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# Jackson and Inverse Inequalities in Rearrangement Invariant Banach Function Spaces on Dini-Smooth Domains

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**Abstract.** In this work we obtain some Jackson type direct theorem and converse theorem of polynomial approximation with respect to fractional order moduli of smoothness in rearrangement invariant quasi Banach function spaces (RIQBFS) on the unit circle. Later using these results we obtain similar estimates for some RIBFS on sufficiently smooth domains of complex plane.

**Key Words and Phrases**: direct theorem, inverse theorem, modulus of smoothness, rearrangement invariant space.

**2010 Mathematics Subject Classifications**: 42A10, 46E30

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## 1. Introduction

The main inequalities of approximation such as Jackson and inverse inequalities in the different subspaces of the rearrangement invariant quasi Banach function spaces (RIQBFS) on $[0, 2\pi]$ or on rectifiable Jordan curves $\Xi$ were investigated by several mathematicians. For example, the degree of polynomial approximation in the Smirnov spaces $E^p(G)$ and Lebesgue spaces $L^p(\Xi)$ on $\Xi$, have been estimated in [1-13] under various restrictions on the boundary $\Xi$ of $G$. The similar problems in weighted Smirnov and Lebesgue spaces (on $\Xi$) were studied in [14] and [15]. The appropriate inverse theorems and a constructive characterization of generalized Lipschitz class in the weighted Smirnov spaces were obtained in [16]. Some inverse theorems in Smirnov-Orlicz spaces were proved in [17]. Some direct theorems of approximation theory were obtained in [18], [19] and [20] by using algebraic and interpolating polynomials. Generalized Faber polynomials are important apparatus of approximation in various functional classes (see, e.

 

g., [15, 21]). Basicity of systems of generalized Faber polynomials was considered in [22]. The general solution of the homogeneous Riemann problem in the weighted Smirnov classes was investigated in [23]. Approximation problems in Morrey-Smirnov classes were considered in [24].

On the other hand, moduli of smoothness of fractional order were defined by Taberski [25] and Butzer, Dychoff, Gorlich, Stens [26] and used for various approximation problems on Lebesgue spaces. Fractional smoothness is necessary [27] for some Ulyanov type inequalities. Using fractional smoothness, direct and inverse theorems of trigonometric approximation in RIQBFS $\mathbb{X}([0,2\pi])$ on $[0,2\pi]$ were obtained in [28]. In this work, we obtain some Jackson type direct theorem and converse theorem of polynomial approximation with respect to fractional order moduli of smoothness in the spaces RIQBFS $\mathbb{X}(\mathsf{T})$ on complex unit circle $\mathsf{T}$. Later, using these results we obtain that similar estimates for RIBFS $\mathbb{X}(\Gamma)$ on sufficiently smooth domains of complex plane hold.

The paper is organized as follows. In Section 2, the central theorems of trigonometric approximation in the spaces RIQBFS $\mathbb{X}(\mathsf{T})$ are proved. Sections 3 and 4 include some necessary lemmas and proofs of the results of Section 2. In Sections 5 and 6 we give the main inequalities of algebraic approximation in some RIBFS $\mathbb{X}(\Gamma)$ on sufficiently smooth domains of complex plane.

Throughout this work, by $C, c, c_i$ we denote the constants which are absolute or depend only on the parameters given in their brackets.

## 2. Approximation on the unit circle

Let $\mathcal{M}$ be the set of all measurable functions defined on $\mathsf{T} := \left\{e^{i\theta} : \theta \in [0,2\pi)\right\}$ and let $\mathcal{M}^+$ be the subset of functions from $\mathcal{M}$ whose values lie in $[0,\infty]$. By $\chi_E$ we denote the characteristic function of a measurable set $E \subset \mathsf{T}$. A mapping $\rho : \mathcal{M}^+ \to [0,\infty]$ is called a *function norm* if for all constants $a \geq 0$, for all functions $f, g, f_n \quad (n = 1,2,3,\ldots)$, and for all measurable subsets $E$ of $\mathsf{T}$, the following properties hold:

(i) $\rho(f) = 0$ iff $f = 0$ a.e.; $\rho(af) = a\rho(f)$; $\rho(f+g) \leq \rho(f) + \rho(g)$,

(ii) if $0 \leq g \leq f$ a.e., then $\rho(g) \leq \rho(f)$, (iii) if $0 \leq f_n \uparrow f$ a.e., then $\rho(f_n) \uparrow \rho(f)$,

(iv) $\rho(\chi_E) < \infty$ holds for every set $E \subset \mathsf{T}$ having a finite Lebesgue measure $|E| < \infty$,

(v) $\int_E |f(w)|\,|dw| \leq C_E\ \rho(f)$ holds for every set $E \subset \mathsf{T}$ having a finite Lebesgue measure $|E| < \infty$, with a constant $C_E \in (0,\infty)$ depending on $E$ and $\rho$ but independent of $f$.

If $\rho$ is a function norm, its *associate norm* $\rho'$ is defined on $\mathcal{M}^+$ by

$$\rho'(g) := \sup\left\{\int_{\mathsf{T}} |f(w)\, g(w)|\, |dw| : f \in \mathcal{M}^+,\, \rho(f) \le 1\right\}, \quad g \in \mathcal{M}^+.$$

If $\rho$ is a function norm, then $\rho'$ is itself a function norm [29, p. 8, Th. 2.2]. Let $\rho$ be a function norm. The collection of functions

$$X(\mathsf{T}) := X(\mathsf{T}, \rho) := \{f \in \mathcal{M} : \rho(|f|) < \infty\}$$

is called *Banach function space* (shortly BFS) on $\mathsf{T}$. For each $f \in X(\mathsf{T})$ we define

$$\|f\|_{X(\mathsf{T})} := \rho(|f|).$$

A Banach function space $X$ equipped with the norm $\|\cdot\|_{X(\mathsf{T})}$ is a Banach space [29, p. 3-5, Theorems. 1.4 and 1.6]. Let $\rho'$ be the associate norm of a function norm $\rho$. The Banach function space $X(\mathsf{T}, \rho')$ determined by the function norm $\rho'$ is called the *associate space* of $X(\mathsf{T}) = X(\mathsf{T}, \rho)$ and is denoted by $X'(\mathsf{T})$. It is well-known [29, p. 9] that

$$\|f\|_{X(\mathsf{T})} = \sup\left\{\int_{\mathsf{T}} |f(w)\, g(w)|\, |dw| : g \in X'(\mathsf{T}),\ \|g\|_{X'(\mathsf{T})} \le 1\right\}, \tag{1}$$

holds. The distribution function $\mu_f$ of a measurable function $f$ is defined as

$$\mu_f(\lambda) = measure\{w \in \mathsf{T} : |f(w)| > \lambda\},\ \lambda \ge 0.$$

A Banach function norm is *rearrangement invariant* if $\rho(f) = \rho(g)$ for every pair of functions $f$, $g$ which are equimeasurable, that is $\mu_f(\lambda) = \mu_g(\lambda)$.

Given a Banach function space $X(\mathsf{T})$, we define $X_r(\mathsf{T}) := \{f \in \mathcal{M} : f^r \in X(\mathsf{T})\}$, $r \in (0, \infty)$ and $r$-norm as

$$\|f\|_{Xr(\mathsf{T})} := \||f|^r\|_{X(\mathsf{T})}^{1/r}.$$

A *quasi Banach function norm* is a mapping $\rho : \mathcal{M}^+ \to [0, \infty]$ such that it satisfies (ii)-(iv) of above definition of function norm, but satisfies (i) as a quasinorm, namely, $\rho(f) = 0$ if and only if $f = 0$ a.e.; $\rho(af) = a\rho(f)$; $\rho(f + g) \le c(\rho(f) + \rho(g))$. If a quasi Banach function norm $\rho$ is rearrangement invariant, then the collection of functions $X(\mathsf{T}, \rho) = \{f \in \mathcal{M} : \rho(|f|) < \infty\}$ will be called *rearrangement invariant quasi Banach function space* (shortly RIQBFS). A quasi BFS $X(\mathsf{T})$ is said to be $p$-convex for some $p \in (0, 1]$ if there is a $c$ such that for all $f_1, \dots, f_N \in X(\mathsf{T})$ we have

$$\left\|\left(\sum_{i=1}^N |f_i|^p\right)^{1/p}\right\|_{X(\mathsf{T})} \le c\left(\sum_{i=1}^N \|f_i\|_{X(\mathsf{T})}^p\right)^{1/p}. \tag{2}$$

In this case the condition (2) is equivalent to the fact that $X_{1/p}(\mathsf{T})$ is a rearrangement invariant BFS. From (1) one can see that $\|\cdot\|_{X(\mathsf{T})}$ can be equivalently represented [30] as

$$\|f\|_{X(\mathsf{T})} \asymp \sup\left\{\left(\int_{\mathsf{T}} |f(w)|^p |g(w)| |dw|\right)^{1/p} : \|g\|_{Y'(\mathsf{T})} \leq 1\right\}, \tag{3}$$

where $Y'(\mathsf{T})$ is the associate space of the rearrangement invariant BFS $Y(\mathsf{T}) = X_{1/p}(\mathsf{T})$. $A(x) \asymp B(x)$ will mean that there exist constants $c$ and $C$ such that $cA(x) \leq B(x) \leq CA(x)$. There are examples [31] of quasi BFS which are not $p$-convex for any $p > 0$.

Let $X(\mathsf{T})$ be a quasi BFS. A function $f \in X(\mathsf{T})$ is said to have *absolutely continuous norm* if

$$\lim_{n\to\infty} \|f\chi_{A_n}\|_{X(\mathsf{T})} = 0$$

for every decreasing sequence of measurable sets $(A_n)$ with $\chi_{A_n} \to 0$ a.e. If every $f \in X(\mathsf{T})$ has this property, we will say $X(\mathsf{T})$ has *absolutely continuous norm.*

Hereafter throughout this work we will assume that $\mathbb{X}(\mathsf{T}) := X(\mathsf{T}, AC, p)$ is a RIQBFS which has absolutely continuous norm and is $p$-convex for some $p \in (0,1]$. These assumptions on the function space are not very restrictive. For example, Orlicz spaces on $\mathsf{T}$, classical Lorentz spaces $L^{pq}(\mathsf{T})$, $p, q \in (0,\infty)$, Zygmund spaces $L^p(\log L)^\alpha(\mathsf{T})$, $p \in (0,\infty)$, $\alpha \in \mathbb{R}$, Lorentz $\Lambda(\mathsf{T})$ spaces and Marcinkiewicz spaces on $\mathsf{T}$ satisfy [30] these conditions. For a complete treatise of rearrangement invariant BFS and RIQBFS we refer to [29, 32, 33, 34].

**Remark 1.** *Let $X(\mathsf{T})$ be a RIQBFS The following conditions are equivalent:*

*(i) The set $\mathcal{P}_n(\mathsf{T}) := \left\{P : P\left(e^{i\theta}\right) = \sum_{j=-n}^{n} c_j e^{ij\theta}, c_j \in \mathbb{C}\right\}$ of polynomials is dense in $X(\mathsf{T})$.*

*(ii) The set of continuous functions on $\mathsf{T}$ is dense in $X(\mathsf{T})$.*

*(iii) Rotation operator $R_h f(w) := f\left(we^{ih}\right)$ is a bounded operator in $X(\mathsf{T})$, namely,*

$$\|R_h f\|_{X(\mathsf{T})} \leq c\,\|f\|_{X(\mathsf{T})}$$

*for every $f \in X(\mathsf{T})$, $h \in [0,2\pi)$ and $w \in \mathsf{T}$.*

*(iv)$X(\mathsf{T})$ has absolutely continuous norm.*

These properties are proved for rearrangement invariant BFS in [29, p. 157, Lemma 6.3] and they hold also for RIQBFS $X(\mathsf{T})$ which has absolutely continuous norm.

Let $w \in \mathsf{T}$, $h \in [0,2\pi)$, $\alpha \in \mathbb{R}^+ := (0,\infty)$, $f \in \mathbb{X}(\mathsf{T})$ and set

$$\Delta_h^\alpha f(w) := (I - R_h)^\alpha f(w) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f\left(we^{ikh}\right)$$

with Binomial coefficients $\binom{\alpha}{k} := \frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!}$ for $k \geq 1$ and $\binom{\alpha}{0} := 1$, where $I$ is identity operator.

If $\frac{1}{\alpha+1} < p$, then using [35, p.14] we have

$$\left|\binom{\alpha}{k}\right| \leq \frac{c(\alpha)}{k^{\alpha+1}}, \quad k \in \mathbb{Z}^+,$$

and hence we obtain

$$\sum_{k=1}^{\infty}\left|\binom{\alpha}{k}\right|^p \leq c(\alpha,p)\sum_{k=1}^{\infty}\frac{c(\alpha)}{k^{p(\alpha+1)}} < \infty. \tag{4}$$

On the other hand, if $g$ belongs to $Y'(\mathsf{T})$, the associate space of the rearrangement invariant BFS $Y(\mathsf{T}) = X_{1/p}(\mathsf{T})$, then using Levi Monotone Convergence Theorem and Remark 1 (iii) we have

$$\int_{\mathsf{T}}\left|\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k} f\left(we^{ikh}\right)\right|^p |g(w)|\,|dw|$$

$$\leq \lim_{j\to\infty}\left(\int_{\mathsf{T}}\sum_{k=0}^{j}\left|\binom{\alpha}{k} f\left(we^{ikh}\right)\right|^p |g(w)|\,|dw|\right)$$

$$\leq c\left(\sum_{k=0}^{\infty}\left|\binom{\alpha}{k}\right|^p\right)\left(\sup_{\|g\|_{Y'(\mathsf{T})}\leq 1}\int_{\mathsf{T}}|f(w)|^p\,|g(w)|\,|dw|\right)$$

and hence from (3) and (4)

$$\|\Delta_h^{\alpha} f\|_{X(\mathsf{T})} \leq c\,\|f\|_{X(\mathsf{T})}.$$

This last inequality implies that if $f \in \mathbb{X}(\mathsf{T})$, $\alpha \in \mathbb{R}^+$, $(\alpha+1)^{-1} < p$ and $h \in [0, 2\pi)$, then $\Delta_h^{\alpha} f \in X(\mathsf{T})$.

Now, if $\alpha \in \mathbb{R}^+$, $f \in \mathbb{X}(\mathsf{T})$, $(\alpha+1)^{-1} < p$ and $h \in [0, 2\pi)$, then we can define the *$\alpha$-th modulus of smoothness* of a function $f$ as

$$\omega_{\alpha}(f,\delta)_{X(\mathsf{T})} := \sup_{0<h\leq\delta}\|\Delta_h^{\alpha} f\|_{X(\mathsf{T})}, \quad \delta \geq 0.$$

**Remark 2.** *The $\alpha$-th modulus of smoothness $\omega_{\alpha}(f,\delta)_{X(\mathsf{T})}$,$\alpha \in \mathbb{R}^+$, $(\alpha+1)^{-1} < p$, of $f \in \mathbb{X}(\mathsf{T}) = X(\mathsf{T}, AC, p)$ has the following properties:*

*(i) $\omega_{\alpha}(f,\delta)_{X(\mathsf{T})}$ is a non-negative and non-decreasing function of $\delta \geq 0$.*

*(ii)$\omega_{\alpha}^p(f_1+f_2,\cdot)_{X(\mathsf{T})} \leq \omega_{\alpha}^p(f_1,\cdot)_{X(\mathsf{T})} + \omega_{\alpha}^p(f_2,\cdot)_{X(\mathsf{T})}$.*

*(iii) $\lim_{\delta\to 0^+}\omega_{\alpha}(f,\delta)_{X(\mathsf{T})} = 0$.*

By means of Remark 1 (ii) and (iv) let

$$E_n(f)_{X(\mathsf{T})} := \inf_{P\in\mathcal{P}_n} \|f-P\|_{X(\mathsf{T})}, \quad f\in\mathbb{X}(\mathsf{T}), n=0,1,2,\ldots.$$

We denote by $X^{\alpha}(\mathsf{T})$, $\alpha\in\mathbb{R}^{+}$, the linear space of functions $f\in\mathbb{X}(\mathsf{T})$ such that $f^{(\alpha)}\in X(\mathsf{T})$.

We say that a function $g=f^{(\alpha)}$, $\alpha\in\mathbb{R}^{+}$, is the *αth derivative* of $f\in X^{\alpha}(\mathsf{T})$ if there is a function $g\in X(\mathsf{T})$ such that

$$\lim_{h\to 0^{+}} \left\| \frac{\Delta_h^{\alpha}(f)}{h^{\alpha}} - g \right\|_{X(\mathsf{T})} = 0.$$

The *αth Weyl's derivative* ($\alpha\in\mathbb{R}^{+}$) of a polynomial

$$T_n(\theta)=\sum\nolimits_{v=-n}^{n}\gamma_v e^{iv\theta},\ \ n\geq 1,\ \theta\in[0,2\pi),\ \gamma_v\in\mathbb{C},$$

of class $\mathcal{P}_n(\mathsf{T})$ is defined as

$$T_n^{\{\alpha\}}(\theta)=\sum\nolimits_{v\in\mathbb{Z}_n^{*}}\gamma_v(iv)^{\alpha}e^{iv\theta},$$

a.e. on $[0,2\pi)$, where $\mathbb{Z}_n^{*}:=\{\pm1,\pm2,\ldots,\pm n\}$ and $(iv)^{-\alpha}:=|v|^{-\alpha}e^{(-1/2)\pi i\alpha sign v}$ as principal value.

**Remark 3.** *([28])Let*

$$T_n(\theta)=\sum\nolimits_{v=-n}^{n}\gamma_v e^{iv\theta},(n\geq 1)$$

*be a polynomial of class $\mathcal{P}_n(\mathsf{T})$. Then for every $\alpha\in\mathbb{R}^{+}$ and $\theta\in[0,2\pi)$ we have*

$$T_n^{\{\alpha\}}(\theta)=T_n^{(\alpha)}(\theta).$$

Let us denote by $[x]$ the integer part of a real number $x$ and $\{x\}:=x-[x]$. Let $q_{\mathbb{X}(\mathsf{T})}$ be the Boyd's upper index of $\mathbb{X}(\mathsf{T})$.

For $\alpha,t\in\mathbb{R}^{+}$ and $f\in\mathbb{X}(\mathsf{T})$ we define for $n=1,2,3,\ldots$, the realization functional

$$R_{\alpha}(f,1/n,X(\mathsf{T}),X^{\alpha}(\mathsf{T})) := \|f-T\|_{X(\mathsf{T})} + n^{-\alpha}\left\|T^{(\alpha)}\right\|_{X(\mathsf{T})}.$$

**Theorem 1.** *If $\alpha\in\mathbb{R}^{+}$, $f\in\mathbb{X}$, $p^{-1}<\min\{\alpha+1,2-\{\alpha\}\}$ and $q_{\mathbb{X}}<\infty$, then the equivalence*

$$\omega_{\alpha}(f,1/n)_X \asymp R_{\alpha}(f,1/n,X(\mathsf{T}),X^{\alpha}(\mathsf{T})),$$

*holds.*

In particular, the following Jackson type direct theorems of trigonometric approximation hold.

**Corollary 1.** *If* $\alpha \in \mathbb{R}^{+}$, $f \in \mathbb{X}(\mathsf{T})$, $p^{-1} < \min\{\alpha+1, 2-\{\alpha\}\}$ *and* $q_{\mathbb{X}(\mathsf{T})} < \infty$, *then there exists a constant* $c > 0$ *dependent only on* $\alpha$ *and* $\mathbb{X}$ *such that for* $n = 1, 2, 3, \ldots$

$$E_n(f)_{X(\mathsf{T})} \leq c\omega_\alpha\left(f, \frac{1}{n}\right)_{X(\mathsf{T})}$$

*holds.*

The following converse estimate of trigonometric approximation holds.

**Theorem 2.** *If* $\alpha \in \mathbb{R}^{+}$, $f \in \mathbb{X}(\mathsf{T})$, $(\alpha+1)^{-1} < p$ *and* $q_{\mathbb{X}(\mathsf{T})} < \infty$, *then for* $n = 1, 2, 3, \ldots$

$$\omega_\alpha\left(f, \frac{\pi}{n}\right)_{X(\mathsf{T})} \leq \frac{c}{n^\alpha}\left(\sum\nolimits_{\nu=0}^{n} (\nu+1)^{p\alpha-1} E_\nu^p(f)_{X(\mathsf{T})}\right)^{1/p}, \tag{5}$$

*holds, where the constant* $c > 0$ *depends only on* $\alpha$ *and* $\mathbb{X}$.

**Corollary 2.** *Under the conditions of Theorems 1 and 2 the estimate*

$$E_n(f)_{X(\mathsf{T})} = \mathcal{O}\left(n^{-\sigma}\right),\ 1 > \sigma > 0,\ n = 1, 2, \ldots,$$

*holds if and only if*

$$\omega_\alpha(f, \delta)_{X(\mathsf{T})} = \mathcal{O}(\delta^\sigma).$$

**Theorem 3.** *Let* $f \in \mathbb{X}(\mathsf{T})$ *and* $q_{\mathbb{X}(\mathsf{T})} < \infty$. *If* $\beta \in (0, \infty)$, *and*

$$\sum_{\nu=1}^{\infty} \nu^{p\beta-1} E_\nu^p(f)_{X(\mathsf{T})} < \infty, \tag{6}$$

*then the derivative* $f^{(\beta)}$ *exists. Further, denoting by* $T_n \in \mathcal{T}_n$, $n \geq 1$, *the best approximating polynomial of* $f$ *in* $\|\cdot\|_{X(\mathsf{T})}$ *metric, we have*

$$\left\|f^{(\beta)} - \mathrm{T}_n^{(\beta)}\right\|_{X(\mathsf{T})} \leq c\left(n^\beta E_n(f)_{X(\mathsf{T})} + \left(\sum_{\nu=n+1}^{\infty} \nu^{p\beta-1} E_\nu^p(f)_{X(\mathsf{T})}\right)^{1/p}\right),$$

*where the constant* $c > 0$ *depends only on* $\beta$ *and* $\mathbb{X}$.

As a corollary of Theorems 3 and 2, we have

**Corollary 3.** *Let $f \in \mathbb{X}(\mathsf{T})$, $\beta \in (0,\infty)$, $q_{\mathbb{X}(\mathsf{T})} < \infty$ and*

$$\sum_{\nu=1}^{\infty} \nu^{p\alpha-1} E_{\nu}^{p}(f)_{X(\mathsf{T})} < \infty,$$

*for some $\alpha > 0$. In this case for $n = 1,2,\ldots$, there exists a constant $c > 0$ dependent only on $\alpha$, $\beta$ and $\mathbb{X}$ such that*

$$\omega_{\beta}\left(f^{(\alpha)}, \frac{1}{n}\right)_{X(\mathsf{T})} \leq c\left\{\frac{1}{n^{\beta}}\left(\sum_{\nu=0}^{n}(\nu+1)^{p(\alpha+\beta)-1} E_{\nu}^{p}(f)_{X(\mathsf{T})}\right)^{1/p} + \right.$$

$$\left. + \left(\sum_{\nu=n+1}^{\infty} \nu^{p\alpha-1} E_{\nu}^{p}(f)_{X(\mathsf{T})}\right)^{1/p}\right\},$$

*holds.*

## 3. Auxiliary results

The following lemma was proved in [28] (as Lemma 2.4 and Lemma 2.5). The Extrapolation Theorem 2.3 in [28] does not hold in that format. So we will give here its complete proof without using extrapolation.

**Lemma 1.** *Let $X$ be a RIQBFS which is $p$-convex for some $p \in (0,1], T_n \in \mathcal{T}_n, n \geq 1, \alpha \in \mathbb{R}^{+}, q_{\mathbb{X}} < \infty$ and $0 < h \leq \frac{\pi}{n}$. Then there exist constants $c, C > 0$ such that*

$$\left\|T_n^{(\alpha)}\right\|_X \leq c\left(\frac{n}{2\sin(nh/2)}\right)^{\alpha} \left\|\Delta_h^{\alpha} T_n\right\|_X, \tag{7}$$

$$\left\|\Delta_h^{\alpha} T_n\right\|_X \leq C h^{\alpha} \left\|T_n^{(\alpha)}\right\|_X, \tag{8}$$

*hold when $X$ is RIBFS and*

$$\left\|T_n^{(\alpha)}\right\|_X \leq c n^{\alpha} \left\|\Delta_{(\pi/n)}^{\alpha} T_n\right\|_X, \tag{9}$$

$$\left\|\Delta_h^{\alpha} T_n\right\|_X \leq C h^{\alpha} \left\|T_n^{(\alpha)}\right\|_X, \tag{10}$$

*hold when $X$ is RIQBFS.*

*Proof.* When $X$ is RIBFS, the inequalities (7) and (8) were proved in [36] (cf. 1.1 Theorem). When $X$ is RIQBFS, we will use the method of Kolomoitsev

[37]. The proof method of Theorem 1 of [37] suits well this case. Let $v \in C^{\infty}(\mathbb{R}), v(x) = 1$ for $|x| \leq \pi$ and $v(x) = 0$ for $|x| \geq (3\pi/2)$. We set

$$K_{\delta}(x) = \sum_{k} \frac{(ik)^{\beta}}{\left(2i \sin \frac{k\delta}{2}\right)^{\beta}} v(\delta k) e^{ikx},$$

with $\frac{0}{0} = 1$. Since the convolution equality

$$T_n^{(\beta)}(x) = \left(K_{\delta} * \Delta_{\delta}^{\beta} T_n\right)\left(x - \frac{\delta\beta}{2}\right),$$

holds, using (3) we get

$$\left\|T_n^{(\beta)}\right\|_X \leq \delta^{1-\frac{1}{p}} \|K_{\delta}\|_p \left\|\Delta_{\delta}^{\beta} T_n\right\|_X.$$

Lemma 1 of [37] gives

$$\delta^{1-\frac{1}{p}} \|K_{\delta}\|_p \leq C\delta^{-\beta},$$

and hence we have

$$\left\|T_n^{(\alpha)}\right\|_X \leq c\delta^{-\beta} \left\|\Delta_{\delta}^{\beta} T_n\right\|_X.$$

Now taking $\delta = \frac{\pi}{n}$ we have (9). For the inequality (10) we set

$$\widetilde{K_h}(x) = \sum_{k} \frac{\left(2i \sin \frac{kh}{2}\right)^{\beta}}{(ik)^{\beta}} v(hk) e^{ikx}.$$

Then

$$\Delta_h^{\beta} T_n(x) = \left(\widetilde{K_h} * T_n^{\beta}\right)\left(x + \frac{h\beta}{2}\right).$$

The same method now gives (10). ◄

## 4. Proofs of the results in Section 2

*Proof.* [of Theorem 1] Let $t \in (0, 2\pi)$. Then there exists $n \in \mathbb{Z}^+$ such that $\pi/n < t \leq 2\pi/n$. Let $t_n^*$ be the best approximating polynomial to $f \in \mathbb{X}(\mathsf{T})$. Using Corollary 1 of [28] we get

$$\begin{aligned} \|f - t_n^*\|_{X(\mathsf{T})} &= E_n(f)_{X(\mathsf{T})} = E_n\left(f\left(e^{i\theta}\right)\right)_{X([0,2\pi))} \\ &\leq c\omega_{\alpha}\left(f\left(e^{i\theta}\right), \frac{\pi}{n}\right)_{X([0,2\pi))} \leq c\omega_{\alpha}\left(f, \frac{\pi}{n}\right)_{X(\mathsf{T})}. \end{aligned} \tag{11}$$

From (11) we have

$$\left\| t_n^{*(\alpha)} \right\|_{X(\mathsf{T})} \leq cn^{\alpha} \left\| \Delta_{\pi/n}^{\alpha} t_n^* \right\|_{X(\mathsf{T})} \leq$$

$$\leq c\,(\pi/t)^{\alpha} \left\{ c \left\| f - t_n^* \right\|_{X(\mathsf{T})} + \left\| \Delta_{\pi/n}^{\alpha} f \right\|_{X(\mathsf{T})} \right\} \leq ct^{-\alpha} \omega_{\alpha} \left( f, \frac{\pi}{n} \right)_{X(\mathsf{T})},$$

and therefore

$$K_{\alpha}\left(f, t, X\left(\mathsf{T}\right), X^{\alpha}\left(\mathsf{T}\right)\right) \leq \left\| f - t_n^* \right\|_{X(\mathsf{T})} + t^{\alpha} \left\| t_n^{*(\alpha)} \right\|_{X(\mathsf{T})} \leq c\omega_{\alpha}\left(f, t\right)_{X(\mathsf{T})}.$$

The opposite direction of the last inequality is easy to obtain. ◀

*Proof.* [of Theorem 2] Let $T_n \in \mathcal{P}_n\left(\mathsf{T}\right)$ be the best approximating polynomial of $f$ and let $m \in \mathbb{Z}^+$. If we set

$$U_0 = T_1 - T_0,\, U_{\upsilon} = T_{2^{\upsilon}} - T_{2^{\upsilon-1}},\, \upsilon \geq 1, \tag{12}$$

then

$$T_{2^m} = T_0 + \sum_{\nu=0}^{m} U_{\upsilon}.$$

In this case

$$\omega_{\alpha}^{p}\left(f, \pi/n\right)_{X(\mathsf{T})} \leq \omega_{\alpha}^{p}\left(f - T_{2^m}, \pi/n\right)_{X(\mathsf{T})} + \omega_{\alpha}^{p}\left(T_{2^m}, \pi/n\right)_{X(\mathsf{T})},$$

$$\omega_{\alpha}^{p}\left(f - T_{2^m}, \pi/n\right)_{X(\mathsf{T})} \leq cE_{2^m}^{p}\left(f\right)_{X(\mathsf{T})}$$

and

$$\omega_{\alpha}^{p}\left(T_{2^m}, \pi/n\right)_{X(\mathsf{T})} \leq \omega_{\alpha}^{p}\left(U_0, \pi/n\right)_{X(\mathsf{T})} + \sum_{\nu=1}^{m} \omega_{\alpha}^{p}\left(U_{\upsilon}, \pi/n\right)_{X(\mathsf{T})}$$

$$\leq c\left(\frac{\pi}{n}\right)^{p\alpha} \left( \left\| U_0 \right\|_{X(\mathsf{T})}^{p} + \sum_{\nu=1}^{m} 2^{\upsilon p\alpha} \left\| U_{\upsilon} \right\|_{X(\mathsf{T})}^{p} \right).$$

On the other hand

$$\left\| U_0 \right\|_{X(\mathsf{T})}^{p} \leq cE_0^{p}\left(f\right)_{X(\mathsf{T})}$$

and

$$\left\| U_{\upsilon} \right\|_{X(\mathsf{T})}^{p} \leq 2E_{2^{\upsilon-1}}^{p}\left(f\right)_{X(\mathsf{T})}.$$

Since

$$2^{\upsilon p\alpha} E_{2^{\upsilon-1}}^{p}\left(f\right)_{X(\mathsf{T})} \leq c \sum_{\mu=2^{\upsilon-2}+1}^{2^{\upsilon-1}} \mu^{p\alpha-1} E_{\mu}^{p}\left(f\right)_{X(\mathsf{T})}, \quad \nu = 2, 3, \ldots$$

we get

$$\omega_\alpha^p (T_{2^m}, \pi/n)_{X(\mathsf{T})} \le c \left(\frac{\pi}{n}\right)^{p\alpha} \left\{ E_0^p (f)_{X(\mathsf{T})} + \sum_{\upsilon=1}^{m} 2^{\upsilon p\alpha} E_{2^{\upsilon-1}}^p (f)_{X(\mathsf{T})} \right\}.$$

If we choose $m$ so that $2^{m-1} \le n < 2^m$, then

$$\omega_\alpha^p (T_{2^m}, \pi/n)_{X(\mathsf{T})} \le \frac{c}{n^{p\alpha}} \sum_{\upsilon=0}^{n} (\nu+1)^{p\alpha-1} E_\upsilon^p (f)_{X(\mathsf{T})}$$

and

$$E_{2^m}^p (f)_{X(\mathsf{T})} \le E_{2^{m-1}}^p (f)_{X(\mathsf{T})} \le \frac{c}{n^{\alpha p}} \sum_{\upsilon=0}^{n} (\nu+1)^{p\alpha-1} E_\upsilon^p (f)_{X(\mathsf{T})} .$$

These inequalities yield the result. ◀

*Proof.* [of Theorem 3] By Levi's theorem and (12)

$$\left\| T_0(\cdot) + \sum_{\upsilon=0}^{\infty} U_\upsilon(\cdot) \right\|_{X(\mathsf{T})}^p = \lim_{r\to\infty} \left\| T_0(\cdot) + \sum_{\upsilon=0}^{r} U_\upsilon(\cdot) \right\|_{X(\mathsf{T})}^p$$

$$\le c \left\| T_0(\cdot) \right\|_{X(\mathsf{T})}^p + c \lim_{r\to\infty} \sum_{\upsilon=0}^{r} \left\| U_\upsilon(\cdot) \right\|_{X(\mathsf{T})}^p$$

$$\le c E_0^p (f)_{X(\mathsf{T})} + c \sum_{\upsilon=1}^{\infty} E_{2^{\upsilon-1}}^p (f)_{X(\mathsf{T})} < \infty.$$

From (6) it follows that the last series converges and therefore

$$f(w) = \lim_{r\to\infty} T_{2^r}(w) = T_0(w) + \sum_{\upsilon=0}^{\infty} U_\upsilon(w) \ \text{ a.e.}$$

Analogously, using Levi's Theorem

$$\left\| \sum_{\upsilon=0}^{\infty} U_\upsilon^{(\beta)} \right\|_X^p \le c \sum_{\upsilon=0}^{\infty} \left\| U_\upsilon^{(\beta)} \right\|_X^p \le c \sum_{\upsilon=0}^{\infty} 2^{\upsilon p\beta} \left\| U_\upsilon \right\|_X^p$$

$$\le c \left( E_0^p (f)_{X(\mathsf{T})} + \sum_{\upsilon=1}^{\infty} 2^{\upsilon p\beta} E_{2^{\upsilon-1}}^p (f)_{X(\mathsf{T})} \right) < \infty,$$

and the series

$$\sum_{\upsilon=0}^{\infty} U_\upsilon^{(\beta)} (\cdot) ,$$

converges a.e., its sum $g$ is of class $X$. Now let's prove that $g=f^{(\beta)}$ a.e.

For $0\neq h\in\mathbb{R}$ we have

$$\left\Vert \frac{\Delta_h^\beta f}{h^\beta}-g\right\Vert_{X(\mathsf{T})}^p\leq c\left\Vert \frac{1}{h^\beta}\sum_{k=0}^{N}(-1)^k\binom{\beta}{k}f\left(we^{ikh}\right)-g(w)\right\Vert_{X(\mathsf{T})}^p$$

$$+c\left\Vert \frac{1}{h^\beta}\sum_{k=N+1}^{\infty}(-1)^k\binom{\beta}{k}f\left(we^{ikh}\right)\right\Vert_{X(\mathsf{T})}^p:=c\left(I_1+I_2\right).$$

In this case

$$I_2\leq\frac{1}{|h|^{\beta p}}\sum_{k=N+1}^{\infty}\left|\binom{\beta}{k}\right|^p\|f\|_{X(\mathsf{T})}^p$$

and hence

$$\lim_{N\to\infty}I_2=0.$$

Now by Levi's theorem

$$I_1=\left\Vert \frac{1}{h^\beta}\sum_{\upsilon=0}^{\infty}\sum_{k=0}^{N}(-1)^k\binom{\beta}{k}U_\upsilon\left(we^{ikh}\right)-g(w)\right\Vert_{X(\mathsf{T})}^p$$

$$\leq c\sum_{\upsilon=0}^{\infty}\left\Vert \frac{1}{h^\beta}\sum_{k=0}^{N}(-1)^k\binom{\beta}{k}U_\upsilon\left(we^{ikh}\right)-U_\upsilon^{(\beta)}(w)\right\Vert_{X(\mathsf{T})}^p=:Y_N.$$

The last series converges uniformly in $N\geq 1$, because its $\upsilon$th term doesn't exceed

$$\frac{1}{|h|^{\beta p}}\sum_{k=0}^{\infty}\left|\binom{\beta}{k}\right|^p\left(\|U_\upsilon\|_{X(\mathsf{T})}^p+\left\Vert U_\upsilon^{(\beta)}\right\Vert_{X(\mathsf{T})}^p\right)\leq c\left(\frac{1}{|h|^{\beta p}}+1\right)2^{\upsilon p\beta}E_{2^{\upsilon-1}}^p(f)_{X(\mathsf{T})}.$$

From Lebesgue Dominated convergence theorem we have

$$\lim_{N\to\infty}Y_N=\sum_{\upsilon=0}^{\infty}\left\Vert \frac{\Delta_h^\beta U_\upsilon}{h^\beta}-U_\upsilon^{(\beta)}\right\Vert_{X(\mathsf{T})}^p$$

and then

$$\left\Vert \frac{\Delta_h^\beta f}{h^\beta}-g\right\Vert_{X(\mathsf{T})}^p\leq c\sum_{\upsilon=0}^{\infty}\left\Vert \frac{\Delta_h^\beta U_\upsilon}{h^\beta}-U_\upsilon^{(\beta)}\right\Vert_{X(\mathsf{T})}^p$$

$$\leq c\sum_{\upsilon=0}^{s}\left\Vert \frac{\Delta_h^\beta U_\upsilon}{h^\beta}-U_\upsilon^{(\beta)}\right\Vert_{X(\mathsf{T})}^p+c\sum_{\upsilon=s+1}^{\infty}\left\Vert \frac{\Delta_h^\beta U_\upsilon}{h^\beta}\right\Vert_{X(\mathsf{T})}^p+c\sum_{\upsilon=s+1}^{\infty}\left\Vert U_\upsilon^{(\beta)}\right\Vert_{X(\mathsf{T})}^p$$

$$\leq c\sum_{\upsilon=0}^{s}\left\|\frac{\Delta_h^\beta U_\upsilon}{h^\beta}-U_\upsilon^{(\beta)}\right\|_{X(\mathsf{T})}^p+c\sum_{\upsilon=s+1}^{\infty}2^{\upsilon p\beta}E_{2^{\upsilon-1}}^p(f)_{X(\mathsf{T})}.$$

For given positive $\varepsilon$, the last term is less than $\varepsilon$ for sufficiently large $s$. By Remark 3 we get

$$\lim_{h\to 0^+}\left\|\frac{\Delta_h^\beta U_\upsilon}{h^\beta}-U_\upsilon^{(\beta)}\right\|_{X(\mathsf{T})}^p=0$$

and therefore

$$\lim_{h\to 0^+}\left\|\frac{\Delta_h^\beta f}{h^\beta}-g\right\|_{X(\mathsf{T})}^p<\varepsilon.$$

This implies that $g=f^{(\beta)}$ a.e.

Let $m\in\mathbb{Z}^+$ be such that $2^{m-1}\leq n<2^m$. Then we have

$$\left\|T_n^{(\beta)}-f^{(\beta)}\right\|_{X(\mathsf{T})}^p=\left\|T_n^{(\beta)}-\sum_{\upsilon=0}^{\infty}U_\upsilon^{(\beta)}\right\|_{X(\mathsf{T})}^p$$

$$\leq c\left\|T_n^{(\beta)}-T_{2^m}^{(\beta)}\right\|_{X(\mathsf{T})}^p+c\sum_{\upsilon=m+1}^{\infty}\left\|U_\upsilon^{(\beta)}\right\|_{X(\mathsf{T})}^p$$

$$\leq c\left(2^{mp\beta}E_n^p(f)_{X(\mathsf{T})}+\sum_{\upsilon=m+1}^{\infty}2^{\upsilon p\beta}E_{2^{\upsilon-1}}^p(f)_{X(\mathsf{T})}\right)$$

$$\leq c\left(n^{\beta p}E_n^p(f)_{X(\mathsf{T})}+\sum_{\mu=n+1}^{\infty}\mu^{p\beta-1}E_\mu^p(f)_{X(\mathsf{T})}\right)$$

and the result is proved. ◄

## 5. Approximation on Dini-smooth domains

Let $\Gamma$ be a closed rectifiable Jordan curve and let $G:=int\Gamma$, $G^-:=ext\Gamma$, $\mathbb{D}:=\{w\in\mathbb{C}:|w|<1\}$, $\mathsf{T}:=\partial\mathbb{D}$, $\mathbb{D}^-:=ext\mathsf{T}$. Without loss of generality we may assume $0\in G$.

Let $w=\varphi(z)$ and $w=\varphi_1(z)$ be the conformal mapping of $G^-$ and $G$ onto $\mathbb{D}^-$ normalized by the conditions

$$\varphi(\infty)=\infty,\qquad \lim_{z\to\infty}\varphi(z)/z>0,$$

and

$$\varphi_1(0) = \infty, \qquad \lim_{z\to 0} z\varphi_1(z) > 0,$$

respectively. We denote by $\psi$ and $\psi_1$ the inverse mappings of $\varphi$ and $\varphi_1$, respectively. A smooth Jordan curve $\Gamma$ will be called *Dini-smooth* if the function $\theta(s)$, the angle between the tangent line and the positive real axis expressed as a function of arclength $s$, has modulus of continuity $\Omega(\theta, s)$ satisfying the Dini condition

$$\int_0^\delta \frac{\Omega(\theta, s)}{s} ds < \infty, \quad \delta > 0.$$

If $\Gamma$ is Dini-smooth, then [38]

$$0 < c < |\psi'(w)| < C < \infty, \quad |w| \geq 1, \tag{13}$$

with some constants $c$, $C$. Similar inequalities hold also for $\psi_1'$ and $\varphi_1'$, in case of $|w| = 1$ and $z \in \Gamma$, respectively.

Let $\mathbb{X}(\Gamma) := \mathbb{X}(\Gamma, AC, p)$ , $1 \leq p$, be the collection of all measurable functions $f : \Gamma \to \mathbb{C}$ such that $f \circ \psi$ and $f \circ \psi_1$ belong to the RIBFS $\mathbb{X}(\mathsf{T}, AC, p)$ defined in Section 2.

For each $f \in \mathbb{X}(\Gamma)$ we define

$$\|f\|_{X(\Gamma)} := \|f \circ \psi\|_{X(\mathsf{T})}, \quad f \in \mathbb{X}(\Gamma).$$

For definitions and fundamental properties of general rearrangement invariant spaces we refer to [29].

Let $f \in L^1(\Gamma)$. Then, the functions $f^+$ and $f^-$ defined by

$$f^+(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\varsigma)}{\varsigma - z} d\varsigma, \quad z \in G,$$

$$f^-(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\varsigma)}{\varsigma - z} d\varsigma, \quad z \in G^-,$$

are analytic in $G$ and $G^-$, respectively, and $f^-(\infty) = 0$.

Let $\Gamma$ be a Dini-smooth curve and let $f_0 := f \circ \psi$, $f_1 := f \circ \psi_1$ for $f \in \mathbb{X}(\Gamma, AC, p)$. Then from (13) we have $f_0 \in \mathbb{X}(\mathsf{T}, AC, p)$ and $f_1 \in \mathbb{X}(\mathsf{T}, AC, p)$ for $f \in \mathbb{X}(\Gamma, AC, p)$. Using the nontangential boundary values of $f_0^+$ and $f_1^+$ on $\mathsf{T}$ we define

$$\omega^r_{\Gamma,X}(f, \delta) := \omega_r\left(f_0^+, \delta\right)_{X(\mathsf{T})},$$

$$\tilde{\omega}^r_{\Gamma,X}(f,\delta) := \omega_r\left(f_1^+,\delta\right)_{X(\mathbb{T})}$$

for $r > 0$ and $\delta > 0$.

We define the following rearrangement invariant Smirnov function classes: $E_X(G)$ will denote the class of functions $f \in E^1(G)$ such that the boundary values of $f$ belong to $\mathbb{X}(\Gamma, AC, p)$. Similarly, $E_X(G^-)$ will denote the class of functions $f \in E^1(G^-)$ such that the boundary values of $f$ belong to $\mathbb{X}(\Gamma, AC, p)$. Also, $\widetilde{E}_X(G^-)$ will denote the class of functions $f \in E_X(G^-)$ such that $f^-(\infty) = 0$.

Since the Luxemburg norm of Orlicz space is a R.I. function norm, every Orlicz space is a R.I. space and therefore every Smirnov-Orlicz space is a R.I. Smirnov space.

We set

$$\mathcal{E}_n(f,G)_{X(\Gamma)} := \inf_{P\in\mathcal{P}_n}\|f-P\|_{X(\Gamma)} \quad \text{and} \quad \tilde{\mathcal{E}}_n\left(g,G^-\right)_{X(\Gamma)} := \inf_{R\in\mathcal{R}_n}\|g-R\|_{X(\Gamma)},$$

where $f \in E_X(G)$, $g \in E_X(G^-)$, $\mathcal{P}_n$ is the set of algebraic polynomials of degree not greater than $n$ and $\mathcal{R}_n$ is the set of rational functions of the form

$$\sum_{k=0}^{n}\frac{a_k}{z^k}.$$

Let $a_k$ and $\tilde{a}_k$ be the Faber-Laurent coefficients [39] for the function $f \in L^1(\Gamma)$ and the rational function

$$R_n(z,f) := \sum_{k=0}^{n} a_k\Phi_k(z) + \sum_{k=1}^{n}\tilde{a}_kF_k(1/z) := P_n(\cdot,f) + \sum_{k=1}^{n}\tilde{a}_kF_k(1/z),$$

be the Faber-Laurent rational function [39] corresponding to $f \in L^1(\Gamma)$, where $\Phi_k$ and $F_k(1/z)$ are Faber polynomials for continuums $\bar{G}$ and $\overline{G^-}$.

The main results of this section is the following

**Theorem 4.** *Let $\Gamma$ be a Dini-smooth curve, $f \in \mathbb{X}(\Gamma)$ and $\mathbb{X}(\Gamma)$ have nontrivial Boyd indices. Then there is a constant $c > 0$ such that for any natural $n$*

$$\|f - R_n(\cdot,f)\|_{X(\Gamma)} \le c\left\{\omega^r_{\Gamma,X}(f,1/n) + \tilde{\omega}^r_{\Gamma,X}(f,1/n)\right\},$$

*where $r > 0$.*

**Corollary 4.** *Let $\Gamma$ be a Dini-smooth curve and $\mathbb{X}(\Gamma)$ have nontrivial Boyd indices. If $f \in E_X(G)$, then there is a constant $c > 0$ such that for every natural $n$*

$$\|f - P_n(\cdot,f)\|_{X(\Gamma)} \le c\,\omega^r_{\Gamma,X}(f,1/n), \quad r > 0.$$

**Corollary 5.** *Let $\Gamma$ be a Dini-smooth curve and $\mathbb{X}(\Gamma)$ have nontrivial Boyd indices. If $f \in \tilde{E}_X(G^-)$, then there is a constant $c > 0$ such that for every natural $n$*

$$\|f - R_n(\cdot, f)\|_{X(\Gamma)} \leq c\, \omega^r_{\Gamma,X}(f, 1/n), \quad r > 0.$$

The following inverse theorem holds.

**Theorem 5.** *Let $\Gamma$ be a Dini-smooth curve, $\mathbb{X}(\Gamma)$ have nontrivial Boyd indices and $X(\mathsf{T})$ be a reflexive R.I. space. Then for $f \in E_X(G)$*

$$\omega^r_{\Gamma,X}(f, 1/n) \leq \frac{c}{n^r}\left\{\sum_{k=0}^{n}(k+1)^{r-1} E_k(f, G)_{X(\Gamma)}\right\}, \quad r > 0,$$

*with a constant $c > 0$.*

The inverse theorem for unbounded domains has the following form.

**Theorem 6.** *Let $\Gamma$ be a Dini-smooth curve, $\mathbb{X}(\Gamma)$ have nontrivial Boyd indices and $X(\mathsf{T})$ be a reflexive R.I. space. Then for $f \in \tilde{E}_X(G^-)$*

$$\tilde{\omega}^r_{\Gamma,X}(f, 1/n) \leq \frac{c}{n^r}\left\{\sum_{k=0}^{n}(k+1)^{r-1} \tilde{\mathcal{E}}_k\left(f, G^-\right)_{X(\Gamma)}\right\}, \quad r > 0,$$

*with a constant $c > 0$.*

## 6. Proofs of the results in Section 5

We define [39] the operators $T : E_X(\mathbb{D}) \to E_X(G, \omega)$ and $\tilde{T} : E_X(\mathbb{D}) \to \tilde{E}_X(G^-)$ by

$$T(g)(z) = \frac{1}{2\pi i}\int_{\mathsf{T}} \frac{g(w)\,\psi'(w)}{\psi(w) - z}dw, \quad z \in G, \quad g \in E_X(\mathbb{D}),$$

$$\tilde{T}(g)(z) = \frac{1}{2\pi i}\int_{\mathsf{T}} \frac{g(w)\,\psi_1'(w)}{\psi_1(w) - z}dw, \quad z \in G^-, \quad g \in E_X(\mathbb{D}).$$

**Theorem 7.** *[39] Let $\Gamma$ be a Dini-smooth curve, $\mathbb{X}(\Gamma)$ have nontrivial Boyd indices and $X(\mathsf{T})$ be a reflexive R.I. space. Then the operators*

$$T : \ E_X(\mathbb{D}) \to E_X(G) \quad \textit{and} \quad \tilde{T} : \ E_X(\mathbb{D}) \to \tilde{E}_X\left(G^-\right),$$

*are one-to-one and onto.*

*Proof.* [of Theorem 4] Let's prove that

$$\left\| f^{-}(z) + \sum_{k=1}^{n} \tilde{a}_k F_k(1/z) \right\|_{X(\Gamma)} \leq c\tilde{\omega}^{r}_{\Gamma,X}(f, 1/n) \tag{14}$$

and

$$\left\| f^{+}(z) - \sum_{k=0}^{n} a_k \Phi_k(z) \right\|_{X(\Gamma)} \leq c\omega^{r}_{\Gamma,X}(f, 1/n). \tag{15}$$

These will give the result. First we deal with (14) and let $f \in X(\Gamma)$. Since

$$f(\varsigma) = f_0^{+}(\varphi(\varsigma)) - f_0^{-}(\varphi(\varsigma)), \tag{16}$$

a.e. on $\Gamma$, we have the inequality

$$f(\varsigma) = f_1^{+}(\varphi_1(\varsigma)) - f_1^{-}(\varphi_1(\varsigma)), \tag{17}$$

a.e. on $\Gamma$.

Let $z' \in G \setminus \{0\}$. Using (17) we have

$$\sum_{k=1}^{n} \tilde{a}_k F_k\left(1/z'\right) = \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k\left(z'\right) - \frac{1}{2\pi i} \int_{\Gamma} \frac{\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(\varsigma)}{\varsigma - z'} d\varsigma$$

$$= \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k\left(z'\right) - \frac{1}{2\pi i} \int_{\Gamma} \frac{\left(\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(\varsigma) - f_1^{+}(\varphi_1(\varsigma))\right)}{\varsigma - z'} d\varsigma$$

$$-\frac{1}{2\pi i} \int_{\Gamma} \frac{f_1^{-}(\varphi_1(\varsigma))}{\varsigma - z'} d\varsigma - \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\varsigma)}{\varsigma - z'} d\varsigma = \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k\left(z'\right)$$

$$-\frac{1}{2\pi i} \int_{\Gamma} \frac{\left(\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(\varsigma) - f_1^{+}(\varphi_1(\varsigma))\right)}{\varsigma - z'} d\varsigma - f_1^{-}\left(\varphi_1\left(z'\right)\right) - f^{-}\left(z'\right).$$

Hence, taking the nontangential limit $z' \to z \in \Gamma$, inside of $\Gamma$, we obtain

$$\begin{aligned} \sum_{k=1}^{n} \tilde{a}_k F_k(1/z) \quad &= \quad \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z) - \frac{1}{2}\left(\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z) - f_1^{+}(\varphi_1(z))\right) \\ &-S_{\Gamma}\left[\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k - \left(f_1^{+} \circ \varphi_1\right)\right] - f_1^{-}(\varphi_1(z)) - f^{+}(z) \end{aligned}$$

a.e. on $\Gamma$.

Using (17), the boundedness [40] of $S_\Gamma$ in $\mathbb{X}(\Gamma)$, we get

$$\begin{aligned}\left\| f^{-}(z) + \sum_{k=1}^{n} \tilde{a}_k F_k(1/z') \right\|_{X(\Gamma)} &= \left\| \frac{1}{2}\left( \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z) - f_1^{+}(\varphi_1(z)) \right) \right. \\ &\quad \left. - S_\Gamma \left[ \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k - \left(f_1^{+} \circ \varphi_1\right) \right](z) \right\|_{X(\Gamma)} \\ \leq c \left\| \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z) - f_1^{+}(\varphi_1(z)) \right\|_{X(\Gamma)} &\leq c \left\| f_1^{+}(w) - \sum_{k=1}^{n} \tilde{a}_k w^k \right\|_{X(\mathsf{T})}.\end{aligned}$$

On the other hand, the Faber-Laurent coefficients $\tilde{a}_k$ of the function $f$ and the Taylor coefficients of the function $f_1^{+}$ at the origin are the same. We conclude that

$$\left\| f^{-} + \sum_{k=1}^{n} \tilde{a}_k F_k(1/z) \right\|_{X(\Gamma)} \leq c\omega_X^r\left(f_1^{+}, 1/n\right) = c\tilde{\omega}_{\Gamma,X}^r(f, 1/n)$$

and (14) is proved. The proof of relation (15) goes similarly. ◀

*Proof.* [of Theorem 5] Let $f \in E_X(G)$. Then we have $T\left(f_0^{+}\right) = f$. Since by Theorem 7 the operator $T : E_X(\mathbb{D}) \to E_X(G)$ is linear, bounded, one-to-one and onto, the operator $T^{-1} : E_X(G) \to E_X(\mathbb{D})$ is also linear and bounded. We take a $p_n^* \in \mathcal{P}_n$ as the best approximating algebraic polynomial to $f$ in $E_X(G)$, i.e.

$$\mathcal{E}_n(f, G)_{X(\Gamma)} = \|f - p_n^*\|_{X(\Gamma)}.$$

Then $T^{-1}(p_n^*) \in \mathcal{P}_n(\mathbb{D})$ and therefore

$$\begin{aligned}E_n\left(f_0^{+}\right)_{X(\mathsf{T})} &\leq \left\|f_0^{+} - T^{-1}(p_n^*)\right\|_{X(\mathsf{T})} = \left\|T^{-1}(f) - T^{-1}(p_n^*)\right\|_{X(\mathsf{T})} \\ &= \left\|T^{-1}(f - p_n^*)\right\|_{X(\mathsf{T})} \leq \left\|T^{-1}\right\| \|f - p_n^*\|_{X(\Gamma)} = \left\|T^{-1}\right\| \mathcal{E}_n(f, G)_{X(\Gamma)}, \qquad (18)\end{aligned}$$

because the operator $T^{-1}$ is bounded.

On the other hand, we have

$$\omega_r\left(f_0^{+}, 1/n\right)_{X(\mathsf{T})} \leq \frac{c}{n^r}\left\{ \sum_{k=0}^{n} (k+1)^{r-1} E_k\left(f_0^{+}\right)_{X(\mathsf{T})} \right\}, \quad r > 0.$$

The last inequality and (18) imply that

$$\omega_{\Gamma,X}^r(f, 1/n) = \omega_r\left(f_0^{+}, 1/n\right)_{X(\mathsf{T})} \leq \frac{c}{n^r}\left\{ \sum_{k=0}^{n} (k+1)^{r-1} E_k\left(f_0^{+}\right)_{X(\mathsf{T})} \right\}$$

$$\leq \frac{c\left\|T^{-1}\right\|}{n^{r}}\left\{\sum_{k=0}^{n}(k+1)^{r-1}\mathcal{E}_{k}\left(f,G\right)_{X(\Gamma)}\right\}, \quad r>0.$$

◀

*Proof.* [of Theorem 6] The proof goes similarly to that of previous one. ◀

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## References

[1] J.L. Walsh, H.G. Russel, *Integrated continuity conditions and degree of approximation by polynomials or by bounded analytic functions*, Trans. Amer. Math. Soc., **92**, 1959, 355-370.

[2] S.Y. Al'per, *Approximation in the mean of analytic functions of class $E^p$*, In: Investigations on the modern problems of the function theory of a complex variable, Gos. Izdat. Fiz.-Mat. Lit., Moscow, 1960, 272-286 (In Russian).

[3] V.M. Kokilashvili, *A direct theorem for the approximation in the mean of analytic functions by polynomials*, Dokl. Akad. Nauk SSSR, **185**, 1969, 749-752 (In Russian).

[4] I.I. Ibragimov, D.I. Mamedhanov, *A constructive characterization of a certain class of functions*, Dokl. Akad. Nauk SSSR, **223**, 1975, 35-37; Soviet Math. Dokl., **4**, 1976, 820-823.

[5] J.E. Andersson, *On the degree of polynomial approximation in $E^p(D)$*, J. Approx. Theory, **19**, 1977, 61-68.

[6] D.M. Israfilov, *Approximate properties of the generalized Faber series in an integral metric*, Izv. Akad. Nauk Az. SSR, Ser. Fiz.-Tekh. Math. Nauk, **2**, 1987, 10-14 (In Russian).

[7] A. Çavuş and D. M. Israfilov, *Approximation by Faber-Laurent rational functions in the mean of functions of the class $L_p(\Gamma)$ with $1<p<\infty$*, Approx. Theory Appl., **11**, 1995, 105-118.

[8] S.Z. Jafarov, *Approximation by Fejer Sums of Fourier Trigonometric Series in Weighted Orlicz Space*, Hacet. J. Math. Stat., **42(3)**, 2013, 259-268.

[9] S.Z. Jafarov, *The inverse theorem of approximation of the function in Smirnov- Orlicz classes*, Math. Inequal. Appl., **12(4)**, 2012, 835-844.

[10] S.Z. Jafarov, *On approximation in weighted Smirnov-Orlicz Classes*, Complex Var. Elliptic Equ., **57(5)**, 2012, 567-577.

[11] S.Z. Jafarov, *Approximation of functions by rational functions on closed curves of the complex plane*, Arab. J. Sci. Eng., **36**, 2011, 1529-1534.

[12] S.Z. Jafarov, *Approximations of Harmonic Functions Classes with Singularities on Quasiconformal Curves*, Taiwanese J. Math., **12(3)**, 2008, 829–840.

[13] S.Z. Jafarov, *On moduli of smoothness in Orlicz classes*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb., **33(51)**, 2010, 85-92.

[14] D.M. Israfilov, *Approximation by p-Faber polynomials in the weighted Smirnov class $E^p(G,\omega)$ and the Bieberbach polynomials*, Constr. Approx., **17**, 2001, 335-351.

[15] D.M. Israfilov, *Approximation by p-Faber-Laurent rational functions in the weighted Lebesgue spaces*, Czechoslovak Math. J., **54**, 2004, 751-765.

[16] D.M. Israfilov, A. Guven, *Approximation in weighted Smirnov classes*, East J. Approx., **11**, 2005, 91-102.

[17] V.M. Kokilashvili, *On analytic functions of Smirnov-Orlicz classes*, Studia Math., **31**, 1968, 43-59.

[18] A. Guven and D.M. Israfilov, *Polynomial approximation in Smirnov-Orlicz classes*, Comput. Methods Funct. Theory, **2**, 2002, 509-517.

[19] D.M. Israfilov, B. Oktay, R. Akgun, *Approximation in Smirnov-Orlicz classes*, Glas. Mat., **40**, 2005, 87-102.

[20] R. Akgün, D.M. Israfilov, *Approximation by interpolating polynomials in Smirnov-Orlicz class*, J. Korean Math. Soc., **43**, 2006, 412-424.

[21] A. Guven, D.M. Israfilov, *Approximation in rearrangement invariant spaces on Carleson curves*, East J. Approx., **12(4)**, 2006, 381-395.

[22] B.T. Bilalov, T.I. Najafov, *On basicity of systems of generalized Faber polynomials*, Jaen J. Approx., **5(1)**, 2013, 19-34.

[23] S.R. Sadigova, *The general solution of the homogeneous Riemann problem in the weighted Smirnov classes*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb., **40(2)**, 2014, 115-124.

[24] D.M. Israfilov, N.P. Tozman, *Approximation in Morrey-Smirnov classes*, Azerbaijan J. Math. **1(1)**, 2011, 99-113.

[25] R. Taberski, *Differences, moduli and derivatives of fractional orders*, Comment. Math. Prace Mat., **19(2)**, 1977, 389-400.

[26] P.L. Butzer, H. Dyckhoff, E. Görlich and R.L. Stens, *Best trigonometric approximation, fractional order derivatives and Lipschitz classes*, Canad. J. Math., **29(4)**, (1977), 781-793.

[27] B. Simonov, S. Tikhonov, *Sharp Ul'yanov-type inequalities using fractional smoothness*, J. Approx. Theory, **162(9)**, 2010, 1654-1684.

[28] R. Akgün, *Approximation by polynomials in rearrangement invariant quasi Banach spaces*, Banach J. Math. Anal., **6(2)**, 2012, 113-131.

[29] C. Benneth, R. Sharpley, *Interpolation of operators, Pure and Applied Mathematics*, **129**, Academic Press, Boston, 1988.

[30] G.P. Gurbea, J. G-Cuerva, J.M. Martell, C. Perez, *Extrapolation with weights, rearrangement-invariant function spaces, modular inequalities and applications to singular integrals*, Adv. Math., **203**, 2006, 256-318.

[31] W.B. Jahnson, G. Schechtman, *Sums of independent random variables in rearrangement invariant function spaces*, Ann. Probab., **17**, 1987, 789-808.

[32] L. Grafakos, N.J. Kalton, *Some remarks on multilinear maps and interpolation*, Math. Ann., **319**, 2001, 151-180.

[33] N.J. Kalton, *Convexity conditions on non-locally convex lattices*, Glasgow J. Math., **25**, 1984, 141-152.

[34] S.J. Montgomery-Smith, *The Hardy operator and Boyd Indices*, In: Lecture Notes in Pure and Applied Mathematics, **175**, 1996, 359-364, Dekker, New York.

[35] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional integrals and derivatives, Theory and applications*, Gordon and Breach Science Publishers, 1993.

[36] R. Akgün, *Inequalities for one sided approximation in Orlicz spaces*, Hacet. J. Math. Stat., **40(2)**, 2011, 231-240.

[37] Y.S. Kolomoitsev, *An inequality of Nikol'skii-Stechkin-Boas type with a fractional derivative in $L_p$*, $0 < p < 1$, Tr. Inst. Prikl. Mat. Mekh., **15**, 2007, 115-119 (In Russian).

[38] S.E. Warschawski, *Über das ranverhalten der Ableitung der Abildungsfunktion bei konformer Abbildung*, Math. Z., **35**, 1932, 321-456.

[39] D.M. Israfilov, R. Akgün, *Approximation by polynomials and rational functions in weighted rearrangement invariant spaces*, J. Math. Anal. Appl., **346(2)**, 2008, 489-500.

[40] A.Y. Karlovich, *Algebras of singular integral operators with PC coefficients in rearrangement-invariant spaces with Muckenhoupt weights*, J. Oper. Theory, **47**, 2002, 303-323.

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