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The Existence and Nonexistence of Global Solutions of the Cauchy Problem for Systems of Three Semilinear Klein–Gordon Equations

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Abstract. In this paper we consider the Cauchy problem for the systems of three Klein-Gordon equations with a week bond with the masses and damping term. We study qualitative characteristics of the family of potential wells, the existence and nonexistence of global solutions, the instability of standing waves, and the behavior of the energy norms of solutions at large time.

Key Words and Phrases: global solution, Klein–Gordon equation, hyperbolic system, existence, nonexistence, stability.

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1. Introduction

The Klein–Gordon equation is distinguished among other nonlinear hyperbolic equations b its theoretical and practical significance. The nonlinear Klein–Gordon equation appears in the study of some problems of mathematical physics. For example, this equation arises in general relativity, nonlinear optics (e.g., in the study of instability phenomena such as self-focusing), plasma physics, fluid mechanics, radiation theory or in the theory of spin waves [1, 2, 3].

The Cauchy problem for nonlinear Klein-Gordon equation

$$u_{tt} - \Delta u + mu + u_t = f(u), \quad t > 0, \ x \in R^n, \tag{1}$$

$$u(0, x) = u_0(x), u_t(0, x) = u_1(x), x \in R^n, \tag{2}$$

has been studied by many authors (see e.g. [4]).

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Existence and nonexistence of global solutions are the main points of study for the problem (1), (2) in the case $m = 0, \quad f(u) \sim |u|^p$ (see e.g. [5, 6]).

In [7, 8], the problem (1), (2) has been investigated in the case $m = 0, \quad f(u) \sim |u|^p$, where $1 < p \leq p_c = 1 + \frac{2}{n}$, and the existence of sufficiently small initial data (u_0, u_1) was proved for which the corresponding Cauchy problem has no global solution. In [7, 8] the Klein-Gordon equation has been investigated in the case $m = 0, \quad f(u) \sim |u|^p$ when $p > p_c = 1 + \frac{2}{n}$, and the existence of a global solution for the problem (1), (2) has been proved for sufficiently small (u_0, u_1).

In the case $m > 0$, i.e for the Klein-Gordon equation with mass, the above effects do not occur. In this case, the main objects of study are the corresponding potential well and stability or instability of standing wave. There is a series of works devoted to that problem [9, 10, 11, 12].

In [12], the Cauchy problem (1), (2) has been studied in the case $f(u) = |u|^{p-1}u$, where $p > 1$, if $n = 2$ and $1 < p < \frac{n+2}{n-2}$ if $n \geq 3$. By investigating the family of potential wells, the set of initial data for which the corresponding Cauchy problem has no global solution has been found . The exponential decay of energetic norms corresponding to global solutions was also established in [12]. More information on the Cauchy problem for the system of Klein-Gordon equations can be found in [13, 14, 15, 16].

In this paper, we consider the Cauchy problem for the systems of three Klein-Gordon equations with a weak bond with the masses and damping term.

Consider the Cauchy problem

$$\begin{cases} u_{1tt} - \Delta u_1 + m_1 u_1 + \gamma_1 u_{1t} = |u_1|^{p_1-1} |u_2|^{p_2+1} |u_3|^{p_3+1} u_1 \\ u_{2tt} - \Delta u_2 + m_2 u_2 + \gamma_2 u_{2t} = |u_1|^{p_1+1} |u_2|^{p_2-1} |u_3|^{p_3+1} u_2 \\ u_{3tt} - \Delta u_3 + m_3 u_3 + \gamma_3 u_{3t} = |u_1|^{p_1+1} |u_2|^{p_2+1} |u_3|^{p_3-1} u_3 \end{cases} \tag{3}$$

in the domain $[0, \infty) \times R^n$with the initial conditions

$$u_i(0, x) = u_{i0}(x), \quad u_{it}(0, x) = u_{i1}(x), \qquad x \in R^n, i = 1, 2, 3, \tag{4}$$

where u_1, u_2, u_3 are real functions depending on $t \in R_+, x \in R^n$;

$$n \geq 2, \ p_j > 0, \ j = 1, 2, 3, \tag{5}$$

and additionally

$$p_1 + p_2 + p_3 \leq 1 \text{if } n = 3. \tag{6}$$

We study the family of potential wells and the existence and nonexistence of global solutions.

In the case $m_j > 0,\ j = 1, 2, 3$, the system (3) determines a model of interaction between three fields with masses m_1, m_2 and m_3 with interaction constants λ_1, λ_2 and λ_3. As in [13], in this paper we examine a quality characteristics of a family of potential wells, the existence and nonexistence of global solutions, the unstable standing waves, the behavior of the energy norms of solution for a large values of time. For the systems of two Klein-Gordon equations the similar problems have been studied in [15].

In the sequel, by $|\,.\,|_q$ we denote the usual $L_q(R^n)$-norm. For simplicity, we write $|\,.\,|_q$ instead of $|\,.\,|$. The scalar product in $L_2(R^n)$ will be denoted by $\langle .,. \rangle$. The norm in the Sobolev space $H^1 = W_2^1(R^n)$, will be denoted by $\|u\| = \left[\left||\nabla u|^2 + |u|^2\right|\right]^{1/2}$, where ∇ is the gradient. The constants C and c used throughout this paper are positive generic constants throughout this paper and can may be different in different occasions.

For simplicity, hereafter we will assume $m_1 = m_2 = m_3 = 1$.

2. Structure of potential well and the existence of a vacuum zone

Consider the system of equations

$$\begin{cases} -\Delta\phi_1 + \phi_1 = |\phi_1|^{p_1-1}\,|\phi_2|^{p_2+1}\,|\phi_3|^{p_3+1}\,\phi_1, \\ -\Delta\phi_2 + \phi_2 = |\phi_1|^{p_1+1}\,|\phi_2|^{p_2-1}\,|\phi_3|^{p_3+1}\,\phi_2, \\ -\Delta\phi_3 + \phi_3 = |\phi_1|^{p_1+1}\,|\phi_2|^{p_2+1}\,|\phi_3|^{p_3-1}\,\phi_3. \end{cases} \tag{7}$$

Suppose $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ is a solution of system (1). Then $(u_1(t,x), u_2(t,x), u_3(t,x)) = (\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$, is the solution of system (3) with initial conditions

$$u_1(0,x) = \overline{\phi_1}(x), \quad u_2(0,x) = \overline{\phi_2}(x), \quad u_3(0,x) = \overline{\phi_3}(x).$$

Then $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ is called a standing solution of the problem (3), (4).
We define the following functionals

$$J(\phi_1, \phi_2, \phi_3) = \sum_{j=1}^{3} \frac{p_j + 1}{2}\,\|\phi_j\|^2 - G,$$

$$I(\phi_1, \phi_2, \phi_3) = \sum_{j=1}^{3} \frac{p_j + 1}{p_1 + p_2 + p_3 + 3}\,\|\phi_j\|^2 - G,$$

where

$$G = G(\phi_1, \phi_1, \phi_1) = \int_{R^n} |\phi_1(x)|^{p_1+1}\,|\phi_2(x)|^{p_2+1}\,|\phi_3(x)|^{p_3+1}\,dx.$$

Lemma 1. *Let* $(\phi_1, \phi_2, \phi_3) \in H^1 \times H^1 \times H^1 \backslash \{(0,0,0)\}$. *Then*
(i) $\lim_{\lambda \to 0} J(\lambda\phi_1, \lambda\phi_2, \lambda\phi_3) = 0$, $\lim_{\lambda \to +\infty} J(\lambda\phi_1, \lambda\phi_2, \lambda\phi_3) = -\infty$;
(ii) there is a single point $\lambda^* = \lambda^*(\phi_1, \phi_2, \phi_3)$ *in the interval* $0 < \lambda < +\infty$, *where*

$$\frac{d}{d\lambda} J(\lambda\phi_1, \lambda\phi_2, \lambda\phi_3)\,|_{\lambda=\lambda^*} = 0;$$

(iii) $J(\lambda\phi_1, \lambda\phi_2, \lambda\phi_3)$ *is not decreasing on* $0 \le \lambda \le \lambda^*$, *not increasing on* $\lambda^* \le \lambda < +\infty$ *and it reaches its maximum at the point* $\lambda = \lambda^*$;
(iv) $I(\lambda\phi_1, \lambda\phi_2, \lambda\phi_3) > 0$ *for* $0 < \lambda < \lambda^*$; $I(\lambda\phi_1, \lambda\phi_2, \lambda\phi_3) < 0$ *for* $\lambda^* < \lambda < +\infty$ *and*

$$I(\lambda^*\phi_1, \lambda^*\phi_2, \lambda^*\phi_3) = 0.$$

We define the set

$$N = \left\{(\phi_1, \phi_2, \phi_3) : (\phi_1, \phi_2, \phi_3) \in H^1 \times H^1 \times H^1 \backslash \{(0,0,0)\},\ I(\phi_1, \phi_2, \phi_3) = 0\right\}.$$

Suppose $(\phi_1, \phi_2, \phi_3) \in N$. Then

$$J(\phi_1, \phi_2, \phi_3) = \left(1 - \frac{2}{p_1 + p_2 + p_3 + 3}\right) \sum_{j=1}^{3} \frac{p_j + 1}{2} \|\phi_j\|^2 > 0,$$

i.e J is bounded from below on the set N. Consider the variation problem

$$d = \inf_{(\phi_1, \phi_2, \phi_3) \in N} J(\phi_1, \phi_2, \phi_3).$$

Lemma 2. *There is* $\left(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3\right) \in N$ *such that*
(i) $J\left(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3\right) = \inf_{(\phi_1, \phi_2, \phi_2) \in N} J(\phi_1, \phi_2, \phi_2) = d > 0$;
(ii) $\left(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3\right)$ *is the standing solution of the problem* (3), (4).

For $\delta > 0$ we define also

$$I_\delta(\phi_1, \phi_2, \phi_3) = \delta \sum_{j=1}^{3} \frac{p_j + 1}{p_1 + p_2 + p_3 + 3} \|\phi_j\|^2 - \int_{R^n} |\phi_1|^{p_1+1} \cdot |\phi_2|^{p_2+1} |\phi_3|^{p_3+1}\, dx,$$

and

$$r(\delta) = r(\delta, p_1, p_2, p_3) = \left(\frac{\delta}{C^{p_1+p_2+p_3+3}}\right)^{\frac{2}{p_1+p_2+p_3+1}},$$

where $C = C_{p_1+p_2+p_3+3} = \sup_{\|u\| \neq 0} \frac{|u|_{p_1+p_2+p_3+3}}{\|u\|}$.

Lemma 3. *If* $(u_1, u_2, u_3) \in H^1 \times H^1 \times H^1 \setminus \{(0,0,0)\}$ *and* $\sum_{j=1}^{3} \frac{p_j+1}{p_1+p_2+p_3+3} \times \|u_j\|^2 < r(\delta)$, *then* $I_\delta(u_1, u_2, u_3) > 0$.

Lemma 4. *If* $(u_1, u_2, u_3) \in H^1 \times H^1 \times H^1$ *and* $I_\delta(u_1, u_2, u_3) < 0$, *then* $\sum_{j=1}^{3} \frac{p_j+1}{p_1+p_2+p_3+3} \|u_j\|^2 > r(\delta)$.

Lemma 5. *If* $(u_1, u_2, u_3) \in H^1 \times H^1 \times H^1 \setminus \{(0,0,0)\}$ *and* $I_\delta(u_1, u_2, u_3) = 0$, *then*

$$\sum_{j=1}^{3} \frac{p_j+1}{p_1+p_2+p_3+3} \|u_j\|^2 \ge r(\delta).$$

Lemma 6. *Suppose that conditions* (5), (6) *are fulfilled. Then*

$$d(\delta) \ge a(\delta)\, r(\delta), \tag{8}$$

where

$$d(\delta) = \delta^{\frac{2}{p_1+p_2+p_3+1}} \frac{p_1+p_2+p_3+3-2\delta}{p_1+p_2+p_3+1} d, \tag{9}$$

$$a(\delta) = \frac{p_1+p_2+p_3+3}{2}\delta. \tag{10}$$

It's obvious that

$$\lim_{\delta \to +0} d(\delta) = 0, \tag{11}$$

$$d\left(\frac{p_1+p_2+p_3+3}{2}\right) = 0, \tag{12}$$

$$d(1) = d, \tag{13}$$

$$d'(\delta) > 0, \delta \in (0,1), \tag{14}$$

$$d'(\delta) < 0, \delta \in \left(1, \frac{p_1+p_2+p_3+3}{2}\right). \tag{15}$$

Let the conditions (5) and (6) be satisfied. Then for arbitrary $(u_{10}, u_{20}, u_{30}) \in H^1 \times H^1 \times H^1$, $(u_{11}, u_{12}, u_{13}) \in L_2(R^n) \times L_2(R^n) \times L_2(R^n)$, there exists $T' > 0$ such that problem (3), (4) has a unique solution $(u_1(\cdot), u_2(\cdot), u_3(\cdot)) \in C([0,T'); H^1 \times H^1 \times H^1) \cap C^1([0,T'); L_2(R^n) \times L_2(R^n) \times L_2(R^n))$. If $T_{max} = supT'$, i.e. T_{max} is the length of the maximal existence interval of the solution $(u_1(\cdot), u_2(\cdot), u_3(\cdot)) \in C([0,T_{max}); H^1 \times H^1 \times H^1) \cap C^1([0,T_{max}); L_2(R^n) \times L_2(R^n) \times L_2(R^n))$, then either $T_{max} = +\infty$, or $\limsup_{t \to T_{\max}-0} \sum_{i=1}^{3} [||u_i(t,\cdot)|| + |\dot{u}_i(t,\cdot)|] = +\infty$.

We denote by $E(t)$ the following energy function:

$$E(t)=\sum_{j=1}^{3}\frac{p_j+1}{2}\left[|u_{jt}(t,\cdot)|^2+\|u_j(t,\cdot)\|^2\right]-$$

$$-\int_{R^n}|u_1(t,x)|^{p_1+1}\cdot|u_2(t,x)|^{p_2+1}|u_3(t,x)|^{p_3+1}\,dx,$$

and we define the following sets

$$W_\delta=\left\{(u_1,u_2,u_3)\in H^1\times H^1\times H^1:\, I_\delta(u_1,u_2,u_3)>0,\right.$$
$$\left.J(u_1,u_2,u_3)<d(\delta)\,\right\}\cup\{(0,0,0)\},$$
$$V_\delta=\left\{(u_1,u_2,u_3)\in H^1\times H^1\times H^1:\, I_\delta(u_1,u_2,u_3)<0,\, J(u_1,u_2,u_3)<d(\delta)\right\},$$

where $0<\delta<r_0$. From (11)-(15) it follows that for every $e\in(0,d)$ the equation $d(\delta)=e$ has two roots δ_1,δ_2, so that $\delta_1<1<\delta_2$.

Theorem 1. *Suppose that* $(u_{10},u_{20},u_{30})\in H^1\times H^1\times H^1$,$(u_{11},u_{12},u_{13})\in L_2(R^n)\times L_2(R^n)\times L_2(R^n)$, *and conditions* (5),(6) *hold. If* $0<e<d$ *and* $\delta_1<\delta_2$ *are the roots of the equation* $d(\delta)=e$, *then the following assertions are valid:*
(a) if $I(u_{10},u_{20},u_{30})>0$ *or* $\|u_{10}\|=\|u_{20}\|=\|u_{30}\|=0$, *then, for all solutions* $(u_1(t,\cdot),u_2(t,\cdot),u_3(t,\cdot))$ *of problem* (3), (4) *with initial energy* $0<E(0)\le e$, $(u_1(t,\cdot),u_2(t,\cdot),u_3(t,\cdot))\in W_\delta$, *where* $\delta_1<\delta<\delta_2$;
(b) if $I(u_{10},u_{20},u_{30})<0$, *then, for all solutions* $(u_1(t,\cdot),u_2(t,\cdot),u_3(t,\cdot))$ *of problem* (3), (4) *with initial energy* $0<E(0)\le e$, $(u_1(t,\cdot),u_2(t,\cdot),u_3(t,\cdot))\in V_\delta$, *where* $\delta_1<\delta<\delta_2$.

Proof. **a**) Let $(u_{10},u_{20},u_{30})\in H^1\times H^1\times H^1$, $(u_{11},u_{12},u_{13})\in L_2(R^n)\times L_2(R^n)\times L_2(R^n)$ and

$$0<E(0)\le e. \tag{16}$$

Let

$$I(u_{10},u_{20},u_{30})>0\ \text{ or }\ \|u_{10}\|=\|u_{20}\|=\|u_{30}\|=0. \tag{17}$$

It follows from (3), (4) that the following energy equality holds

$$E(t)+\sum_{j=1}^{3}\frac{\gamma_j(p_j+1)}{2}\int_0^t|\dot u_j(s,.)|^2\,ds=E(0). \tag{18}$$

By virtue of (16) and (18), $J(u_1(t,\cdot),u_2(t,\cdot),u_3(t,\cdot))<e$. On the other hand, for $\delta_1<\delta<\delta_2$ we have $e<d(\delta)$. Therefore

$$J(u_1(t,\cdot),u_2(t,\cdot),u_3(t,\cdot))<d(\delta). \tag{19}$$

Suppose that the assertion a) does not hold. Then in view of (17) and (19) there exists $\bar{t} \in (0, \infty)$ such that

$$I_\delta\left(u_1(t,\cdot), u_2(t,\cdot), u_3(t,\cdot)\right) > 0, t \in (0, \bar{t}), \tag{20}$$

$$I_\delta\left(u_1(\bar{t},\cdot), u_2(\bar{t},\cdot), u_3(\bar{t},\cdot)\right) = 0. \tag{21}$$

Thus, $(u_1(\bar{t},\cdot), u_2(\bar{t},\cdot), u_3(\bar{t},\cdot)) \in N_\delta$, therefore, by the definition of $d(\delta)$ we have

$$d(\delta) \le J\left(u_1(\bar{t},\cdot), u_2(\bar{t},\cdot), u_3(\bar{t},\cdot)\right),$$

which contradicts (9).

Now we prove the assertion b). Let $(u_{10}, u_{20}, u_{30}) \in H^1 \times H^1 \times H^1$, $(u_{11}, u_{12}, u_{13}) \in L_2(R^n) \times L_2(R^n) \times L_2(R^n)$, $0 < E(0) \le e$ and $I(u_{10}, u_{20}, u_{30}) < 0$. Similar to the case a), we obtain the existence of $\bar{t} \in [0, T]$ such that for any $t \in [0, \bar{t})$ the inequality

$$I\left(u_1(t,\cdot), u_2(t,\cdot), u_3(t,\cdot)\right) < 0,$$

holds and $I\left(u_1(\bar{t},\cdot), u_2(\bar{t},\cdot), u_3(\bar{t},\cdot),\right) = 0$.

Then we again have a contradiction:

$$d(\delta) \le J\left(u_1(\bar{t},\cdot), u_2(\bar{t},\cdot), u_3(\bar{t},\cdot),\right) \le e < d(\delta). \blacktriangleleft$$

By Theorem 1, we have the following theorem.

Theorem 2. *Suppose that* $(u_{10}, u_{20}, u_{30}) \in H^1 \times H^1 \times H^1$, $(u_{11}, u_{12}, u_{13}) \in L_2(R^n) \times L_2(R^n) \times L_2(R^n)$, *and conditions* (5),(6) *hold. If* $0 < E(0) \le e$ *and* δ_1 *and* δ_2 *are the roots of the equation* $d(\delta) = e$, *then the sets* $W_{\delta_1\delta_2} = \bigcup\limits_{\delta_1<\delta<\delta_2} W_\delta$ *and* $V_{\delta_1\delta_2} = \bigcup\limits_{\delta_1<\delta<\delta_2} V_\delta$ *are invariant on the trajectories of the dynamical system generated by problem* (3), (4).

The next theorem is a consequence of Theorem 2 and shows that there is a so-called vacuum zone between the two invariant sets.

Theorem 3. *If the assumptions of Theorem 2 hold, then all solutions of problem* (3), (4) *satisfy the relation* $(u_1(t,\cdot), u_2(t,\cdot), u_3(t,\cdot)) \notin N_{\delta_1,\delta_2} = \bigcup\limits_{\delta_1<\delta<\delta_2} N_\delta$.

Now, consider the case $E(0) \le 0$.

Theorem 4. *Suppose that* $(u_{10}, u_{20}, u_{30}) \in H^1 \times H^1 \times H^1$, $(u_{11}, u_{12}, u_{13}) \in L_2(R^n) \times L_2(R^n) \times L_2(R^n)$, *and conditions* (5),(6) *hold. If* $E(0) = 0$, $\|u_{10}\| \neq 0$, $\|u_{20}\| \neq 0$ *and* $\|u_{30}\| \neq 0$, *then the solution of problem* (3), (4) *satisfies the inequality*

$$\sum_{j=1}^{3} \frac{p_j + 1}{2} \|u_j(t, \cdot)\|^2 \geq r_0, \tag{22}$$

where $r_0 = \left(\frac{p_1+p_2+p_3+3}{2C^2}\right)^{\frac{p_1+p_2+p_3+3}{p_1+p_2+p_2+1}}$.

Proof. Let $(u_1(t, \cdot), u_2(t, \cdot), u_3(t, \cdot))$ be the solution of problem (3), (4) with initial energy $E(0) = 0$, where $\|u_{10}\| \neq 0$, $\|u_{20}\| \neq 0$, $\|u_{30}\| \neq 0$.

Let $T_{\max}$ be the maximal interval of existence of the solution $(u_1(t, \cdot), u_2(t, \cdot), u_3(t, \cdot))$. In view of the definition of $E(t)$, we have

$$E(t) = \sum_{j=1}^{3} \frac{p_j + 1}{2} |\dot{u}_j(t, \cdot)|^2 + J(u_1(t, \cdot), u_2(t, \cdot), u_3(t, \cdot)) = 0, t \in [0, T_{max}). \tag{23}$$

It follows that

$$J(u_1(t, \cdot), u_2(t, \cdot), u_3(t, \cdot)) \leq 0 < d(\delta), \; t \in [0, T_{\max}), \tag{24}$$

and

$$\sum_{j=1}^{3} \frac{p_j + 1}{2} |u_j(t, \cdot)|^2 \leq \int_{R^n} |u_1(t, x)|^{p_1+1} \cdot |u_2(t, x)|^{p_2+1} |u_3(t, x)|^{p_3+1} dx.$$

On the other hand, according to the Hölder's inequality

$$G = \int_{R^n} |u_1(t, x)|^{p_1+1} \cdot |u_2(t, x)|^{p_2+1} |u_3(t, x)|^{p_3+1} dx \leq$$

$$\leq \left(\int_{R^n} |u_1(t, x)|^{p_1+p_2+p_3+3} dx\right)^{\frac{p_{11}+1}{p_1+p_2+p_{31}+3}} \times$$

$$\times \left(\int_{R^n} |u_2(t, x)|^{p_1+p_2+p_3+3} dx\right)^{\frac{p_{12}+1}{p_1+p_2+p_{31}+3}} \times$$

$$\times \left(\int_{R^n} |u_3(t, x)|^{p_1+p_2+p_3+3} dx\right)^{\frac{p_3+1}{p_1+p_2+p_3+3}}. \tag{25}$$

Then, using the embedding theorem [21], we get

$$\sum_{j=1}^{3} \frac{p_j+1}{2} \left| u_j\left(t,\cdot\right) \right|^2 \leq$$

$$\leq C^{p_1+p_2+p_3+3} \left(\frac{2}{p_1+p_2+p_3+3} \right)^{\frac{p_1+p_2+p_3+3}{2}} \left(\sum_{j=1}^{3} \frac{p_j+1}{2} \left| u_j\left(t,\cdot\right) \right|^2 \right)^{\frac{p_1+p_2+p_3+3}{2}} . \tag{26}$$

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Since $(u_{10}, u_{20}, u_{30}) \in H^1 \times H^1 \times H^1$, $(u_{11}, u_{12}, u_{13}) \in L_2\left(R^n\right) \times L_2\left(R^n\right) \times L_2\left(R^n\right)$ and $\|u_{30}\| \neq 0$, there exists a half-interval $[0, t_1)$, where $\|u_1(t,.)\| \neq 0$, $\|u_2(t,.)\| \neq 0$, $\|u_3(t,.)\| \neq 0$. Then from (14) we obtain

$$\sum_{j=1}^{3} \frac{p_j+1}{2} \left\| u_j\left(t,\cdot\right) \right\|^2 \geq \left(\frac{p_1+p_2+p_2+3}{2C^2} \right)^{\frac{p_1+p_2+p_2+3}{p_1+p_2+p_2+1}} = r_0, t \in [0, t_1) . \tag{27}$$

It follows that $\|u_1(t,.)\| \neq 0$, $\|u_2(t,.)\| \neq 0$, $\|u_3(t,.)\| \neq 0$, therefore (26) is also valid on the half-open interval $\delta_1 < \delta < \delta_2$, and so on. Thus, (22) is true on $0 \leq t < T_{max}$.

Theorem 5. *Suppose that* $(u_{10}, u_{20}, u_{30}) \in H^1 \times H^1 \times H^1 \backslash \{0,0,0\}$, $(u_{11}, u_{12}, u_{13}) \in L_2\left(R^n\right) \times L_2\left(R^n\right) \times L_2\left(R^n\right)$, *and conditions* (5),(6) *hold. If* $E\left(0\right) < 0$ *or* $E\left(0\right) = 0$ *and* $(u_{10}, u_{20}, u_{30}) \neq (0,0,0)$, *then* $\left(u_1\left(t,\cdot\right), u_2\left(t,\cdot\right), u_3\left(t,\cdot\right)\right) \in V_\delta$ *for* $t \in [0, T_{max})$, *where* $0 < \delta < \frac{p_1+p_2+p_3+3}{2}$.

Proof. If $E\left(0\right) < 0$, then from (14) we obtain

$$J\left(u_1\left(t,\cdot\right), u_2\left(t,\cdot\right), u_3\left(t,\cdot\right)\right) \leq E\left(0\right) < 0 < d\left(\delta\right) . \tag{28}$$

On the other hand

$$J\left(u_1\left(t,\cdot\right), u_2\left(t,\cdot\right), u_3\left(t,\cdot\right)\right) =$$

$$= \frac{p_1+p_2+p_3+3-2\delta}{p_1+p_2+p_3+3} \sum_{j=1}^{3} \frac{p_j+1}{2} \left| u_j(t,.) \right|^2 + I_\delta(u_1(t,.), u_2(t,.), u_3(t,.)),$$

therefore

$$I_\delta(u_1(t,.), u_2(t,.), u_3(t,.)) < 0, \ t \in [0, T_{max}), \tag{29}$$

if

$$0 < \delta < \frac{p_1+p_2+p_3+3}{2}.$$

If $E(0) = 0$, then in view of Theorem 4, from (27), (28) we find that the inequality (29) is true in this case also if $0 < \delta < \frac{p_1+p_2+p_3+3}{2}$.

Thus

$$(u_1(t,.), u_2(t,.), u_3(t,.)) \in V_\delta,$$

where $0 < \delta < \frac{p_1+p_2+p_3+3}{2}$. ◀

Theorems 3-5 imply the following result.

Theorem 6. *If $E(0) < 0$, then W_1 and V_1 are invariant with respect to the dynamical system generated by problem* (3), (4).

3. Existence and asymptotes of global solutions

Theorem 6 implies the following theorem on global solvability

Theorem 7. *Suppose that $(u_{10}, u_{20}, u_{30}) \in H^1 \times H^1 \times H^1$, $(u_{11}, u_{12}, u_{13}) \in L_2(R^n) \times L_2(R^n) \times L_2(R^n)$, $(u_1(t,\cdot), u_2(t,\cdot)) \in W_\delta$, and conditions* (5),(6) *hold. If $(u_1(t_0,.), u_2(t_0,.), u_3(t_0,.)) \in W_1$ at some moment of time $t_0 \in [0, T_{\max})$, then $T_{max} = +\infty$ and $(u_1(t,.), u_2(t,.), u_3(t,.))$ satisfies a priori estimate*

$$\sum_{j=1}^{3} (p_j + 1) \left[\| u_j(t,.)\|^2 + |\dot{u}_j(t,.)|^2 \right] \le \frac{2d(p_1+p_2+p_3+3)}{p_1+p_2+p_3+1}, \ t \in [0, T_{\max}) . \tag{30}$$

Proof. By Theorem 5, $(u_1(t,.), u_2(t,.), u_3(t,.)) \in W_1,\ t \in [0, T_{\max})$, therefore $I(u_1(t,.), u_2(t,.), u_3(t,.)) > 0,\ \ 0 < t < T_{\max}$. Then from (23) it follows that for $0 \le t < T_{max}$ the a priori estimate (30) is true, therefore $T_{max} = +\infty$, i.e. the problem (3), (4) has a global solution. ◀

Theorem 7 implies the following

Theorem 8. *Suppose that $(u_{10}, u_{20}, u_{30}) \in H^1 \times H^1 \times H^1$, $(u_{11}, u_{12}, u_{13}) \in L_2(R^n) \times L_2(R^n) \times L_2(R^n)$, and conditions* (5), (6) *hold. If $0 < E(0) < d$ and $I_{\delta_2}(u_{10}, u_{20}, u_{30}) > 0$ or $\|u_{10}\| = \|u_{20}\| = \|u_{30}\| = 0$, where $\delta_1 < \delta_2$ are the roots of the equation $d(\delta) = E(0)$, then problem* (3), (4) *has a unique solution $(u_1(\cdot), u_2(\cdot), u_3(\cdot)) \in C([0,\infty); H^1 \times H^1 \times H^1) \bigcap C^1([0,\infty); L_2(R^n) \times L_2(R^n) \times L_2(R^n))$ and*

$$(u_1(t,\cdot), u_2(t,\cdot), u_3(t,\cdot)) \in W_\delta, \delta_1 < \delta < \delta_2, 0 \le t < +\infty.$$

Proof. It is easy to see that $I(u_{10}, u_{20}, u_{30}) > 0$. Indeed, otherwise there would exist $\bar{\delta} \in [1, \delta_2)$ such that $I_{\bar{\delta}}(u_{10}, u_{20}, u_{30}) = 0$. Then $J(u_{10}, u_{20}, u_{30}) \le$

$d(\delta)$, which contradicts the inequality $J(u_{10}, u_{20}, u_{30}) \leq E(0) < d(\delta)$, for $\delta_1 < \delta < \delta_2$.

If $(u_{10}, u_{20}, u_{30}) \in H^2 \times H^2 \times H^2$, $(u_{11}, u_{12}, u_{13}) \in H^1 \times H^1 \times H^1$, then for the solution of problem (3), (4) we have the following identity:

$$I(u_1, u_2, u_3) = (p_1 + p_2 + p_3 + 3)^{-1} \left\{ \sum_{j=1}^{3} (p_j + 1) |\dot{u}_j(t, \cdot)|^2 - \right.$$

$$\left. - \frac{d}{dt} \sum_{j=1}^{3} (p_j + 1) \left[\langle u_j(t, \cdot), \dot{u}_j(t, \cdot) \rangle + \frac{\gamma_j}{2} |u_j(t, \cdot)|^2 \right] \right\}, \tag{31}$$

and the following estimation is valid

$$I(u_1, u_2, u_3) > (1 - \delta_1) \sum_{j=1}^{3} \frac{p_j + 1}{p_1 + p_2 + p_3 + 3} \|u_j(t, \cdot)\|^2, \tag{32}$$

where δ_1 is the lowest root of equation $d(\delta) = E(0)$. ◄

The following theorem on asymptotic behavior of energetic function for $t \to +\infty$ can be proved using (31) and (32).

Theorem 9. *Suppose that* $(u_{10}, u_{20}, u_{30}) \in H^1 \times H^1 \times H^1$, $(u_{11}, u_{12}, u_{13}) \in L_2(R^n) \times L_2(R^n) \times L_2(R^n)$, $0 < E(0) < d$, $I(u_{10}, u_{20}, u_{30}) > 0$ *or* $\|u_{10}\| = \|u_{20}\| = \|u_{30}\| = 0$, *and conditions* (5),(6) *hold. Then there exist* $K > 0$ *and a* $k > 0$ *such that* $E(t) \leq Ke^{-kt}$ *for* $t \geq 0$.

4. Absence of global solutions and instability of standing waves

In this section, we investigate the nonexistence of global solution.

Theorem 10. *Suppose that* $\delta_1 = \delta_2 = \delta_3 \geq 0$, $(u_{10}, u_{20}, u_{30}) \in H^s \times H^s \times H^s$ *and* $(u_{11}, u_{12}, u_{13}) \in H^{s-1} \times H^{s-1} \times H^{s-1}$, *where* $s > \frac{n}{2}$. *Suppose also that conditions* (5),(6) *and one of the following conditions hold:*

a) $E(0) < 0$;

b) $0 \leq E(0) < d$, $I(u_{10}, u_{20}, u_{30}) < 0$ *and* $0 \leq \gamma < \lambda_1(p_1 + p_2 + p_3)$, *where* $\lambda_1 = \frac{1}{c_0}$ *and* c_0 *is the norm of the embedding operator* $W_2^1(R^n) \subset L_2(R^n)$. *Then* $T_{\max} < +\infty$ *and* $\lim\limits_{t \to T_{\max} - 0} \sum_{j=1}^{3} ||u_j(t, \cdot)||^2 = +\infty$.

Proof. a) If $E(0) < 0$, then using the proof given in [16], we obtain the assertion of the theorem.

b) Let $0 \leq E(0) < d$, $I(u_{10}, u_{20}, u_{30}) < 0$ and $0 \leq \gamma^2 < \lambda_1 (p_1 + p_2 + p_3 + 1)$, where $\lambda_1 = \frac{1}{c_0}$. Denoting

$$F(t) = \sum_{j=1}^{3} (p_j + 1) |u_j(t,.)|^2 \ , \ t \in [0, T_{\max}) ,$$

we obtain

$$\dot{F}(t) = 2 \sum_{j=1}^{3} (p_j + 1) \langle u_j(t,.), \dot{u}_j(t,.) \rangle \ , \ t \in [0, T_{\max}) . \ \blacktriangleleft \tag{33}$$

Assume that the assertion of Theorem 10 is not true, i.e $T_{\max} = +\infty$. Since $(u_{10}, u_{20}, u_{30}) \in H^s \times H^s \times H^s$ and $(u_{11}, u_{12}, u_{13}) \in H^{s-1} \times H^{s-1} \times H^{s-1}$, where $s > \frac{n}{2}$, we have

$$(u_1(t,x), u_2(t,x), u_3(t,x)) \in$$
$$\in C([0,\infty), H^s \times H^s \times H^s) \cap C^1 \left([0,\infty), H^{s-1} \times H^{s-1} \times H^{s-1}\right),$$

and obviously $\ddot{F}(t) \in C[0,\infty)$.

Taking into account (3), by a simple calculation we obtain

$$\frac{d}{dt}[e^{\gamma t}\dot{F}(t)] = 2\gamma e^{\gamma t} \sum_{j=1}^{3} (p_j + 1) \langle u_j(t,.), \dot{u}_j(t,.) \rangle +$$

$$+2e^{\gamma t} \sum_{j=1}^{3} (p_j + 1) \ [|\dot{u}_j(t,.)|^2 - \|u_j(t,.)\|^2 - \gamma \langle u_j(t,.), \dot{u}_j(t,.) \rangle] +$$

$$+2(p_1 + p_2 + p_3 + 3)e^{\gamma t} \int_{R^n} |\dot{u}_j(t,x)|^{p_1+1} |\dot{u}_j(t,x)|^{p_2+1} |\dot{u}_j(t,x)|^{p_3+1} dx =$$

$$= 2e^{\gamma t} \sum_{j=1}^{3} (p_j + 1) \ |\dot{u}_j(t,.)|^2 + 2(\delta - 1)e^{\gamma t} \sum_{j=1}^{3} (p_j + 1) \ \|u_j(t,.)\|^2 -$$

$$-2e^{\gamma t} I_\delta(u_1(t,.), u_1(t,.), u_1(t,.)). \tag{34}$$

Since $E(0) < d$, there exist δ_1, δ_2 such that $\delta_1 < 1 < \delta_2$ and

$$d(\delta_i) = E(0) \, , \ \ i = 1, 2.$$

In (33), we take $\delta = \delta_2$. According to Theorem 5

$$I_{\delta_2}(u_1(t,.), u_1(t,.), u_1(t,.)) \leq 0, \tag{35}$$

therefore, from (34), (35) we get

$$\frac{d}{dt}[e^{\gamma t}\dot{F}(t)] \geq 2(\delta_2 - 1)e^{\gamma t}\sum_{j=1}^{3}(p_j+1)\ \|u_j(t,.)\|^2 . \qquad (36)$$

On the other hand, applying Lemma 4, we have the following estimate:

$$\sum_{j=1}^{3}\frac{p_j+1}{p_1+p_2+p_2+3}\ \|u_j\,(t,\cdot)\|^2 > r(\delta_2) \qquad (37)$$

From (36) and (37) it follows that

$$\frac{d}{dt}[e^{\gamma t}\dot{F}(t)] \geq e^{\gamma t}c(\delta_2), \qquad (38)$$

where $c(\delta_2) = 2(\delta_2 - 1)r(\delta_2)\,(p_1 + p_2 + p_2 + 3)$.

From (38) we find that for sufficiently large t_0

$$\dot{F}(t) \geq \frac{A(\delta_2)}{2\gamma}, \quad t \geq t_0, \qquad (39)$$

where $A\,(\delta_2) > 0$. Thus

$$\lim_{t\to+\infty} F(t) = +\infty.$$

On the other hand

$$\ddot{F}(t) = 2\sum_{j=1}^{3}(p_j+1)\ [|\dot{u}_j(t,.)|^2 - \|u_j(t,.)\|^2] - 2\gamma\sum_{j=1}^{3}(p_j+1)\,\langle u_j(t,.),\dot{u}_j(t,.)\rangle +$$

$$+2(p_1+p_2+p_3+3)\int_{R^n}|\dot{u}_j(t,x)|^{p_1+1}\,|\dot{u}_j(t,x)|^{p_2+1}\,|\dot{u}_j(t,x)|^{p_3+1}\,dx =$$

$$= (p_1+p_2+p_3+5)\sum_{j=1}^{3}(p_j+1)\ |\dot{u}_j(t,.)|^2 +$$

$$+ (p_1+p_2+p_3+1)\sum_{j=1}^{3}(p_j+1)\,\|u_j(t,.)\|^2 -$$

$$-2\gamma\sum_{j=1}^{3}(p_j+1)\,\langle u_j(t,.),\dot{u}_j(t,.)\rangle + (p_1+p_2+p_3+3)\sum_{j=1}^{3}\int_0^t|\dot{u}_j(s,.)|^2\,ds -$$

$$-2(p_1+p_2+p_3+3)E(0) \geq (4+\varepsilon)\sum_{j=1}^{3}(p_j+1)\ [|\dot{u}_j(t,.)|^2 + \psi(t), \qquad (40)$$

where

$$\psi(t) = (p_1 + p_2 + p_3 + 1 - \varepsilon) \sum_{j=1}^{3} (p_j + 1)\ |\dot{u}_j(t,.)|^2 +$$

$$+\lambda_1 (p_1 + p_2 + p_3 + 1) \sum_{j=1}^{3} (p_j + 1)\ |u_j(t,.)|^2 -$$

$$-2\gamma \sum_{j=1}^{3} (p_j + 1) \langle u_j(t,.), \dot{u}_j(t,.)\rangle - 2(p + p + p + 3)E(0). \tag{41}$$

Using Hölder's and Young's inequalities, we have

$$\left| 2\gamma \sum_{>=1}^{3} (p_j + 1) \langle u_j(t,.), \dot{u}_j(t,.)\rangle \right| \leq (p_1 + p_2 + p_3 + 1 - \varepsilon) \sum_{j=1}^{3} (p_j + 1)\ |\dot{u}_j(t,.)|^2 +$$

$$+\frac{\gamma^2}{p_1 + p_2 + p_3 + 1 - \varepsilon} \sum_{j=1}^{3} (p_j + 1)\ |u_j(t,.)|^2 . \tag{42}$$

From (40)- (42) it follows that for sufficiently large $t \geq t_0$ the estimate

$$\ddot{F}(t) \geq (4 + \varepsilon) \sum_{j=1}^{3} (p_j + 1)\ [|\dot{u}_j(t,.)|^2 . \tag{43}$$

is true. It follows from (33) and (41) that

$$\ddot{F}(t)F(t) - (1 + \frac{\varepsilon}{4})\dot{F}^2(t) \geq (4 + \varepsilon) \sum_{j=1}^{3} (p_j + 1)\ |\dot{u}_j(t,.)|^2 \cdot \sum_{j=1}^{3} (p_j + 1)\ |u_j(t,.)|^2 -$$

$$-\left(1 + \frac{\varepsilon}{4}\right) \left[\sum_{j=1}^{3} (p_j + 1) \langle u_j(t,.), \dot{u}_j(t,.)\rangle \right]^2 , \quad t \geq t_1.$$

Using Hölder's inequality, we obtain

$$\ddot{F}(t)F(t) - (1 + \frac{\varepsilon}{4})\dot{F}^2(t) \geq 0,\ \ t \geq t_1. \tag{44}$$

From (39) and (44) we obtain the following inequalities

$$\left(F^{-\left(1+\frac{\varepsilon}{4}\right)}(t) \right)'' \leq 0, \quad t \geq t_1,$$

whence it follows that

$$\left(F^{-\left(1+\frac{\varepsilon}{4}\right)}(t)\right)' = \frac{-\left(1+\frac{\varepsilon}{4}\right)F'(t)}{F^{2+\frac{\varepsilon}{2}}(t)} < 0, \quad t \geq t_1. \tag{45}$$

In view of (39) and (45), there exists a $t^* \in (0, t_1)$ such that $\lim\limits_{t\to t^*} F^{-1}(t) = 0$, i.e. $\lim\limits_{t\to t^*} F(t) = +\infty$.

This contradiction shows that $T_{\max} < +\infty$.

Remark 1. *Under the assumptions of Theorem 10*

$$\overline{\underset{t\to T_{\max}-0}{Lim}} \sum_{j=1}^{3}\left[|||u_j(t,\cdot)|||^2 + ||\dot{u}_j(t,\cdot)||^2\right] = +\infty.$$

Theorem 11. *Suppose that conditions* (5),(6) *hold and*

$$E(0) > 0, I(u_{10}, u_{20}, u_{30}) < 0, \sum_{j=1}^{3} \frac{p_j+1}{2} |||u_{j0}|||^2 > \frac{p_1+p_2+p_3+3}{p_1+p_2} E(0).$$

Then the solution of the Cauchy problem (8), (9) *blow up in finite time (see [17]).*

Remark 2. *Under the conditions* (5), (6), *from Theorems 10 and 11 it follows that the standing waves generated by problems* (3), (4) *are unstable.*

5. Proofs of Lemmas

Proof of Lemma 1. Properties (i) follow directly from

$$J(\lambda\Phi_1, \lambda\Phi_2, \lambda\Phi_3) = \lambda^2 \sum_{j=1}^{3} \frac{p_j+1}{2}\left(|\nabla\Phi_j|^2 + |\Phi_j|^2\right) -$$

$$-\lambda^{p_1+p_2+p_3+2} \int_{R^n} |\Phi_1(x)|^{p_1+1} \cdot |\Phi_2(x)|^{p_2+1} |\Phi_3(x)|^{p_3+1} dx.$$

(ii) Elementary computation shows that

$$\frac{d}{d\lambda} J(\lambda\Phi_1, \lambda\Phi, \lambda\Phi_3) = \lambda \sum_{j=1}^{3} (p_j+1) \|\Phi_j\|^2 -$$

$$-\left(p_1+p_2+p_3+3\right)\lambda^{p_1+p_2+p_3+2}\cdot\int_{R^n}\left|\Phi_1(x)\right|^{p_1+1}\left|\Phi_2(x)\right|^{p_2+1}\left|\Phi_3(x)\right|^{p_3+1}dx. \tag{46}$$

Hence, it is evident that at the point

$$\lambda^*=\left[\frac{\sum_{j=1}^3\left(p_j+1\right)\|\Phi_j\|^2}{\left(p_1+p_2+p_3+3\right)\int_{R^n}\left|\Phi_1(x)\right|^{p_1+1}\left|\Phi_2(x)\right|^{p_2+1}\left|\Phi_2(x)\right|^{p_3+1}dx}\right]^{\frac{1}{p_1+p_2+p_3+1}},$$

the following equality holds

$$\frac{d}{d\lambda}J\left(\lambda\Phi_1,\lambda\Phi_2,\lambda\Phi_3\right)|_{\lambda=\lambda^*}=0.$$

(iii) From (46) it is clear that

$$\frac{d}{d\lambda}J\left(\lambda\Phi_1,\lambda\Phi_2,\lambda\Phi_3\right)>0 \text{ for the } 0<\lambda<\lambda^*,$$

and

$$\frac{d}{d\lambda}J\left(\lambda\Phi_1,\lambda\Phi_2,\lambda\Phi_3\right)<0 \text{ for the } \lambda^*<\lambda<+\infty,$$

i.e. the assertion (iii) is true.

(iv) From definitions of functionals J and I, it also follows from (46) that

$$I\left(\lambda\Phi_1,\lambda\Phi_2,\lambda\Phi_3\right)=\frac{\lambda}{p_1+p_2+p_3+3}\frac{d}{d\lambda}J\left(\lambda\Phi_1,\lambda\Phi_2,\lambda\Phi_3\right).$$

We define the set

$$N=\left\{\left(\phi_1,\phi_2,\phi_3\right):\ \left(\phi_1,\phi_2,\phi_3\right)\in H^1\times H^1\times H^1\setminus\{(0,0,0)\},\ I\left(\phi_1,\phi_2,\phi_3\right)=0\right\}.$$

Suppose $(\phi_1,\phi_2,\phi_3)\in N$, Then

$$J\left(\phi_1,\phi_2,\phi_3\right)=\left(1-\frac{2}{p_1+p_2+p_3+3}\right)\sum_{j=1}^3\frac{p_j+1}{2}\|\phi_j\|^2>0, \tag{47}$$

i.e. J is bounded from below on the set N. Thus J is bounded from below on the set N. Let us consider the variation problem

$$d=\inf_{(\phi_1,\phi_2,\phi_3)\in N}J\left(\phi_1,\phi_2,\phi_3\right). \blacktriangleleft$$

Proof of Lemma 2. From (47) it follows that if $(u_1,u_2,u_3)\in N$, then

$$J\left(u_1,u_2,u_3\right)=\frac{p_1+p_2+p_3+1}{p_1+p_2+p_3+3}\sum_{j=1}^3\frac{p_j+1}{2}\|u_j\|^2.$$

Let (u_{1m}, u_{2m}, u_{3m}) be a minimizing sequence, i.e.

$$\underset{m\to\infty}{Lim}\, J(u_{1m}, u_{2m}, u_{3m}) = \inf_{(u_1,u_2,u_3)\in N} J(u_1, u_2, u_3) = d.$$

Let's denote $u_{j\lambda} = \lambda u_j,\ j = 1,2,3$ and denote by $v_{jm} = (u^*_{jm})_{\mu_m}$ the Schwarz symmetrization [18, 19, 20] of the function $y_{jm} = \mu_m u_{jm}$ with respect to the variable x, where μ_m is chosen so that $(v_{1m}, v_{2m}, v_{3m}) \in N$

We denote by $v_{jm} = (u^*_{jm})_{\mu_m}$ the Schwartz symmetrization [18, 19, 20] of the function $y_{jm} = \mu_m u_{jm}$, where μ_m is chosen so that $(v_{1m}, v_{2m}, v_{3m}) \in N$.
By virtue of (47)

$$J(v_{1m}, v_{2m}, v_{3m}) = (1 - \frac{2\delta}{p_1+p_2+p_3+3}) \cdot \sum_{j=1}^{3} \frac{p_j+1}{2} \|v_{jm}\|^2 , \qquad (48)$$

On the other hand

$$\int_{R^n} |\nabla v_{jm}|^2 dx = \int_{R^n} \left|\nabla (u^*_{jm})_{\mu_m}\right|^2 dx =$$
$$= \int_{R^n} |(\nabla (u_{jm})_{\mu_m})^*|^2 dx \ \le \int_{R^n} |\nabla (u_{jm})_{\mu_m}|^2 dx, \qquad (49)$$

From (48),(49) it follows that

$$J(v_{1m}, v_{2m}, v_{3m}) \le J((u_{1m})_{\mu_m}, (u_{2m})_{\mu_m}, (u_{3m})_{\mu_m}). \qquad (50)$$

On the other hand, by the choice of μ_n, we have

$$J((u_{1m})_{\mu_m}, (u_{2m})_{\mu_m}, (u_{3m})_{\mu_m}) \le J(u_{1m}, u_{2m}, u_{3m}). \qquad (51)$$

Consequently, $\underset{m\to\infty}{Lim}\, J(v_{1m}, v_{2m}, v_{3m}) = d$.

It follows that

$$\|\nabla v_{jm}\| \le c. \qquad (52)$$

Then we conclude that there exists $(v_{1\infty} v_{2\infty} v_{3\infty}) \in H^1 \times H^1 \times H^1$ such that, possibly taking $m \to +\infty$, along a subsequence,

$$v_{jm} \to v_{j\infty} \text{ weakly in } H^1\ \ j = 1,2,3. \qquad (53)$$

Then, by virtue of the compactness of the embedding $H^1_{ralial} \subset L_{p_1+p_2+p_3+3}(R^n)$ (see [21]), where $p_1+p_2+p_3+3 \le \frac{2n}{n-2}$, we obtain

$$v_{jm} \to v_{j\infty} \text{ in } L_{p_1+p_2+p_3+3}(R^n) \text{ as } m \to +\infty,\ j = 1,2,3. \qquad (54)$$

Let us prove that $(v_{1\infty}, v_{2\infty}, v_{3\infty}) \ne (0,0,0)$.

Assume the opposite, i.e. suppose that

$$(v_{1\infty}, v_{2\infty\infty}, v_{3\infty}) = (0,0,0). \tag{55}$$

Then, by (25), (54) and (55), we obtain

$$\int_{R^n} |v_{1m}(x)|^{p_1+1} |v_{2m}(x)|^{p_2+1} |v_{3m}(x)|^{p_3+1} dx \to 0 \ \text{ as } m \to +\infty.$$

On the other hand, $I(v_{1m}, v_{1m}, v_{1m}) = 0$, so from (54) it follows that

$$v_{jm} \to 0 \text{ in } H^1 \text{ as } m \to +\infty,\ j = 1,2,3. \tag{56}$$

Then, using again the condition $(v_{1m}, v_{1m}, v_{1m}) \in N$, Hölder's inequality and embedding theorem $H^1 \subset L_{p_1+p_2+p_3+3}(R^n)$ (see [22]), we obtain

$$\sum_{j=1}^{3} \frac{p_j+1}{p_1+p_2+p_3+3} \|v_{jm}\|^2 = \int_{R^n} |v_{1m}|^{p_1+1} \cdot |v_{2m}|^{p_2+1} |v_{3m}|^{p_3+1} dx \le$$

$$\le \|v_{1m}\|^{p_1+1}_{L_{p_1+p_2+p_3+3}(R^n)} \|v_{2m}\|^{p_2+1}_{L_{p_1+p_2+p_3+3}(R^n)} \|v_{3m}\|^{p_{31}+1}_{L_{p_1+p_2+p_3+3}(R^n)}.$$

By the Gagliardo–Nirenberg type multiplicative inequality, we have

$$\|v_{jm}\|^{p_j+1}_{L_{p_1+p_2+p_3+3}(R^n)} \le |\nabla v_{jm}|^{(p_j+1)\theta} |v_{jm}|^{(p_j+1)(1-\theta)} \ (see\ [21]), \tag{57}$$

where

$$\theta = n\left(\frac{1}{2} - \frac{1}{p_1+p_2+p_3+3}\right), j = 1,2,3.$$

From (52) and (57) we have

$$\|v_{jm}\|^{p_j+1}_{L_{p_1+p_2+p_3+3}(R^n)} \le c\, |\nabla v_{jm}|^{(p_j+1)\theta}, j = 1,2,3.$$

Consequently

$$\sum_{j=1}^{3} \frac{p_j+1}{p_1+p_2+p_3+3} \|v_{jm}\|^2 \le c^3 \left(\sum_{j=1}^{3} \|v_{jm}\|^2\right)^{\frac{n}{2}(p_1+p_2+p_3+1)}.$$

It follows that

$$\sum_{j=1}^{3} \|v_{jm}\|^2 \ge c_1 > 0.$$

Therefore our assumption isn't correct. Thus $d > 0$. ◀

Proof of Lemma 3. Using the inequality (25), embedding $H^1 \subset L_{p_1+p_2+p_3+3}(R^n)$ (see [22]) and Young's inequality, we get

$$\int_{R^n} |u_1(x)|^{p_1+1} |u_2(x)|^{p_2+1} |u_3(x)|^{p_3+1} dx \leq$$

$$\leq C^{p_1+p_2+p_3+3} \left[\sum_{j=1}^{3} \frac{p_j+1}{p_1+p_2+p_3+3} \|u_j\|^2 \right]^{\frac{p_1+p_2+p_3+1}{2}+1}.$$

If $\sum_{j=1}^{3} \frac{p_j+1}{p_1+p_2+p_3+3} \|u_j\|^2 < r(\delta)$, then we get

$$\int_{R^n} |u_1(x)|^{p_1+1} |u_2(x)|^{p_2+1} |u_3(x)|^{p_3+1} dx \leq \delta \sum_{j=1}^{3} \frac{p_j+1}{p_1+p_2+p_3+3} \|u_j\|^2.$$

From the definition of $I_\delta(u_1, u_2, u_3)$, we have $I_\delta(u_1, u_2, u_3) > 0$. ◀

Proof of Lemma 4. If $(u_1, u_2, u_3) \in H^1 \times H^1 \times H^1$, $\|u_1\| \neq 0$, $\|u_2\| \neq 0$, $\|u_3\| \neq 0$ and $I_\delta(u_1, u_2, u_3) < 0$, then we have the following inequality

$$\delta \sum_{j=1}^{3} \frac{p_j+1}{p_1+p_2+p_3+3} \|u_j\|^2 \leq$$

$$\leq C^{p_1+p_2+p_3+3} \left(\sum_{j=1}^{3} \frac{p_j+1}{p_1+p_2+p_3+3} \|u_j\|^2 \right)^{\frac{p_1+p_2+p_3+1}{2}+1},$$

whence the required inequality follows. ◀

Proof of Lemma 5. If $\|u_1\| \neq 0$, $\|u_2\| \neq 0$, $\|u_3\| \neq 0$, then from $I_\delta(u_1, u_2, u_3) = 0$ we get

$$\delta \sum_{j=1}^{3} \frac{p_j+1}{p_1+p_2+p_3+2} \|u_j\|^2 \int_{R^n} |u_1(x)|^{p_1+1} \cdot |u_2(x)|^{p_2+1} |u_3(x)|^{p_3+1} dx \leq$$

$$\leq C^{p_1+p_2+p_3+3} \left(\sum_{j=1}^{3} \frac{p_j+1}{p_1+p_2+p_3+3} \|u_j\|^2 \right)^{\frac{p_1+p_2+p_3+1}{2}+1}.$$

Thus

$$\sum_{j=1}^{3} \frac{p_j+1}{p_1+p_2+p_3+3} \|u_j\|^2 \geq r(\delta) = \left(\frac{\delta}{C^{p_1+p_2+p_3+3}} \right)^{\frac{2}{p_1+p_2+p_3+1}}. \ ◀$$

Proof of Lemma 6. In view of Lemma 5, for each $(u_1, u_2, u_3) \in N$ we have

$$\sum_{j=1}^{3} \frac{p_j + 1}{p_1 + p_2 + p_3 + 2} \|u_j\|^2 \geq r(\delta).$$

Therefore

$$J(u_1, u_2, u_3) = \left(\frac{p_1 + p_2 + p_3 + 3}{2} - \delta\right) \sum_{j=1}^{3} \frac{p_j + 1}{p_1 + p_2 + p_3 + 3} \|u_j\|^2 \geq a(\delta) r(\delta),$$

where $0 < \delta < \frac{p_1+p_2+p_3+3}{2}$, $a(\delta) = \frac{p_1+p_2+p_3+3}{2} - \delta$. It follows that $d(\delta) \geq a(\delta) r(\delta)$. Suppose that $(\bar{u}_1, \bar{u}_2, \bar{u}_3) \in N$ is the minimizing element, i.e. $d = J(\bar{u}_1, \bar{u}_2, \bar{u}_3)$

For any $\delta > 0$ we choose $\lambda = \lambda(\delta)$ such that

$$\delta \sum_{j=1}^{3} \frac{p_j + 1}{p_1 + p_2 + p_3 + 3} \|\lambda \bar{u}_j\|^2 = \int_{R^n} |\lambda \bar{u}_1|^{p_1+1} \cdot |\lambda \bar{u}_2|^{p_2+1} |\lambda \bar{u}_3|^{p_3+1} dx. \tag{58}$$

Hence

$$\lambda(\delta) = \left[\frac{\delta \sum_{j=1}^{3} (p_j + 1) \left\|\underset{j}{\bar{u}}\right\|^2}{(p_1 + p_2 + p_2 + 3) \int_{R^n} |\bar{u}_1|^{p_1+1} \cdot |\bar{u}_2|^{p_2+1} |\bar{u}_3|^{p_3+1} dx}\right]^{\frac{1}{p_1+p_2+p_3+1}} =$$

$$= \delta^{\frac{1}{p_1+p_2+p_3+1}}$$

In view of (58), $(\lambda(\delta) \bar{u}_1, \lambda(\delta) \bar{u}_2, \lambda(\delta) \bar{u}_3) \in N_\delta$, therefore, by definition of $d(\delta)$, we have the following inequality

$$d(\delta) \leq J(\lambda(\delta) \bar{u}_1, \lambda(\delta) \bar{u}_2, \lambda(\delta) \bar{u}_3) =$$

$$= \delta^{\frac{2}{p_1+p_2+p_2+1}} \sum_{j=1}^{3} \frac{p_j + 1}{2} \|\bar{u}_j\|^2 - \delta^{1+\frac{2}{p_1+p_2+p_2+1}} \int_{R^n} |\bar{u}_1|^{p_1+1} \cdot |\bar{u}_2|^{p_2+1} |\bar{u}_3|^{p_3+1} dx. \tag{59}$$

On the other hand

$$(\bar{u}_1, \bar{u}_2, \bar{u}_3) \in N. \tag{60}$$

Therefore

$$\int_{R^n} |\bar{u}_1|^{p_1+1} \cdot |\bar{u}_2|^{p_2+1} |\bar{u}_3|^{p_3+1} dx = \sum_{j=1}^{3} \frac{p_j + 1}{p_1 + p_2 + p_3 + 3} \|\bar{u}_j\|^2. \tag{61}$$

It follows from (59) and (61) that

$$d(\delta) \leq \delta^{\frac{2}{p_1+p_2+p_2+1}} \left(1 - \frac{2\delta}{p_1+p_2+p_3+3}\right) \sum_{j=1}^{3} \frac{p_j+1}{2} \|\bar{u}_j\|^2 . \tag{62}$$

Since $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ is the minimizing element, we have

$$d = J(\bar{u}_1, \bar{u}_2, \bar{u}_3) = \frac{p_1+p_2+p_2+1}{p_1+p_2+p_3+3} \sum_{j=1}^{3} \frac{p_j+1}{2} \|\bar{u}_j\|^2 ,$$

i.e.

$$\sum_{j=1}^{3} \frac{p_j+1}{2} \|\bar{u}_j\|^2 = \frac{p_1+p_2+p_3+3}{p_1+p_2+p_3+1} d. \tag{63}$$

It follows from (62) and (63) that

$$d(\delta) \leq \frac{p_1+p_2+p_3+3-2\delta}{p_1+p_2+p_3+1} \delta^{\frac{2}{p_1+p_2+p_3+1}} d. \tag{64}$$

Let $(\bar{v}_1, \bar{v}_2, \bar{v}_3) \in N_\delta$ be the minimizing element of the functional $J(u_1, u_2, u_3)$, i.e.

$$J(\bar{v}_1, \bar{v}_2, \bar{v}_3) = \min_{(\bar{v}_1, \bar{v}_{2,-}, \bar{v}_3) \in N_\delta} J(v_1, v_2, v_3) = d(\delta) .$$

The parameter $\mu = \mu(\delta)$ is chosen so that

$$(\mu v_1, \mu v_2, \mu v_3) \in N, I(\mu \bar{v}_1, \mu \bar{v}_2, \mu \bar{v}_3) = 0. \tag{65}$$

Then

$$\mu = \mu(\delta) = \left[\frac{\sum_{j=1}^{3} (p_j+1) \|\bar{v}_j\|^2}{(p_1+p_2+p_3+3) \int_{R^n} |\bar{v}_1|^{p_1+1} |\bar{v}_2|^{p_2+1} |\bar{v}_3|^{p_3+1} dx} \right]^{\frac{1}{p_1+p_2+p_3+1}} =$$

$$= \left(\frac{1}{\delta}\right)^{\frac{1}{p_1+p_2+p_3+1}} .$$

By the definition of d, we have

$$d \leq J(\mu \bar{v}_1, \mu \bar{v}_2, \mu \bar{v}_3) = \left(\frac{1}{\delta}\right)^{\frac{1}{p_1+p_2+p_3+1}} \sum_{j=1}^{3} \frac{p_j+1}{2} \|\bar{v}_j\|^2 -$$

$$- \left(\frac{1}{\delta}\right)^{\frac{p_1+p_2+p_3+3}{p_1+p_2+p_3+1}} \int_{R^n} |\bar{v}_1|^{p_1+1} \cdot |\bar{v}_2|^{p_2+1} |\bar{v}_3|^{p_3+1} dx =$$

$$= \left(\frac{1}{\delta}\right)^{\frac{1}{p_1+p_2+p_3+1}} \frac{p_1+p_2+p_3+1}{p_1+p_2+p_3+3} \sum_{j=1}^{3} \frac{p_j+1}{2} \|\bar{v}_j\|^2 . \qquad (66)$$

On the other hand, from (65) and (66) we get

$$J(\bar{v}_1, \bar{v}_2, \bar{v}_3) = (1 - \frac{2\delta}{p_1+p_2+p_3+3}) \cdot \sum_{j=1}^{3} \frac{p_j+1}{2} \|v_j\|^2 .$$

Hence we have

$$\sum_{j=1}^{3} \frac{p_j+1}{2} \|\bar{v}_j\|^2 = \frac{p_1+p_2+p_3+3}{p_1+p_2+p_3+3-2\delta} J(\bar{v}_1, \bar{v}_2, \bar{v}_3) =$$

$$= \frac{p_1+p_2+p_3+3}{p_1+p_2+p_3+3-2\delta} d(\delta) . \qquad (67)$$

From (66) and (67) it follows that

$$d \leq \left(\frac{1}{\delta}\right)^{\frac{1}{p_1+p_2+p_3+1}} \frac{p_1+p_2+p_3+1}{p_1+p_2+p_3+3-2\delta} d(\delta)$$

i.e.

$$d(\delta) \geq \frac{p_1+p_2+p_3+3-2\delta}{p_1+p_2+p_3+1} \delta^{\frac{1}{p_1+p_2+p_3+1}} d. \qquad (68)$$

Comparing (64) and (68), we obtain

$$d(\delta) = \frac{p_1+p_2+p_3+3-2\delta}{p_1+p_2+p_3+1} \delta^{\frac{1}{p_1+p_2+p_3+1}} d.$$

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