac{|f_2(y)|}{|\tilde{x} - y|^n}\, dy \\
&\le C \int_{(2\mathcal{B}^+_r)^c} \frac{|f(y)|}{|x^0 - y|^n} dy \le C \int_{(2\mathcal{B}^+_r)^c} |f(y)| dy \int_{|x^0-y|}^\infty \frac{dt}{t^{n+1}} \\
&\le C \int_{2r}^\infty \left(\int_{2r \le |x^0 - y| < t} |f(y)| dy \right) \frac{dt}{t^{n+1}} \le C \int_{2r}^\infty \left(\int_{\mathcal{B}^+_t} |f(y)| dy \right) \frac{dt}{t^{n+1}}.
\end{aligned}$$

Applying the Hölder's inequality (see, Lemma 10), we get

$$\begin{aligned}\int_{(2\mathcal{B}_r^+)^c} \frac{|f(y)|}{|x^0-y|^n} dy &\lesssim \int_{2r}^{\infty} \|f\|_{L^\Phi(\mathcal{B}_t^+)} \|1\|_{L^{\tilde\Phi}(\mathcal{B}_t^+)} \frac{dt}{t^{n+1}} \\ &= \int_{2r}^{\infty} \|f\|_{L^\Phi(\mathcal{B}_t^+)} \frac{1}{\widetilde{\Phi}^{-1}(|\mathcal{B}_t^+|^{-1})} \frac{dt}{t^{n+1}} \\ &\approx \int_{2r}^{\infty} \|f\|_{L^\Phi(\mathcal{B}_t^+)} \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}. \end{aligned} \tag{20}$$

Direct calculations give

$$\|\widetilde{T} f_2\|_{L^\Phi(\mathcal{B}_r^+)} \le C \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L^\Phi(\mathcal{B}_t^+)} \, \Phi^{-1}\big(t^{-n}\big) \, \frac{dt}{t} \tag{21}$$

and the last estimate holds for all $f \in L^\Phi(\mathbb{R}^n_+)$ satisfying (16). Thus

$$\|\widetilde{T} f\|_{L^\Phi(\mathcal{B}_r^+)} \le C \left(\|f\|_{L^\Phi(2\mathcal{B}_r^+)} + \frac{1}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L^\Phi(\mathcal{B}_t^+)} \, \Phi^{-1}\big(t^{-n}\big) \, \frac{dt}{t} \right). \tag{22}$$

On the other hand,

$$\begin{aligned}\|f\|_{L^\Phi(2\mathcal{B}_r)} &= \frac{C}{\Phi^{-1}\big(r^{-n}\big)} \|f\|_{L^\Phi(2\mathcal{B}_r)} \int_{2r}^{\infty} \Phi^{-1}\big(t^{-n}\big) \, \frac{dt}{t} \\ &\le \frac{C}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \|f\|_{L^\Phi(\mathcal{B}_t^+)} \, \Phi^{-1}\big(t^{-n}\big) \, \frac{dt}{t} \end{aligned} \tag{23}$$

which unified with (22) gives (17).

ii) Let now $f \in L^\Phi(\mathbb{R}^n_+)$. Then the weak (Φ,Φ)-boundedness of $\widetilde{T}$ (see Theorem 16) implies

$$\|\widetilde{T} f_1\|_{WL^\Phi(\mathcal{B}_r^+)} \le \|\widetilde{T} f_1\|_{WL^\Phi(\mathbb{R}^n_+)} \le C\|f_1\|_{L^\Phi(\mathbb{R}^n_+)} = C\|f\|_{L^\Phi(2\mathcal{B}_r^+)}.$$

The estimate (18) follows by (21).◄

For proving our main results, we need the following estimate.

Lemma 20. *If $\mathcal{B}_0^+ := \mathcal{B}^+(x_0, r_0)$, then $C \le \widetilde{T}\chi_{\mathcal{B}_0^+}(x)$ for every $x \in \mathcal{B}_0^+$.*

Proof. If $x, y \in \mathcal{B}_0^+$, then $|\tilde{x}-y| \le |\tilde{x}-x_0| + |y-x_0| < 2r_0$. We get $Cr_0^{-n} \le |\tilde{x}-y|^{-n}$. Therefore

$$\widetilde{T}\chi_{\mathcal{B}_0^+}(x) = \int_{\mathbb{R}^n} \chi_{\mathcal{B}_0^+}(y)|\tilde{x}-y|^{-n} dy = \int_{\mathcal{B}_0^+} |\tilde{x}-y|^{-n} dy \ge Cr_0^{-n}|\mathcal{B}_0^+| = C.$$

◄

Theorem 21. *Let Φ be any Young function, and $\varphi_1, \varphi_2 : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable functions satisfying* (1).
i) If $\Phi \in \Delta_2 \bigcap \nabla_2$, then it is bounded from $M^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ to $M^{\Phi,\varphi_2}(\mathbb{R}^n_+)$ and

$$\|\widetilde{T}f\|_{M^{\Phi,\varphi_2}(\mathbb{R}^n_+)} \le C\|f\|_{M^{\Phi,\varphi_1}(\mathbb{R}^n_+)}. \tag{24}$$

ii) If $\Phi \in \Delta_2$, then it is bounded from $M^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ to $WM^{\Phi,\varphi_2}(\mathbb{R}^n_+)$ and

$$\|\widetilde{T}f\|_{M^{\Phi,\varphi_2}(\mathbb{R}^n_+)} \le C\|f\|_{WM^{\Phi,\varphi_1}(\mathbb{R}^n_+)}$$

with constants independent of f.

Proof. Let $\widetilde{T}$ be bounded in $L^\Phi(\mathbb{R}^n_+)$. Then by Lemma 19 we have

$$\|\widetilde{T}f\|_{M^{\Phi,\varphi_2}(\mathbb{R}^n_+)} \le C \sup_{x^0,\, r>0} \varphi_2(x^0,r)^{-1} \int_r^\infty \|f\|_{L^\Phi(\mathcal{B}^+(x^0,t))}\, \Phi^{-1}\big(t^{-n}\big)\, \frac{dt}{t}.$$

Applying Theorem 17 to the above integral with

$$w(r) = \Phi^{-1}\big(r^{-n}\big),\ v_2(x^0,r) = \varphi_2(x^0,r)^{-1}, \qquad v_1(x^0,r) = \varphi_1(x^0,r)^{-1}\, \Phi^{-1}\big(r^{-n}\big),$$

$$g(x^0,r) = \|f\|_{L^\Phi(\mathcal{B}^+(x^0,r))}, \qquad H^*_w g(x^0,r) = \int_r^\infty \|f\|_{L^\Phi(\mathcal{B}^+(x^0,t))} w(t)dt,$$

where the condition (15) is equivalent to (1), we get

$$\|\widetilde{T}f\|_{M^{\Phi,\varphi_2}(\mathbb{R}^n_+)} \le C \sup_{x\in\mathbb{R}^n_+,\, r>0} \varphi_1(x^0,r)^{-1}\, \Phi^{-1}\big(r^{-n}\big)\, \|f\|_{L^\Phi(\mathcal{B}^+(x^0,r))} = C\|f\|_{M^{\Phi,\varphi_1}(\mathbb{R}^n_+)}.$$

The case $p=1$ is treated in the same manner making use of (18) and (15):

$$\begin{aligned}\|\widetilde{T}f\|_{WM^{1,\varphi_2}(\mathbb{R}^n_+)} &\le C \sup_{x^0\in\mathbb{R}^n_+,\, r>0} \varphi_2(x^0,r)^{-1} \int_r^\infty \|f\|_{L^\Phi(\mathcal{B}^+(x^0,t))}\, \Phi^{-1}\big(t^{-n}\big)\, \frac{dt}{t} \\ &= C \sup_{x^0,\, r>0} \varphi_1(x^0,r)^{-1}\, \Phi^{-1}\big(r^{-n}\big)\, \|f\|_{L^\Phi(\mathcal{B}^+(x^0,r))} = C\|f\|_{M^{\Phi,\varphi_1}(\mathbb{R}^n_+)}.\end{aligned}$$

◀

Proof of Theorem 1. The first part of the theorem follows from Lemma 19 and Theorem 21. We shall now prove the second part. Let $\mathcal{B}_0^+ = \mathcal{B}^+(x_0,r_0)$ and $x \in \mathcal{B}_0^+$. It is easy to see that $\widetilde{T}\chi_{\mathcal{B}_0^+}(x) = 1$ for every $x \in \mathcal{B}_0^+$. Therefore, by Lemmas 9 and 20

$$1 = \Phi^{-1}(w(\mathcal{B}_0^+)^{-1})\|\widetilde{T}\chi_{\mathcal{B}_0^+}\|_{L^\Phi(\mathcal{B}_0^+)} \le \varphi_2(\mathcal{B}_0^+)\|\widetilde{T}\chi_{\mathcal{B}_0^+}\|_{M^{\Phi,\varphi_2}}$$

$$\leq C\varphi_2(\mathcal{B}_0^+)\|\chi_{\mathcal{B}_0^+}\|_{M^{\Phi,\varphi_1}} \leq C\frac{\varphi_2(\mathcal{B}_0^+)}{\varphi_1(\mathcal{B}_0^+)}.$$

Since this is true for every $\mathcal{B}_0^+$, we are done.

The third statement of the theorem follows from the other statements of the theorem.◀

5. Commutators of nonsingular integrals in the space $M^{\Phi,\varphi}(\mathbb{R}_+^n)$

For a function $b \in BMO$ define the commutator $[b,\widetilde{T}] = \widetilde{T}[b,f] = b\widetilde{T}f - \widetilde{T}(bf)$. Our aim is to show the boundedness of $[b,\widetilde{T}]$ in $M^{\Phi,\varphi}(\mathbb{R}_+^n)$. For this goal we recall some well known properties of the BMO functions.

Lemma 22. (John-Nirenberg lemma, [19]) *Let $b \in BMO$ and $p \in (1,\infty)$. Then for any ball $\mathcal{B}$ there holds*

$$\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}}|b(y) - b_{\mathcal{B}}|^p dy\right)^{\frac{1}{p}} \leq C(p)\|b\|_*. \tag{25}$$

Definition 23. *A Young function Φ is said to be of upper type p (resp. lower type p) for some $p \in [0,\infty)$, if there exists a positive constant C such that, for all $t \in [1,\infty)$(resp. $t \in [0,1]$) and $s \in [0,\infty)$,*

$$\Phi(st) \leq Ct^p\Phi(s).$$

Remark 24. *We know that if Φ is lower type p_0 and upper type p_1 with $1 < p_0 \leq p_1 < \infty$, then $\Phi \in \Delta_2 \cap \nabla_2$. Conversely if $\Phi \in \Delta_2 \cap \nabla_2$, then Φ is lower type p_0 and upper type p_1 with $1 < p_0 \leq p_1 < \infty$ (see [22]).*

Before proving the main theorems, we need the following lemma.

Lemma 25. *[20] Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that*

$$|b_{\mathcal{B}_r} - b_{\mathcal{B}_t}| \leq C\|b\|_* \ln\frac{t}{r} \quad for \quad 0 < 2r < t,$$

where C is independent of b, x, r, and t.

In the following lemma which was proved in [15] we provide a generalization of the property (25) from L^p-norms to Orlicz norms.

Lemma 26. *Let $b \in BMO$ and Φ be a Young function. Let Φ be lower type p_0 and upper type p_1 with $1 \leq p_0 \leq p_1 < \infty$. Then*

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(r^{-n}\big)\left\|b(\cdot) - b_{B(x,r)}\right\|_{L^\Phi(B(x,r))}.$$

Remark 27. *Note that, the Lemma 26 for the variable exponent Lebesgue space $L^{p(\cdot)}$ case was proved in [21].*

Definition 28. *Let Φ be a Young function. Define*

$$b_\Phi := \inf_{t\in(0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}, \qquad b_\Phi := \sup_{t\in(0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}.$$

Remark 29. *It is known that $\Phi \in \Delta_2 \cap \nabla_2$ if and only if $1 < b_\Phi \le b_\Phi < \infty$ (See, for example, [23]).*

Remark 30. *Remark 29 and Remark 24 show us that a Young function Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$ if and only if $1 < b_\Phi \le b_\Phi < \infty$.*

To estimate the commutator, we use the same idea which we used in the proof of Lemma 19.

Lemma 31. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$ and $b \in BMO(\mathbb{R}^n_+)$. Suppose that for all $f \in L^\Phi_{\rm loc}(\mathbb{R}^n_+)$ and $r > 0$ it holds*

$$\int_1^\infty \Big(1+\ln\frac{t}{r}\Big)\, \|f\|_{L^\Phi(\mathcal{B}^+_t(x^0,t))}\, \Phi^{-1}\big(t^{-n}\big)\frac{dt}{t} < \infty. \tag{26}$$

Then

$$\|[b,\widetilde{T}]f\|_{L^\Phi(\mathcal{B}^+_r)} \le \frac{C\|b\|_*}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L^\Phi(\mathcal{B}^+(x^0,t))}\, \Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.$$

Proof. Decompose f as $f = f\chi_{2\mathcal{B}^+_r} + f\chi_{(2\mathcal{B}^+_r)^c} = f_1 + f_2$. From the boundedness of $[b,\widetilde{T}]$ in $L^\Phi(\mathbb{R}^n_+)$ it follows that

$$\|[b,\widetilde{T}]f_1\|_{L^\Phi(\mathcal{B}^+_r)} \le \|[b,\widetilde{T}]f_1\|_{L^\Phi(\mathbb{R}^n_+)} \le C\|b\|_*\, \|f_1\|_{L^\Phi(\mathbb{R}^n_+)} = C\|b\|_*\, \|f\|_{L^\Phi(2\mathcal{B}^+_r)}.$$

On the other hand, because of (19) we can write

$$\begin{aligned}
\|[b,\widetilde{T}]f_2\|_{L^\Phi(\mathcal{B}^+_r)} &\le C\left(\int_{\mathcal{B}^+_r}\left(\int_{(2\mathcal{B}^+_r)^c} \frac{|b(x)-b(y)||f(y)|}{|x^0-y|^n}dy\right)^p dx\right)^{\frac{1}{p}} \\
&\le C\left(\int_{\mathcal{B}^+_r}\left(\int_{(2\mathcal{B}^+_r)^c} \frac{|b(y)-b_{\mathcal{B}^+_r}||f(y)|}{|x^0-y|^n}dy\right)^p dx\right)^{\frac{1}{p}} \\
&+ C\left(\int_{\mathcal{B}^+_r}\left(\int_{(2\mathcal{B}^+_r)^c} \frac{|b(x)-b_{\mathcal{B}^+_r}||f(y)|}{|x^0-y|^n}dy\right)^p dx\right)^{\frac{1}{p}} = I_1 + I_2.
\end{aligned}$$

We estimate I_1 as follows:

$$\begin{aligned}
I_1 &\le C\frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{(2\mathcal{B}_r^+)^c}\frac{|b(y)-b_{\mathcal{B}_r^+}||f(y)|}{|x^0-y|^n}\,dy\\
&= C\frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{(2\mathcal{B}_r^+)^c}|b(y)-b_{\mathcal{B}_r^+}||f(y)|\int_{|x^0-y|}^{\infty}\frac{dt}{t^{n+1}}\,dy\\
&= C\frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}\int_{2r\le|x^0-y|\le t}|b(y)-b_{\mathcal{B}_r^+}||f(y)|dy\,\frac{dt}{t^{n+1}}\\
&\le C\frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}\int_{\mathcal{B}_t^+}|b(y)-b_{\mathcal{B}_r^+}||f(y)|dy\,\frac{dt}{t^{n+1}}.
\end{aligned}$$

Applying Hölder's inequality, Lemma 22 and (25), we get

$$\begin{aligned}
I_1 \le{}& C\Big(\frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}\int_{\mathcal{B}_t^+}|b(y)-b_{\mathcal{B}_t^+}||f(y)|dy\,\frac{dt}{t^{n+1}}\\
&+\frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}|b_{\mathcal{B}_r^+}-b_{\mathcal{B}_t^+}|\int_{\mathcal{B}_t^+}|f(y)|dy\,\frac{dt}{t^{n+1}}\Big)\\
\le{}& C\Big(\frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}\Big\|b(\cdot)-b_{\mathcal{B}_t^+}\Big\|_{L^{\widetilde{\Phi}}(\mathcal{B}_t^+)}\|f\|_{L^{\Phi}(\mathcal{B}_t^+)}\,\frac{dt}{t^{n+1}}\\
&+\frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}|b_{\mathcal{B}_r^+}-b_{\mathcal{B}_t^+}|\|f\|_{L^{\Phi}(\mathcal{B}_t^+)}\,\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}\Big)\\
\le{}& C\|b\|_*\,\frac{1}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L^{\Phi}(\mathcal{B}_t^+)}\,\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.
\end{aligned}$$

In order to estimate I_2, note that

$$I_2=\Big\|b(\cdot)-b_{\mathcal{B}_r^+}\Big\|_{L^{\Phi}(\mathcal{B}_r^+)}\int_{(2\mathcal{B}_r^+)^c}\frac{|f(y)|}{|x^0-y|^n}dy.$$

By Lemma 22 and (21) we get

$$\begin{aligned}
I_2 &\le \frac{C\|b\|_*}{\Phi^{-1}\big(r^{-n}\big)}\int_{(2\mathcal{B}_r^+)^c}\frac{|f(y)|}{|x^0-y|^n}dy\\
&\le \frac{C\|b\|_*}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}\|f\|_{L^{\Phi}(\mathcal{B}_t^+)}\,\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.
\end{aligned}$$

Summing up I_1 and I_2 we get that for all $p\in(1,\infty)$

$$\|[b,\widetilde{T}]f_2\|_{L^{\Phi}(\mathcal{B}_r^+)}\le\frac{C\|b\|_*}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L^{\Phi}(\mathcal{B}_t^+)}\,\Phi^{-1}\big(t^{-n}\big)\frac{dt}{t}.$$

Finally,

$$\|[b,\widetilde{T}]f\|_{L^\Phi(\mathcal{B}_r^+)} \le C\|b\|_* \, \|f\|_{L^\Phi(2\mathcal{B}_r^+)} + \frac{C\|b\|_*}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L^\Phi(\mathcal{B}_t^+)} \, \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}$$

and the statement follows by (23).◄

Theorem 32. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $b \in BMO(\mathbb{R}^n_+)$ and $\varphi_1, \varphi_2 : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable functions satisfying* (6). *Then $[b,\widetilde{T}]$ is bounded from $M^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ to $M^{\Phi,\varphi_2}(\mathbb{R}^n_+)$ and*

$$\|[b,\widetilde{T}]f\|_{M^{\Phi,\varphi_2}(\mathbb{R}^n_+)} \le C\|b\|_* \, \|f\|_{M^{\Phi,\varphi_1}(\mathbb{R}^n_+)} \tag{27}$$

with a constant independent of f.

The statement of the theorem follows by Lemma 31 and Theorem 17 in the same manner as in the proof of Theorem 21.

Lemma 33. *If $b \in L^1_{\rm loc}(\mathbb{R}^n_+)$ and $\mathcal{B}_0^+ := \mathcal{B}^+(x_0, r_0)$, then*

$$|b(x) - b_{\mathcal{B}_0^+}| \le C|b,\widetilde{T}|\chi_{\mathcal{B}_0^+}(x)$$

for every $x \in \mathcal{B}_0^+$, where $b_{\mathcal{B}_0^+} = \frac{1}{|\mathcal{B}_0^+|}\int_{\mathcal{B}_0^+} b(y)dy$.

Proof. If $x, y \in \mathcal{B}_0^+$, then $|x-y| \le |x-x_0| + |y-x_0| < 2r_0$. We get $Cr_0^{-n} \le |x-y|^{-n}$. Therefore

$$|b,\widetilde{T}|\chi_{\mathcal{B}_0^+}(x) = \int_{\mathcal{B}_0^+} |b(x)-b(y)||x-y|^{-n}dy \ge Cr_0^{-n}\int_{\mathcal{B}_0^+} |b(x)-b(y)|dy$$

$$\ge Cr_0^{-n}\left|\int_{\mathcal{B}_0^+} (b(x)-b(y))dy\right| = C|b(x)-b_{\mathcal{B}_0^+}|.$$

◄

Proof of Theorem 3.

The first part of the theorem follows from Lemma 19 and Theorem 21.

We shall now prove the second part. Let $\mathcal{B}_0^+ = \mathcal{B}^+(x_0, r_0)$ and $x \in \mathcal{B}_0^+$. By Lemma 33 we have $|b(x) - b_{\mathcal{B}_0^+}| \le C|b,\widetilde{T}|\chi_{\mathcal{B}_0^+}(x)$. Therefore, by Lemma 26 and Lemma 14

$$1 \le C\frac{\||b,\widetilde{T}|\chi_{\mathcal{B}_0^+}\|_{L^\Psi(\mathcal{B}_0^+)}}{\|b(\cdot) - b_{\mathcal{B}_0^+}\|_{L^\Psi(\mathcal{B}_0^+)}} \le \frac{C}{\|b\|_*}\||b,\widetilde{T}|\chi_{\mathcal{B}_0^+}\|_{L^\Psi(\mathcal{B}_0^+)}\Psi^{-1}(|\mathcal{B}_0^+|^{-1})$$

$$\leq \frac{C}{\|b\|_*}\varphi_2(r_0)\|[b,\widetilde{T}]\chi_{\mathcal{B}_0^+}\|_{M^{\Psi,\varphi_2}} \leq C\varphi_2(r_0)\|\chi_{\mathcal{B}_0^+}\|_{M^{\Phi,\varphi_1}} \leq C\frac{\varphi_2(r_0)}{\varphi_1(r_0)}.$$

Since this is true for every $r_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem. ◀

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