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An Elementary Proof of the Carathéodory Kernel Convergence Theorem

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Abstract. Our aim is to give an elementary and self-contained proof of the Carathéodory kernel convergence theorem based on some fundamental facts in complex analysis.

Key Words and Phrases: the Carathéodory kernel convergence theorem, kernel of a sequence of domains, the Vitali theorem.

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1. Introduction

The Bieberbach conjecture [1] was one of the most difficult problems in complex analysis. It was initially proved by de Branges [2]. The proof is deeply depending on some major theorems of the theory of univalent functions ([3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]). We focus on the Carathéodory kernel convergence theorem from among them. The origin of this theorem is the paper [14] and we can find the overview of it in several recent books [4, 8, 10, 11, 13]. But there are some unclear points in those five books as below.

Case 1: Gong [13] and Pommerenke [10] have given the Carathéodory kernel convergence theorem in the following form:

Theorem 1. *Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of holomorphic and injective functions on the unit open disk* $\mathbb{D}$ *satisfying* $f_n(0) = 0$ *and* $f_n'(0) > 0$ *for every* $n \in \mathbb{N}$, $D_n := f_n(\mathbb{D})$ *and* D *be the kernel of the sequence of domains* $\{D_n\}_{n=1}^{\infty}$. *Then the following two conditions are equivalent:*

1. *The sequence* $\{f_n\}_{n=1}^{\infty}$ *converges uniformly in the wider sense on* $\mathbb{D}$.

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2. The kernel satisfies $D \neq \mathbb{C}$ and $\{D_n\}_{n=1}^{\infty}$ converges in the kernel sense to D.

Moreover, the limit function of $\{f_n\}_{n=1}^{\infty}$ is a surjection from $\mathbb{D}$ to D. Precise definitions of the kernel and the convergence in the kernel sense are given in Definition 3 below.

Gong [13] and Pommerenke [10] have considered the image domain $D_n := f_n(\mathbb{D})$, where f_n is holomorphic and injective on $\mathbb{D}$ and satisfies $f_n(0) = 0$ and $f_n'(0) > 0$. But it is not obvious that $f_n(\mathbb{D})$ is simply connected (see Remark 1). In addition, they do not comment whether the kernel D is simply connected or not (see Remark 2). But D must be a simply connected domain because the Riemann mapping theorem is applied to D. In addition, we can find the claim that there exist two subsequences $\{f_{m_k}\}_{k=1}^{\infty}, \{f_{n_k}\}_{k=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ such that $\{f_{m_k}\}_{k=1}^{\infty}, \{f_{n_k}\}_{k=1}^{\infty}$ converge uniformly in the wider sense on $\mathbb{D}$ to f, g, respectively, where $f \neq g$ on $\mathbb{D}$, provided that $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly in the wider sense on $\mathbb{D}$. But the claim seems to be not obvious. We also note that any comment on an application of the Vitali theorem is not included in [10, 13].

Remark 1. *Yoshida [15] has commented on the fact that the image $f(\Omega)$ is a simply connected domain whenever f is holomorphic and injective on a simply connected domain Ω. But many other literatures do not comment on it and any detailed proof is not found.*

Remark 2. *The books [10, 11, 13] do not comment on the claim that the kernel should be a simply connected domain. On the other hand, the two books [4, 8] include the claim, however they do not prove it.*

Case 2: Segal [11] has given the Carathéodory kernel convergence theorem in the following form:

Theorem 2. *Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of simply connected domains, D be the kernel of $\{D_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on $\mathbb{D}$ satisfying the following for every $n \in \mathbb{N}$:*

1. $0 \in D_n \subsetneq \mathbb{C}$.
2. *$f_n : \mathbb{D} \to D_n$ is holomorphic and bijective.*
3. *$f_n(0) = 0$ and $f_n'(0) > 0$.*
4. *There exists an open disk $B(0, \rho) := \{z \in \mathbb{C} : |z| < \rho\}$ independent of n such that $B(0, \rho) \subset D_n$.*

Then the following hold:

(A) *If* $\{f_n\}_{n=1}^{\infty}$ *converges uniformly in the wider sense on* $\mathbb{D}$ *to a limit function* f*, then* f *is holomorphic and injective on* $\mathbb{D}$ *and satisfies* $f(0) = 0$, $f'(0) > 0$ *and* $f(\mathbb{D}) = D$*. In addition,* $\{D_n\}_{n=1}^{\infty}$ *converges in the kernel sense to* D*.*

(B) *Conversely, if* $\{D_n\}_{n=1}^{\infty}$ *converges in the kernel sense to* D *and* $D \neq \mathbb{C}$*, then* $\{f_n\}_{n=1}^{\infty}$ *converges uniformly in the wider sense on* $\mathbb{D}$ *to a limit function* f*.*

The case that the kernel is $\{0\}$ is removed by supposing that there exists $B(0,\rho)$ such that $B(0,\rho) \subset D_n$ holds for every $n \in \mathbb{N}$. The fact that $\{f_n\}_{n=1}^{\infty}$ is normal on $\mathbb{D}$ implies the following claim: "If there exist two subsequences $\{f_{n_k}\}_{k=1}^{\infty}$, $\{f_{m_k}\}_{k=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ such that $\{f_{n_k}\}_{k=1}^{\infty}$, $\{f_{m_k}\}_{k=1}^{\infty}$ converge to f, g, respectively, where $f \neq g$ on $\mathbb{D}$, then there exist $\{\tilde{f}_l\}_{l=1}^{\infty} \subset \{f_{n_k}\}_{k=1}^{\infty}$ and $\{\tilde{g}_l\}_{l=1}^{\infty} \subset \{f_{m_k}\}_{k=1}^{\infty}$ converging uniformly in the wider sense on $\mathbb{D}$ to $\tilde{f}$, $\tilde{g}$, respectively, where $\tilde{f} \neq \tilde{g}$ on $\mathbb{D}$." It is not obvious that the kernel of $\{D_{n_k}\}_{k=1}^{\infty}$ does not coincide with the one of $\{D_{m_k}\}_{k=1}^{\infty}$ by using only the claim. It seems that the simply connectedness of the kernel D should be clarified and an argument involving the Riemann mapping theorem should lead to some contradiction.

Case 3: Henrici [8] has given the Carathéodory kernel convergence theorem in the following form:

Theorem 3. *Let* $\{D_n\}_{n=1}^{\infty}$ *be a sequence of simply connected domains,* D *be the kernel of* $\{D_n\}_{n=1}^{\infty}$ *and* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of functions defined on* $\mathbb{D}$ *satisfying the following:*

1. $0 \in D_n \subsetneq \mathbb{C}$ *for every* $n \in \mathbb{N}$*.*
2. $f_n : \mathbb{D} \to D_n$ *is holomorphic and bijective for every* $n \in \mathbb{N}$*.*
3. $f_n(0) = 0$ *and* $f_n'(0) > 0$ *for every* $n \in \mathbb{N}$*.*
4. 0 *is an interior point of* $\displaystyle\bigcap_{n=1}^{\infty} D_n$*.*

Then $\{f_n\}_{n=1}^{\infty}$ *converges uniformly in the wider sense on* $\mathbb{D}$ *to a limit function* f *if and only if* $\{D_n\}_{n=1}^{\infty}$ *converges in the kernel sense to* D*. Moreover,* D *is a simply connected domain and* $\{f_n^{-1}\}_{n=1}^{\infty}$ *converges uniformly in the wider sense on* D *to* f^{-1}*.*

We can find the claim that D is a simply connected domain without proof. In addition, it is not obvious that the normality of $\{f_n\}_{n=1}^{\infty}$ implies the uniformly convergence of it in the wider sense on $\mathbb{D}$. We also note that any comment on applications of the Vitali theorem and the Riemann mapping theorem is not found in [8].

Case 4: Duren [4] has given the Carathéodory kernel convergence theorem in the following form:

Theorem 4. *Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of simply connected domains, D be the kernel of $\{D_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on $\mathbb{D}$ satisfying the following for every $n \in \mathbb{N}$:*

1. $0 \in D_n \subsetneq \mathbb{C}$.

2. $f_n : \mathbb{D} \to D_n$ *is holomorphic and bijective.*

3. $f_n(0) = 0$ *and* $f_n'(0) > 0$.

Then $\{f_n\}_{n=1}^{\infty}$ converges uniformly in the wider sense on $\mathbb{D}$ to a limit function f if and only if $\{D_n\}_{n=1}^{\infty}$ converges in the kernel sense to $D \neq \mathbb{C}$. Moreover, one of the following two conclusions holds:

(A) If $D = \{0\}$, then we have $f \equiv 0$ on $\mathbb{D}$.

(B) If $D \neq \{0\}$, then D is a simply connected domain, $f : \mathbb{D} \to D$ is holomorphic and bijective, and $\{f_n^{-1}\}_{n=1}^{\infty}$ converges uniformly in the wider sense on $\mathbb{D}$ to f^{-1}.

The conclusion of the theorem above includes the claim that the kernel is a simply connected domain. But any proof of it is not found. We also note that an argument involving the Riemann mapping theorem should be needed as is the case with [11] (Case 2).

Keeping the unclear points above in mind, we will give an elementary and self-contained proof of the Carathéodory kernel convergence theorem based on some fundamental facts. We state basic notions in Section 2. In particular, we give straightforward proofs of the Hurwitz theorem and the Vitali theorem. Later we formulate the main theorem and prove it in Section 3. In the present paper, we use the following notation:

1. Given $a \in \mathbb{C}$ and $r > 0$, we write $B(a,r) := \{z \in \mathbb{C} : |z-a| < r\}$ and $B_0(a,r) := \{z \in \mathbb{C} : 0 < |z-a| < r\} = B(a,r) \setminus \{a\}$.

2. We denote the unit open disk by $\mathbb{D} := B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$.

2. Preliminaries

2.1. Fundamental concepts

In this subsection, we give some fundamental facts used in the present paper. For further informations and detailed proofs we refer to [3, 4, 8, 9, 11, 16].

Definition 1. *The set $\mathcal{S}$ consists of all functions f satisfying the following conditions:*

1. *f is holomorphic and injective on $\mathbb{D}$.*
2. *$f(0) = 0$ and $f'(0) = 1$.*

Theorem 5. *Let $\Omega \subset \mathbb{C}$ be a domain and $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on Ω satisfying the following:*

1. *$f_n(z) \neq 0$ for all $z \in \Omega$ and all $n \in \mathbb{N}$.*
2. *$\{f_n\}_{n=1}^{\infty}$ converges uniformly in the wider sense on Ω to a function f.*

Suppose that there exists a point $z_0 \in \Omega$ such that $f(z_0) \neq 0$. Then we have $f(z) \neq 0$ for all $z \in \Omega$.

Remark 3. *In some books (for example [17, 18]) one says that Theorem 5 is the Hurwitz theorem. In the present paper we say that Theorem 9 is the Hurwitz theorem following the books [4, 6, 19].*

Theorem 6 (The Riemann mapping theorem)**.** *For any simply connected domain $\Omega \subsetneq \mathbb{C}$ and any $z_0 \in \Omega$, there exists a unique bijective and holomorphic function $f : \Omega \to \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.*

Theorem 7 (The Koebe 1/4 theorem)**.** *For every $f \in \mathcal{S}$, we have $B(0, 1/4) \subset f(\mathbb{D})$.*

Theorem 8 (The Koebe distortion theorem)**.** *For every $f \in \mathcal{S}$, we have for all $z \in \mathbb{D}$*

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3},$$
$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

Lemma 1. *If a function f is holomorphic and injective on a domain $\Omega \subset \mathbb{C}$, then we have $f'(z) \neq 0$ for all $z \in \Omega$.*

2.2. The three important theorems

In this subsection, we state the three important theorems, that is, Hurwitz, Montel and Vitali theorems. Let us begin with the Hurwitz theorem.

Theorem 9 (The Hurwitz theorem)**.** *Let $\Omega \subset \mathbb{C}$ be a domain, $E \subset \mathbb{C}$, and $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on Ω satisfying the following:*

1. *$\{f_n\}_{n=1}^{\infty}$ converges uniformly in the wider sense on Ω to a function f.*

2. *The limit function f is not any constant function on Ω.*

3. *The closure $\overline{E}$ satisfies $\overline{E} \subset \Omega$ and is a closed domain enclosed by a piece-wise smooth simple closed curve.*

4. *$f(z) \neq 0$ for any $z \in \partial E$.*

Then for sufficiently large $n \in \mathbb{N}$, the number of the zero points included in E of the function f_n equals to the one of f, where we take the order of each zero point into account.

Proof. Following the outline shown by [4, 6, 19] we give a straightforward proof. By virtue of the Weierstrass theorem we see that f is holomorphic on Ω. In particular, f is continuous on ∂E. Thus, the minimum value $m := \min_{z \in \partial E} |f(z)| > 0$ does exist. Namely, we have $|f(z)| \geq m$ for all $z \in \partial E$. On the other hand, $\{f_n\}_{n=1}^{\infty}$ converges uniformly on ∂E to the limit function f. Thus, we can take $n_0 \in \mathbb{N}$ so that $|f_n(z) - f(z)| < m$ holds for every $n \geq n_0$ and $z \in \partial E$. This implies that $|f_n(z) - f(z)| < |f(z)|$. Therefore, the Rouché theorem gives us the desired conclusion. ◄

As a corollary of the Hurwitz theorem we obtain the next result.

Corollary 1. *Let $\Omega \subset \mathbb{C}$ be a domain and $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic and injective functions on Ω. Suppose that $\{f_n\}_{n=1}^{\infty}$ converges uniformly in the wider sense on Ω to a function f and that the limit function f is not any constant function. Then f is also holomorphic and injective on Ω.*

Proof. By virtue of the Weierstrass theorem we see that f is holomorphic on Ω. Assume that f is not injective on Ω. Then there exist z_1, $z_2 \in \Omega$ such that $z_1 \neq z_2$ and $f(z_1) = f(z_2)$. Now we write $\alpha := f(z_1) = f(z_2)$. We note that the function $f - \alpha$ is not any constant function on Ω. Thus, we can take two positive numbers ρ_1', ρ_2' so that for $j = 1$, 2, $B(z_j, \rho_j') \subset \Omega$ and $f - \alpha \neq 0$ on $B_0(z_j, \rho_j')$ hold. In addition, we can take $\rho_1' > \rho_1, \rho_2' > \rho_2$ so that $B(z_1, \rho_1) \cap B(z_2, \rho_2) = \emptyset$ holds. We remark that each function $f_n - \alpha$ is holomorphic and $\{f_n - \alpha\}_{n=1}^{\infty}$ converges uniformly on Ω in the wider sense to the limit function $f-\alpha$. Applying Theorem 9 (the Hurwitz theorem) to $\{f_n-\alpha\}_{n=1}^{\infty}$, for sufficiently large $n \in \mathbb{N}$, we obtain that the number of the zero points included in $B(z_1,\rho_1)$ of the function $f_n-\alpha$ equals to the one of $f-\alpha$, where we take the order of each zero point into account. In particular, the function $f-\alpha$ satisfies $f-\alpha \neq 0$ on $B_0(z_1,\rho_1)$ and $f(z_1)-\alpha=0$. Thus, $f_n-\alpha$ has a unique zero point w_1 in $B(z_1,\rho_1)$. The same argument is valid for $B(z_2,\rho_2)$. That is, there exists a unique point $w_2 \in B(z_2,\rho_2)$ such that $f_n(w_2)-\alpha=0$. Therefore, we obtain $f_n(w_1)=f_n(w_2)=\alpha$ and $w_1 \neq w_2$. This conclusion and the fact that f_n is injective are contradictory. Hence we have proved that f is injective on Ω. ◀

Before stating the next two theorems we need the following definition.

Definition 2. *Let $\mathcal{F}$ be a family of functions defined on an open set $\Omega \subset \mathbb{C}$.*

1. *Let $E \subset \Omega$. The family $\mathcal{F}$ is said to be uniformly bounded on E if there exists a positive constant M such that $|f(z)| \leq M$ holds for every $z \in E$ and $f \in \mathcal{F}$.*

2. *The family $\mathcal{F}$ is said to be uniformly bounded in the wider sense on Ω if $\mathcal{F}$ is uniformly bounded on K for any bounded and closed set $K \subset \Omega$.*

3. *The family $\mathcal{F}$ is said to be normal on Ω if any sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ has a subsequence converging uniformly in the wider sense on Ω.*

Theorem 10 (The Montel theorem)**.** *Let $\mathcal{F}$ be a family of holomorphic functions on an open set Ω. If $\mathcal{F}$ is uniformly bounded in the wider sense on Ω, then $\mathcal{F}$ is normal on Ω.*

We omit the proof of the Montel theorem because some self-contained proofs are well-known. For example, we can find them in [4, 6, 16, 20]. We next state the Vitali theorem.

Theorem 11 (The Vitali theorem)**.** *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on a domain Ω and $A \subset \Omega$ satisfy $A' \cap \Omega \neq \emptyset$, where the set A' consists of all accumulating points of A. Suppose that $\{f_n\}_{n=1}^{\infty}$ is normal on Ω and take a subsequence $\{g_n\}_{n=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ converging uniformly in the wider sense on Ω to a limit function g, and that $\{f_n\}_{n=1}^{\infty}$ converges pointwise on A to g. Then $\{f_n\}_{n=1}^{\infty}$ converges uniformly in the wider sense on Ω.*

Some outlines of the proof can be found in [4, 9, 15, 21], however those books include unclear points. Based on the strategy established by those books we give a detailed proof.

Proof. Assume that $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly in the wider sense on Ω. Then there exist a bounded and closed set $K \subset \Omega$ and $\varepsilon_0 > 0$ such that for any $N \in \mathbb{N}$ we can take $n_0 > N$ and $z_0 \in K$ so that $|f_{n_0}(z_0) - g(z_0)| \geq \varepsilon_0$ holds. Thus, we can take a natural number $n_1 > 1$ and $z_1 \in K$ so that $|f_{n_1}(z_1) - g(z_1)| \geq \varepsilon_0$ holds. We can additionally take a natural number $n_2 > n_1$ and $z_2 \in K$ so that $|f_{n_2}(z_2) - g(z_2)| \geq \varepsilon_0$ holds. Hence we can construct $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$, $\{f_{n_k}\}_{k=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ and $\{z_k\}_{k=1}^{\infty} \subset K$ such that $|f_{n_k}(z_k) - g(z_k)| \geq \varepsilon_0$ and $n_{k+1} > n_k$ hold for every $k \in \mathbb{N}$. Because K is bounded and closed, by virtue of the Bolzano–Weierstrass theorem there exists a subsequence $\{z_{k_m}\}_{m=1}^{\infty} \subset \{z_k\}_{k=1}^{\infty}$ converging to a point $z^* \in K$. Below we rewrite $\{z_k\}_{k=1}^{\infty} := \{z_{k_m}\}_{m=1}^{\infty}$. Noting that $\{f_n\}_{n=1}^{\infty}$ is normal on Ω again, we obtain a subsequence $\{f_{n_{k_l}}\}_{l=1}^{\infty} \subset \{f_{n_k}\}_{k=1}^{\infty}$ converging uniformly in the wider sense on Ω to a limit function h. Take $\alpha \in A$ arbitrarily. Then we have $\lim\limits_{n\to\infty} g_n(\alpha) = g(\alpha)$ and $\lim\limits_{l\to\infty} f_{n_{k_l}}(\alpha) = h(\alpha)$ because both $\{g_n\}_{n=1}^{\infty}$ and $\{f_{n_{k_l}}\}_{l=1}^{\infty}$ converge pointwise on A. On the other hand, the sequence of complex numbers $\{f_n(\alpha)\}_{n=1}^{\infty}$ converges to the limit $g(\alpha)$ because the sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges pointwise on A to the limit function g. Thus, by virtue of the fact that $\{f_{n_{k_l}}(\alpha)\}_{l=1}^{\infty}$ is a subsequence of $\{f_n(\alpha)\}_{n=1}^{\infty}$ we have

$$h(\alpha) = \lim_{l\to\infty} f_{n_{k_l}}(\alpha) = \lim_{n\to\infty} f_n(\alpha) = g(\alpha).$$

We also remark that the Weierstrass theorem implies that both g and h are holomorphic on Ω. Hence by the identity theorem we see that $g(z) = h(z)$ holds for all $z \in \Omega$. On the other hand, the sequence $\{f_{n_{k_l}}\}_{l=1}^{\infty}$ converges uniformly in the wider sense on Ω to the limit function h, that is, for all $\varepsilon > 0$, there exists $L_0 \in \mathbb{N}$ such that $|f_{n_{k_l}}(z_{k_l}) - h(z_{k_l})| < \varepsilon$ holds for every $l \geq L_0$. In addition, the function h is continuous at the point $z^* \in \Omega$, that is, there exists $L_1 > L_0$ such that $|h(z_{k_l}) - h(z^*)| < \varepsilon$ holds for every $l \geq L_1$. Thus, we have

$$\begin{aligned} |f_{n_{k_l}}(z_{k_l}) - h(z^*)| &\leq |f_{n_{k_l}}(z_{k_l}) - h(z_{k_l})| + |h(z_{k_l}) - h(z^*)| \\ &< \varepsilon + \varepsilon = 2\varepsilon, \end{aligned}$$

namely, the sequence of complex numbers $\{f_{n_{k_l}}(z_{k_l})\}_{l=1}^{\infty}$ converges to the limit $h(z^*)$. Since we have obtained that $|f_{n_{k_l}}(z_{k_l}) - g(z_{k_l})| \geq \varepsilon_0$ holds for every $l \in \mathbb{N}$, we see that $h(z^*) - g(z^*) \neq 0$, namely, $h(z^*) \neq g(z^*)$. Because of $z^* \in \Omega$, this contradicts the fact obtained by the identity theorem above. Consequently, we have proved that $\{f_n\}_{n=1}^{\infty}$ converges uniformly in the wider sense on Ω. ◀

2.3. The kernel and the kernel convergence

In this subsection, we define the kernel and convergence in the kernel sense for sequence of domains and give some examples based on the books [10, 13].

Definition 3. *Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of domains such that $0 \in D_n$ for every $n \in \mathbb{N}$. The set D' consists of all $w \in \mathbb{C}$ for which there exists a domain H such that $\{0, w\} \subset H \subset D_n$ holds for all sufficiently large $n \in \mathbb{N}$. Then the kernel D of $\{D_n\}_{n=1}^{\infty}$ is defined by*

$$D := \{0\} \cup D'.$$

The sequence $\{D_n\}_{n=1}^{\infty}$ is said to be converging in the kernel sense to D if all subsequences of $\{D_n\}_{n=1}^{\infty}$ have the same kernel D. In this case we write $D_n \to D$.

Remark 4. *In the case where $D' = \emptyset$, the kernel D is of course defined by $D := \{0\}$.*

Example 1. *Consider some examples of sequences of domains $\{D_n\}_{n=1}^{\infty}$ and their kernels D.*

1. *Let*

$$D_n := \mathbb{C} \setminus \left\{1 + it \,:\, |t| \geq \frac{1}{n}\right\} \;\; (n \in \mathbb{N}).$$

Then the kernel D of $\{D_n\}_{n=1}^{\infty}$ is

$$D = \{z \in \mathbb{C} \,:\, \mathrm{Re} z < 1\}.$$

We see that any subsequence of $\{D_n\}_{n=1}^{\infty}$ has also the same kernel D. Thus we have $D_n \to D$.

2. *Let*

$$D_n := \mathbb{C} \setminus \left\{it \,:\, |t| \geq \frac{1}{n}\right\} \;\; (n \in \mathbb{N}).$$

Then the kernel D of $\{D_n\}_{n=1}^{\infty}$ is

$$D = \{0\},$$

because for all $\varepsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that $B(0, \varepsilon) \not\subset D_{n_0}$. We see that any subsequence of $\{D_n\}_{n=1}^{\infty}$ has also the same kernel D. Thus we have $D_n \to D$.

3. *Let*

$$D_n := \mathbb{C} \setminus \{z \in \mathbb{C} \,:\, \mathrm{Im} z = (-1)^n\} \;\; (n \in \mathbb{N}).$$

Then the kernel D of $\{D_n\}_{n=1}^{\infty}$ is

$$D = \{z \in \mathbb{C} \,:\, \mathrm{Im} z > -1\} \cap \{z \in \mathbb{C} \,:\, \mathrm{Im} z < 1\}.$$

On the other hand, the kernel of subsequence $\{D_{2n}\}_{n=1}^{\infty}$ is

$$\{z \in \mathbb{C} \,:\, \mathrm{Im} z < 1\}.$$

This implies that $\{D_n\}_{n=1}^{\infty}$ does not converge in the kernel sense to D.

3. A proof of the Carathéodory kernel convergence theorem

We formulate the Carathéodory kernel convergence theorem by giving the following two parts in order to make the main result more readable. We also note that we will apply the first theorem to prove the second one.

Theorem 12. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic and injective functions on $\mathbb{D}$ satisfying $f_n(0) = 0$ and $f_n'(0) > 0$ for every $n \in \mathbb{N}$ and D be the kernel of the sequence $\{D_n\}_{n=1}^{\infty}$, where $D_n := f_n(\mathbb{D})$. If the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly in the wider sense on $\mathbb{D}$ to a limit function f, then we have $f(\mathbb{D}) = D$.*

The first theorem itself (Theorem 12) has been essentially proved by Gong [13] and Pommerenke [10]. For readers' convenience we will give a self-contained proof of it.

Theorem 13. *Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of simply connected domains such that $0 \in D_n \subsetneq \mathbb{C}$ for every $n \in \mathbb{N}$, $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions such that $f_n : \mathbb{D} \to D_n$ is holomorphic and bijective, $f_n(0) = 0$ and $f_n'(0) > 0$, and D be the kernel of $\{D_n\}_{n=1}^{\infty}$. Then the following hold:*

- *(i) If $D_n \to D$ and $D = \{0\}$ are true, then $\{f_n\}_{n=1}^{\infty}$ converges uniformly in the wider sense on $\mathbb{D}$ to 0.*
- *(ii) If $D_n \to D$ holds and $D \subsetneq \mathbb{C}$ is a simply connected domain, then $\{f_n\}_{n=1}^{\infty}$ converges uniformly in the wider sense on $\mathbb{D}$ to a limit function f, where f satisfies $f(\mathbb{D}) = D$.*

Remark 5. *The Riemann mapping theorem guarantees that the sequence of functions $\{f_n\}_{n=1}^{\infty}$ does exist in Theorem 13.*

As is pointed out in Introduction, we find the following unclearness when we state the second part (Theorem 13) in the five books [4, 8, 10, 11, 13]:

1. Both the Vitali theorem and the Riemann mapping theorem are indispensable in our self-contained proof of the Carathéodory kernel convergence theorem. We cannot find any comment on an application of the Vitali theorem in [8, 10, 13]. On the other hand, the books [4, 8, 11] do not include any comment on an application of the Riemann mapping theorem.

2. Remarks 1 and 2 say that the five books [4, 8, 10, 11, 13] do not give a detailed explanation on simply connectedness of the image and the kernel. By supposing that every D_n and D are simply connected in Theorem 13, we can avoid the complicated discussion, however we do not give direct proofs of Remarks 1 and 2.

Making up for these unclear points above we have formulated Theorem 13. Applying Theorem 12 we will additionally give an elementary and self-contained proof of Theorem 13.

Proof. [Proof of Theorem 12] (i) We first consider the case where the limit function is not any constant function. By virtue of Corollary 1, f is holomorphic and injective on $\mathbb{D}$. As $f_n(0) = 0$ holds for every $n \in \mathbb{N}$, we have

$$f(0) = \lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0.$$

Step 1: We prove $f(\mathbb{D}) \subset D$. Take $w_0 \in f(\mathbb{D})$ arbitrarily.

If $w_0 = 0$, then by the definition of the kernel we can easily obtain $w_0 \in D$.

Consider the case where $w_0 \neq 0$. There exists $z_0 \in \mathbb{D}$ such that $w_0 = f(z_0)$. We take a constant r so that $|z_0| < r < 1$. We define a set H by $H := f\,(B(0,r))$. We note that f is holomorphic and is not any constant function on $B(0,r)$. Thus, by the principle of domain preservation, H is a domain. It also follows that $\{0,\, w_0\} \subset H$. If the claim that $H \subset D_n$ is true for all sufficiently large $n \in \mathbb{N}$, then the definition of the kernel implies $w_0 \in D$, that is, $f(\mathbb{D}) \subset D$ holds. Thus, we only have to prove the above claim. Assume that for any $n \in \mathbb{N}$, there exists $N > n$ such that $H \not\subset D_N$. Then we can take a natural number $n_1 > 1$ so that $H \not\subset D_{n_1}$. In addition, we can take a natural number $n_2 > n_1$ so that $H \not\subset D_{n_2}$. Hence we can construct $\alpha_k \in H \setminus D_{n_k}$ and $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that $n_{k+1} > n_k > k$ for every $k \in \mathbb{N}$. By virtue of the Archimedes principle, for all $K > 0$, there exists $L \in \mathbb{N}$ such that $L > K$. Thus, for all $l \in \mathbb{N}$ with $l > L$ we have $n_l > n_L > L > K$, that is, $\lim_{k\to\infty} n_k = +\infty$. Because f is continuous on the bounded and closed set $\overline{B(0,r)}$, $M := \max_{z\in\overline{B(0,r)}} |f(z)|$ does exist. Namely, $|f(z)| \leq M$ holds for every $z \in B(0,r)$. This implies that H is a bounded domain. Hence the closure $\overline{H}$ is also bounded. Thus, by the Bolzano–Weierstrass theorem, there exists a subsequence $\{\alpha_{k_l}\}_{l=1}^{\infty} \subset \{\alpha_k\}_{k=1}^{\infty} \subset \overline{H}$ converging to a point $\alpha^* \in \overline{H}$. Below we rewrite $\{\alpha_k\}_{k=1}^{\infty} := \{\alpha_{k_l}\}_{l=1}^{\infty}$. Noting that $\alpha_k \notin D_{n_k}$ and $D_{n_k} = f_{n_k}(\mathbb{D})$ hold for all $k \in \mathbb{N}$, we have $f_{n_k}(z) - \alpha_k \neq 0$ for all $z \in \mathbb{D}$ and all $k \in \mathbb{N}$. On the other hand, $\{f_{n_k} - \alpha_k\}_{k=1}^{\infty}$ converges uniformly in the wider sense on $\mathbb{D}$ to the function $f - \alpha^*$. There exists $z_1 \in \mathbb{D}$ such that $f(z_1) - \alpha^* \neq 0$ because $f - \alpha^*$ is injective on $\mathbb{D}$. Thus, by virtue of Theorem 5, for every $z \in \mathbb{D}$ we have $f(z) - \alpha^* \neq 0$, that is, $f(z) \neq \alpha^*$. This contradicts the fact that $\alpha^* \in \overline{H} \subset f(\mathbb{D})$.

Step 2: We prove $D \subset f(\mathbb{D})$. Take $w_0 \in D$ arbitrarily.

If $w_0 = 0$, then we can easily get $0 = f(0) \in f(\mathbb{D})$.

Consider the case where $w_0 \neq 0$. By virtue of the definition of the kernel, we can take a domain H and a natural number N so that $\{0, w_0\} \subset H \subset D_n$ for all $n \geq N$. We note that $f_n : \mathbb{D} \to D_n$ is holomorphic and bijective for every $n \in \mathbb{N}$. Thus, we can define the holomorphic inverse function $\varphi_n := f_n^{-1} : D_n \to \mathbb{D}$. In particular, $\varphi_n(0) = 0$ holds. We see that $\{\varphi_n\}_{n=N}^{\infty}$ is a sequence of holomorphic and injective functions on H because $H \subset D_n$ holds for every $n \geq N$. In addition, for every $n \geq N$ we have $\varphi_n(H) \subset \varphi_n(D_n) \subset \mathbb{D}$, that is, $\{\varphi_n\}_{n=N}^{\infty}$ is uniformly bounded on H. Hence Theorem 10 (the Montel theorem) implies that $\{\varphi_n\}_{n=N}^{\infty}$ is normal on H. Namely, there exists a subsequence $\{\varphi_{n_k}\}_{k=1}^{\infty} \subset \{\varphi_n\}_{n=N}^{\infty}$ converging uniformly in the wider sense on H to a limit function φ. Applying Corollary 1 to $\{\varphi_{n_k}\}_{k=1}^{\infty}$, we see that φ is holomorphic and injective on H. In particular, $\{\varphi_{n_k}\}_{k=1}^{\infty}$ converges pointwise to φ on H. Because of $0 \in H$, we have

$$\varphi(0) = \lim_{k\to\infty} \varphi_{n_k}(0) = \lim_{k\to\infty} 0 = 0.$$

In addition, fixing $w \in H$ arbitrarily, we have $\lim_{k\to\infty} |\varphi_{n_k}(w)| = |\varphi(w)|$ and $|\varphi_{n_k}(w)| < 1$. Thus, we obtain $|\varphi(w)| \leq 1$. Noting that φ is holomorphic and is not any constant function on H, we have $|\varphi(w)| < 1$ by the maximum modulus principle. In particular, $z_0 := \varphi(w_0) \in \mathbb{D}$ holds. Therefore, f is continuous at the point $z_0 \in \mathbb{D}$, that is, for all $\varepsilon > 0$ there exists $\delta > 0$ satisfying $\overline{B(z_0, \delta)} \subset \mathbb{D}$ and $|f(z) - f(z_0)| < \varepsilon$ for all $z \in B(z_0, \delta)$. Because $\{\varphi_{n_k}\}_{k=1}^{\infty}$ converges at the point w_0, there exists $k_0 \in \mathbb{N}$ such that $|\varphi_{n_k}(w_0) - \varphi(w_0)| < \delta$ holds for all $k \geq k_0$. In addition, because $\{f_{n_k}\}_{k=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ converges uniformly on $\overline{B(z_0, \delta)}$ to f, we can take $k_1 \geq k_0$ so that $|f_{n_k}(z) - f(z)| < \varepsilon$ holds for all $k \geq k_1$ and all $z \in \overline{B(z_0, \delta)}$. Hence we have

$$\begin{aligned}
|f(z_0) - w_0| &= |f(z_0) - f(\varphi_{n_k}(w_0)) + f(\varphi_{n_k}(w_0)) - w_0| \\
&\leq |f(z_0) - f(\varphi_{n_k}(w_0))| + |f(\varphi_{n_k}(w_0)) - w_0| \\
&= |f(z_0) - f(\varphi_{n_k}(w_0))| + |f(\varphi_{n_k}(w_0)) - f_{n_k}(\varphi_{n_k}(w_0))| \\
&< \varepsilon + \varepsilon \\
&= 2\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $w_0 = f(z_0) \in f(\mathbb{D})$, that is, $D \subset f(\mathbb{D})$.

We see that Step 1 and Step 2 imply $f(\mathbb{D}) = D$.

(ii) We next consider the case where the limit function f equals to a constant C . Then we have

$$C = f(0) = \lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0,$$

namely, $f(\mathbb{D}) = \{C\} = \{0\}$. Now we assume that $D \neq \{0\}$. Then we can take an element $w \in D \setminus \{0\}$. By virtue of the definition of the kernel, there exist a domain H and $m \in \mathbb{N}$ such that $\{0, w\} \subset H \subset D_m$. Because H is an open set including 0, we can take $\rho > 0$ so that $B(0, \rho) \subset H \subset D_m$. On the other hand, we can define the holomorphic inverse function $\varphi_m := f_m^{-1} : D_m \to \mathbb{D}$, because $f_m : \mathbb{D} \to D_m$ is holomorphic and bijective. In particular, φ_m is holomorphic and injective on D_m. Thus, by Lemma 1 we have $\varphi_m' \neq 0$ on D_m. On the other hand, φ_m is holomorphic on $B(0, \rho)$ and satisfies $|\varphi_m(w)| < 1$ for all $w \in B(0, \rho)$ and $\varphi_m(0) = 0$. Applying the Schwarz lemma to φ_m, we obtain $|\varphi_m'(0)| \leq 1/\rho$. Because $f_m(\varphi_m(v)) = v$ holds for each $v \in B(0, \rho)$, by the differentiation we have $f_m'(\varphi_m(v))\varphi_m'(v) = 1$. Thus, we obtain

$$|f_m'(0)| = |f_m'(\varphi_m(0))| = \left|\frac{1}{\varphi_m'(0)}\right| \geq \rho > 0.$$

Hence the sequence $\{f_n'(0)\}_{n=1}^{\infty}$ does not converge to 0. On the other hand, $\{f_n\}_{n=1}^{\infty}$ converges uniformly in the wider sense on $\mathbb{D}$ to the limit function $f \equiv 0$. Thus, by virtue of the Weierstrass theorem $\{f_n'\}_{n=1}^{\infty}$ converges uniformly in the wider sense on $\mathbb{D}$ to the limit function 0. In particular, the sequence $\{f_n'(0)\}_{n=1}^{\infty}$ converges to 0. This contradicts the fact above. Consequently, we have proved $D = \{0\}$, that is, $f(\mathbb{D}) = D$. ◀

Now we prove the second part of the Carathéodory kernel convergence theorem (Theorem 13) applying the previous theorem.

Proof. [Proof of Theorem 13] We can define a holomorphic and injective function

$$F_n(z) := \frac{f_n(z)}{f_n'(0)}$$

on $\mathbb{D}$ because $f_n'(0) > 0$ holds for every $n \in \mathbb{N}$. By the definition, we see that $F_n(0) = 0$ and $F_n'(0) = 1$. Namely, $\{F_n\}_{n=1}^{\infty} \subset \mathcal{S}$ holds.

We first prove (i). Suppose that $D = \{0\}$ and $D_n \to \{0\}$.

Step 1: We prove $\lim_{n\to\infty} f_n'(0) = 0$. Assume that $\{f_n'(0)\}_{n=1}^{\infty}$ does not converge to 0. Then there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ we can take $n_0 \geq n$ so that $f_{n_0}'(0) \geq \varepsilon$. Thus, we can take a natural number $n_1 > 1$ so that $f_{n_1}'(0) \geq \varepsilon$. We can additionally take a natural number $n_2 > n_1$ so that $f_{n_2}'(0) \geq \varepsilon$. Hence we can construct $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ and $\{f_{n_k}\}_{k=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ such that $f_{n_k}'(0) \geq \varepsilon$ and $n_{k+1} > n_k$ for each $k \in \mathbb{N}$. Applying Theorem 7 (the Koebe 1/4 theorem) to F_{n_k}, we get $B(0, 1/4) \subset F_{n_k}(\mathbb{D})$. Take $w \in B\left(0, \frac{f_{n_k}'(0)}{4}\right)$ arbitrarily. Then we have $\frac{|w|}{f_{n_k}'(0)} < \frac{1}{4}$, that is, $\frac{w}{f_{n_k}'(0)} \in B(0, 1/4) \subset F_{n_k}(\mathbb{D})$. Hence there exists $z_0 \in \mathbb{D}$ such that

$$\frac{w}{f_{n_k}'(0)} = F_{n_k}(z_0) = \frac{f_{n_k}(z_0)}{f_{n_k}'(0)},$$

namely, $w = f_{n_k}(z_0)$. Thus, we have

$$B\left(0, \frac{\varepsilon}{4}\right) \subset B\left(0, \frac{f'_{n_k}(0)}{4}\right) \subset f_{n_k}(\mathbb{D}) = D_{n_k}.$$

This implies that the kernel of $\{D_{n_k}\}_{k=1}^{\infty}$ includes $B(0, \frac{\varepsilon}{4})$ and contradicts the fact that $D_n \to \{0\}$.

Step 2: We prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly in the wider sense on $\mathbb{D}$ to 0. Take bounded and closed set $K \subset \mathbb{D}$ arbitrarily. The value $M_1 := \max_{z \in K} \frac{|z|}{(1-|z|)^2}$ does exist because the function $\frac{|z|}{(1-|z|)^2}$ is continuous on K. Applying Theorem 8 (the Koebe distortion theorem) to F_n for every $n \in \mathbb{N}$, we obtain that for all $z \in K$

$$|F_n(z)| \leq \frac{|z|}{(1-|z|)^2} \leq M_1.$$

Therefore, we get

$$|f_n(z)| = |F_n(z)|\,|f'_n(0)| \leq M_1 |f'_n(0)| \to 0 \ (n \to \infty).$$

Consequently, we have proved that $\{f_n\}_{n=1}^{\infty}$ converges uniformly in the wider sense on $\mathbb{D}$ to 0.

We next prove (ii). Suppose that $D \subsetneq \mathbb{C}$ is a simply connected domain and $D_n \to D$.

Step 1: We prove that the sequence $\{f'_n(0)\}_{n=1}^{\infty}$ is bounded above. Assume that $\{f'_n(0)\}_{n=1}^{\infty}$ is not bounded above. Then there exists a subsequence $\{f'_{n_k}(0)\}_{k=1}^{\infty} \subset \{f'_n(0)\}_{n=1}^{\infty}$ such that $\lim_{k \to \infty} f'_{n_k}(0) = +\infty$. Applying Theorem 7 (the Koebe 1/4 theorem) to F_{n_k}, the same argument as in the proof of Step 1 in (i) gives us

$$B\left(0, \frac{f'_{n_k}(0)}{4}\right) \subset f_{n_k}(\mathbb{D}) = D_{n_k}.$$

Noting that $f'_{n_k}(0) \to +\infty$ $(k \to \infty)$, we see that $\{D_{n_k}\}_{k=1}^{\infty}$ has the kernel $\mathbb{C}$. This contradicts to $D_n \to D$ and $D \subsetneq \mathbb{C}$.

Step 2: We prove that $\{f_n\}_{n=1}^{\infty}$ is normal on $\mathbb{D}$. By virtue of Step 1, there exists a constant $M_2 > 0$ such that $|f'_n(0)| \leq M_2$ holds for every $n \in \mathbb{N}$. Applying Theorem 8 (the Koebe distortion theorem) to F_n for every $n \in \mathbb{N}$, we obtain that for all $z \in K$

$$|F_n(z)| \leq \frac{|z|}{(1-|z|)^2} \leq M_1.$$

Therefore, we get

$$|f_n(z)| = |F_n(z)|\,|f'_n(0)| \leq M_1 M_2,$$

that is, $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded on K. By virtue of Theorem 10 (the Montel theorem), we have proved that $\{f_n\}_{n=1}^{\infty}$ is normal on $\mathbb{D}$.

Step 3: There exists a subsequence $\{g_n\}_{n=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ converging uniformly in the wider sense on $\mathbb{D}$ to a limit function g because $\{f_n\}_{n=1}^{\infty}$ is normal on $\mathbb{D}$. Below we prove that $\{f_n\}_{n=1}^{\infty}$ converges pointwise on $\mathbb{D}$ to g. Assume that $\{f_n\}_{n=1}^{\infty}$ does not converge pointwise on $\mathbb{D}$. Then there exist $\alpha \in \mathbb{D}$ and $\varepsilon > 0$ such that for any $N \in \mathbb{N}$ we can take $n > N$ so that $|f_n(\alpha) - g(\alpha)| \geq \varepsilon$. Thus, we can take a natural number $n_1 > 1$ so that $|f_{n_1}(\alpha) - g(\alpha)| \geq \varepsilon$. We can additionally take a natural number $n_2 > n_1$ so that $|f_{n_2}(\alpha) - g(\alpha)| \geq \varepsilon$. Therefore, we can construct $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ and $\{f_{n_k}\}_{k=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ such that $n_{k+1} > n_k$ and $|f_{n_k}(\alpha) - g(\alpha)| \geq \varepsilon$ for every $k \in \mathbb{N}$. Noting that $\{f_n\}_{n=1}^{\infty}$ is normal on $\mathbb{D}$ again, there exists a subsequence $\{f_{n_{k_l}}\}_{l=1}^{\infty} \subset \{f_{n_k}\}_{k=1}^{\infty}$ converging uniformly in the wider sense on $\mathbb{D}$ to a limit function f. Since $|f_{n_{k_l}}(\alpha) - g(\alpha)| \geq \varepsilon$ holds for all $l \in \mathbb{N}$, we have $\lim_{l\to\infty} f_{n_{k_l}}(\alpha) - g(\alpha) \neq 0$, that is, $f(\alpha) \neq g(\alpha)$. On the other hand, applying Theorem 12 to $\{f_{n_{k_l}}\}_{l=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$, respectively, we obtain $f(\mathbb{D}) = D$ and $g(\mathbb{D}) = D$. In addition, we have $f(0) = \lim_{l\to\infty} f_{n_{k_l}}(0) = 0$. By virtue of the Weierstrass theorem we see that $f'(0) = \lim_{l\to\infty} f'_{n_{k_l}}(0) \geq 0$. By Corollary 1 we see that f is holomorphic and injective on $\mathbb{D}$. Thus, by Lemma 1 we have $f'(z) \neq 0$ for all $z \in \mathbb{D}$, that is, $f'(0) > 0$. Similarly we see that g is holomorphic and injective on $\mathbb{D}$ and satisfies $g(0) = 0$ and $g'(0) > 0$. We also remark that both f and g are holomorphic on $\mathbb{D}$ and bijections from $\mathbb{D}$ to the simply connected domain $D \subsetneq \mathbb{C}$. By virtue of the uniqueness of the map due to the Riemann mapping theorem we have $f(z) = g(z)$ for all $z \in \mathbb{D}$. This contradicts to $f(\alpha) \neq g(\alpha)$.

Step 4: We prove the conclusion of (ii). By virtue of Step 3, $\{f_n\}_{n=1}^{\infty}$ converges pointwise on $\mathbb{D}$ to the limit function g. Theorem 11 (the Vitali theorem) implies that $\{f_n\}_{n=1}^{\infty}$ converges uniformly in the wider sense on $\mathbb{D}$. Applying Theorem 12 to $\{f_n\}_{n=1}^{\infty}$ we have proved that $g(\mathbb{D}) = D$. ◀

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