Azerbaijan Journal of Mathematics
V. 8, No 2, 2018, July
ISSN 2218-6816

Nonlinear Implicit Hadamard's Fractional Differential Equations with Retarded and Advanced Arguments

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Abstract. In this paper, we establish the existence and uniqueness of solutions for a class of problem for nonlinear implicit fractional differential equations (NIFDE for short) of Hadamard type involving both retarded and advanced arguments. The proofs of our main results are based upon the Banach contraction principle and the Schauder fixed point theorem. We present two examples to show the applicability of our results.

Key Words and Phrases: Hadamard's fractional derivative, implicit fractional differential equations, fractional integral, existence, retarded arguments, advanced arguments, fixed point.

2010 Mathematics Subject Classifications: 34A08, 34K05

1. Introduction

Differential equations of fractional order play a very important role in describing many real world phenomena. The tools of fractional calculus are effectively employed in improving mathematical modeling in several problems of many fields of research in view of its numerous applications in technical and applied sciences, see [1, 2, 29]. The mathematical modeling of many real world phenomena based on fractional-order operators is regarded as better and improved than the one depending on integer-order operators. In particular, fractional calculus has played a significant role in the recent development of special functions and integral transforms, signal processing, control theory, bioengineering and biomedical, viscoelasticity, finance, stochastic processes, wave and diffusion phenomena, plasma physics, social sciences, etc. For further details and applications, see

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[12, 17, 20]. Fractional differential equations involving Riemann-Liouville and Liouville-Caputo type fractional derivatives have been studied extensively by several researchers, such as Benchohra *et al.* [7, 8]. However, the literature on Hadamard type fractional differential equations is not yet as enriched. The fractional derivative due to Hadamard, introduced in 1892, differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains a logarithmic function of arbitrary exponent. A detailed description of the Hadamard fractional derivative and integral can be found in [3, 4, 9, 10, 11, 16, 17]. Another interesting class of problems involving hybrid fractional differential equations have been considered in [6, 13, 18, 21, 28], and the references cited therein.

In this paper, motivated by the work mentioned above, we study boundary value problem of Hadamard-type fractional functional differential equations involving both retarded and advanced arguments

$$D^\alpha y(t) = f(t, y_t, D^\alpha y(t)),\ \text{for each },\ t \in J := [1, e],\ 1 < \alpha \leq 2, \tag{1}$$

$$y(t) = \chi(t),\ t \in [1 - r, 1],\ r > 0, \tag{2}$$

$$y(t) = \psi(t),\ t \in [e, e + h],\ h > 0, \tag{3}$$

where D^α is the Hadamard fractional derivative, $f : J \times C([-r, h], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is a given function, $\chi \in C([1 - r, 1], \mathbb{R})$ with $\chi(1) = 0$ and $\psi \in C([e, e + h], \mathbb{R})$ with $\psi(e) = 0$.

For each function y defined on $[1 - r, e + h]$ and for any $t \in J$, we denote by y_t the element of $C([-r, h], \mathbb{R})$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, h].$$

Integer order differential equations with retardation and anticipation have been considered by many authors, see for example [5, 15, 19, 22, 23, 24, 25, 26].

We present in this work some existence and uniqueness results for a class of problems for implicit fractional differential equations. The present paper is organized as follows. In Section 2, some notations are introduced and we recall some concepts of preliminaries about fractional calculus and auxiliary results. In Section 3, two results for the problem (1)-(3) are presented: the first one is based on the Banach contraction principle, the second one on Schauder's fixed point theorem. Finally, in the last Section, we give two examples to illustrate the applicability of our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C([-r,h],\mathbb{R})$ we denote the Banach space of all continuous functions from $[-r,h]$ into $\mathbb{R}$ equipped with the norm

$$\|y\|_{[-r,h]} = \sup\{|y(t)| : -r \le t \le h\}$$

and $C([1,e],\mathbb{R})$ is the Banach space endowed with the norm

$$\|y\|_{[1,e]} = \sup\{|y(t)| : 1 \le t \le e\}.$$

Also, let $E = C([1-r, e+h],\mathbb{R})$, $E_1 = C([1-r,1],\mathbb{R})$ and $E_2 = C([e,e+h],\mathbb{R})$ be the spaces endowed, respectively, with the norms

$$\|y\|_{[1-r,e+h]} = \sup\{|y(t)| : 1-r \le t \le e+h\},$$

$$\|y\|_{[1-r,1]} = \sup\{|y(t)| : 1-r \le t \le 1\},$$

and

$$\|y\|_{[e,e+h]} = \sup\{|y(t)| : e \le t \le e+h\}.$$

Let $L^1(J,\mathbb{R})$ be the space of Lebesgue-integrable functions $w : J \to \mathbb{R}$ with the norm

$$\|w\|_1 = \int_1^T |w(s)|ds.$$

Definition 1. *([17]). The Hadamard fractional (arbitrary) order integral of the function $h \in L^1(J,\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds,$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \displaystyle\int_0^\infty t^{\alpha-1}e^{-t}dt,\ \alpha > 0$ and $\log(\cdot) = \log_e(\cdot)$.

Definition 2. *([17]) For a function $h : [1,\infty) \to \mathbb{R}$, the Hadamard fractional-order derivative of order α of h is defined by*

$$D^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \frac{h(s)}{s} ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Corollary 1. *([17]) Let $\alpha > 0$ and $n = [\alpha] + 1$. The equality $D^\alpha h(t) = 0$ is valid if and only if*

$$h(t) = \sum_{j=1}^{n} c_j (\log t)^{\alpha - j} \text{ for each } t \in J,$$

where $c_j \in \mathbb{R}$ $(j = 1, \ldots, n)$ are arbitrary constants.

Theorem 1. *([14]) (Banach's fixed point theorem). Let Ω be a non-empty closed subset of a Banach space X. Then any contraction mapping T of Ω into itself has a unique fixed point.*

Theorem 2. *([14]) (Schauder's fixed point theorem.) Let X be a Banach space, and let C be a closed, convex and nonempty subset of X. Let $N : C \longrightarrow C$ be a continuous mapping such that $N(C)$ is a relatively compact subset of X. Then N has at least one fixed point in C.*

3. Existence of solutions

Definition 3. A function $y \in C^2([1-r, e+h], \mathbb{R})$, is said to be a solution of (1)–(3) if y satisfies the equation $D^\alpha y(t) = f(t, y_t, D^\alpha y(t))$ on J, and the conditions $y(t) = \chi(t), \chi(1) = 0$ on $[1-r, 1]$ and $y(t) = \psi(t), \psi(e) = 0$ on $[e, e+h]$.

To prove the existence of solutions to (1)–(3), we need the following auxiliary lemma.

Lemma 1. *Let $1 < \alpha \leq 2$, $\chi \in C([1-r, 1], \mathbb{R})$ with $\chi(1) = 0$, $\psi \in C([e, e+h], \mathbb{R})$ with $\psi(e) = 0$ and $\sigma : J \to \mathbb{R}$ be a continuous function. The linear problem*

$$D^\alpha y(t) = \sigma(t), \quad t \in J,$$

$$y(t) = \chi(t), \ t \in [1-r, 1],$$

$$y(t) = \psi(t), \ t \in [e, e+h],$$

has a unique solution which is given by:

$$y(t) = \begin{cases} \chi(t), \ \text{if} \quad t \in [1-r, 1], \\ \\ -\displaystyle\int_1^e G(t,s) \frac{\sigma(s)}{s} ds, \ \text{if} \quad t \in J, \\ \\ \psi(t), \ \text{if} \quad t \in [e, e+h], \end{cases}$$

where

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}-(\log t-\log s)^{\alpha-1}, & 1\le s\le t\le e,\\ \\ (\log t)^{\alpha-1}(1-\log s)^{\alpha-1}, & 1\le t\le s\le e.\end{cases}\tag{4}$$

We are now in a position to state and prove our existence result for the problem (1)−(3) based on Banach's fixed point.

Theorem 3. *Assume*

(H1) The function $f : J\times C([-r,h],\mathbb{R})\times\mathbb{R}\to\mathbb{R}$ *is continuous.*

(H2) There exist constants $K>0$ *and* $0<L<1$ *such that*

$$|f(t,u,v)-f(t,\bar{u},\bar{v})|\le K\|u-\bar{u}\|_{[-r,h]}+L|v-\bar{v}|$$

for any $u,\bar{u}\in C([-r,h],\mathbb{R}),\ v,\bar{v}\in\mathbb{R}$ *and* $t\in J$*.*

If

$$\frac{2K}{(1-L)\Gamma(\alpha+1)}<1,\tag{5}$$

then there exists a unique solution for the problem (1)–(3)*.*

Proof. Transform the problem (1)–(3) into a fixed point problem.
Consider the operator $N : C([1-r,e+h],\mathbb{R})\to C([1-r,e+h],\mathbb{R})$ defined by

$$Ny(t)=\begin{cases}\chi(t), \text{ if } & t\in[1-r,1],\\ \\ -\displaystyle\int_1^e G(t,s)\frac{g(s)}{s}ds, \text{ if } & t\in J,\\ \\ \psi(t), \text{ if } & t\in[e,e+h],\end{cases}\tag{6}$$

where $g\in C(J,\mathbb{R})$ is such that

$$g(t)=f(t,y_t,g(t)).$$

Clearly, the fixed points of operator N are solutions of problem (1)–(3).

Let $u,w\in C([1-r,e+h],\mathbb{R})$. If $t\in[1-r,1]$, or $t\in[e,e+h]$, then

$$|N(u)(t)-N(w)(t)|=0.$$

For $t \in J$, we have

$$|N(u)(t) - N(w)(t)| \leq \int_1^e |G(t,s)||g(s) - h(s)|\frac{ds}{s},$$

where $g, h \in C(J, \mathbb{R})$ are such that

$$g(t) = f(t, u_t, g(t)),$$

and

$$h(t) = f(t, w_t, h(t)).$$

By $(H2)$ we have

$$\begin{aligned} |g(t) - h(t)| &= |f(t, u_t, g(t)) - f(t, w_t, h(t))| \\ &\leq K\|u_t - w_t\|_{[-r,h]} + L|g(t) - h(t)|. \end{aligned}$$

Then

$$|g(t) - h(t)| \leq \frac{K}{1-L}\|u_t - w_t\|_{[-r,h]}.$$

Therefore, for each $t \in J$:

$$\begin{aligned} |N(u)(t) - N(w)(t)| &\leq \frac{K}{(1-L)}\int_1^e |G(t,s)|\|u_s - w_s\|_{[-r,h]}\frac{ds}{s} \\ &\leq \frac{K}{(1-L)}\|u - w\|_{[1-r,e+h]}\int_1^e |G(t,s)|\frac{ds}{s}. \end{aligned} \tag{7}$$

On the other hand, we have for each $t \in J$:

$$\begin{aligned} \int_1^e |G(t,s)|\frac{ds}{s} &\leq \frac{1}{\Gamma(\alpha)}\left[\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} + (\log t)^{\alpha-1}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{ds}{s}\right] \\ &\leq \frac{2}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{ds}{s} = \frac{2}{\Gamma(\alpha+1)}. \end{aligned} \tag{8}$$

By considering (8) in (7), we have

$$|N(u)(t) - N(w)(t)| \leq \frac{2K}{(1-L)\Gamma(\alpha+1)}\|u - w\|_{[1-r,e+h]}.$$

Thus

$$\|N(u) - N(w)\|_{[1-r,e+h]} \leq \frac{2K}{(1-L)\Gamma(\alpha+1)}\|u - w\|_{[1-r,e+h]}.$$

By (5), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point which is the unique solution of the problem (1)–(3). ◀

Our second result is based on Schauder's fixed point theorem.

Theorem 4. *Assume (H1), (H2) and*

(H3) There exist $p, q, r \in C(J, \mathbb{R}_+)$ *with* $r^* = \sup_{t \in J} r(t) < 1$ *such that*

$$|f(t,u,w)| \leq p(t) + q(t)\|u\|_{[-r,h]} + r(t)|w| \text{ for } t \in J,$$

$$u \in C([-r,h],\mathbb{R}) \text{ and } w \in \mathbb{R}.$$

If

$$\frac{2q^*}{(1-r^*)\Gamma(\alpha+1)} < 1, \tag{9}$$

where $q^* = \sup_{t \in J} q(t)$, *then the problem* (1)–(3) *has at least one solution.*

Proof. We shall use Schauder's fixed point theorem to prove that the operator N defined by (6) has a fixed point. The proof will be given in several steps.

Step 1: *N is continuous.*
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C([1-r, e+h], \mathbb{R})$. If $t \in [1-r, 1]$ or $t \in [e, e+h]$, then

$$|N(u_n)(t) - N(u)(t)| = 0.$$

For $t \in J$, we have

$$|N(u_n)(t) - N(u)(t)| \leq \int_1^e |G(t,s)||g_n(s) - g(s)|\frac{ds}{s}, \tag{10}$$

where $g_n, g \in C(J, \mathbb{R})$ are such that

$$g_n(t) = f(t, u_{nt}, g_n(t)),$$

and

$$g(t) = f(t, u_t, g(t)).$$

By $(H2)$, we have

$$\begin{aligned} |g_n(t) - g(t)| &= |f(t, u_{nt}, g_n(t)) - f(t, u_t, g(t))| \\ &\leq K\|u_{nt} - u_t\|_{[-r,h]} + L|g_n(t) - g(t)|. \end{aligned}$$

Then

$$|g_n(t) - g(t)| \leq \frac{K}{1-L}\|u_{nt} - u_t\|_{[-r,h]}.$$

Since $u_n \to u$, we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$.

Now, let $\eta > 0$ be such that, for each $t \in J$, we have $|g_n(t)| \leq \eta$ and $|g(t)| \leq \eta$. Then, we have

$$\begin{aligned} |G(t,s)||g_n(s) - g(s)| &\leq |G(t,s)|[|g_n(s)| + |g(s)|] \\ &\leq 2\eta|G(t,s)|. \end{aligned}$$

For each $t \in J$, the function $s \to 2\eta|G(t,s)|$ is integrable on J. Then the Lebesgue Dominated Convergence Theorem and (10) imply that

$$|N(u_n)(t) - N(u)(t)| \to 0 \text{ as } n \to \infty,$$

and hence

$$\|N(u_n) - N(u)\|_{[1-r,e+h]} \to 0 \text{ as } n \to \infty.$$

Consequently, N is continuous.

Let

$$R \geq \max \left\{ \frac{2p^*}{(1-r^*)\Gamma(\alpha+1) - 2q^*}, \|\chi\|_{[1-r,1]}, \|\psi\|_{[e,e+h]} \right\},$$

where $p^* = \sup_{t \in J} p(t)$. Define the set

$$D_R = \{u \in C([1-r, e+h], \mathbb{R}) : \|u\|_{[1-r,e+h]} \leq R\}.$$

It is clear that D_R is a bounded, closed and convex subset of $C([1-r, e+h], \mathbb{R})$.

Step 2: $N(D_R) \subset D_R$.

Let $y \in D_R$. We show that $N(y) \in D_R$. For each $t \in J$, we have

$$|N(u)(t)| \leq \int_1^e |G(t,s)||g(s)| \frac{ds}{s}, \tag{11}$$

where $g \in C(J, \mathbb{R})$ is such that

$$g(t) = f(t, u_t, g(t)).$$

By (H3), we have for each $t \in J$,

$$\begin{aligned} |g(t)| &= |f(t, u_t, g(t))| \\ &\leq p(t) + q(t)\|u_t\|_{[-r,h]} + r(t)|g(t)| \\ &\leq p(t) + q(t)\|u\|_{[1-r,e+h]} + r(t)|g(t)| \\ &\leq p(t) + q(t)R + r(t)|g(t)| \\ &\leq p^* + q^*R + r^*|g(t)|, \end{aligned}$$

and then

$$|g(t)| \leq \frac{p^* + q^* R}{1 - r^*} = M.$$

Thus (8), (9) and (11) imply

$$|N(u)(t)| \leq \frac{2(p^* + q^* R)}{(1 - r^*)\Gamma(\alpha + 1)} \leq R.$$

If also $t \in [1 - r, 1]$, then

$$|N(u)(t)| \leq \|\chi\|_{[1-r,1]},$$

and if $t \in [e, e + h]$, then

$$|N(u)(t)| \leq \|\psi\|_{[e,e+h]}.$$

Thus,

$$\|N(u)\|_{[1-r,e+h]} \leq R.$$

Step 3: $N(D_R)$ *is relatively compact.*

Let $t_1, t_2 \in [1, e]$, $t_1 < t_2$, and let $y \in D_R$. Then

$$\begin{aligned} |N(u)(t_2) - N(u)(t_1)| &\leq \int_1^e |G(t_2, s) - G(t_1, s)||g(s)|\frac{ds}{s} \\ &\leq M \int_1^e |G(t_2, s) - G(t_1, s)|\frac{ds}{s}. \end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 \leq 1$ and $t_1 \leq 1 \leq t_2$ is obvious.

As a consequence of Steps 1 to 3, together with the Ascoli-Arzela theorem, we can conclude that $N : C([1 - r, e + h], \mathbb{R}) \to C([1 - r, e + h], \mathbb{R})$ is continuous and compact. As a consequence of Schauder's fixed point theorem, we deduce that N has a fixed point which is a solution of the problem $(1)-(3)$. ◀

4. Examples

Example 1. *Consider the following problem*

$$D^{\frac{3}{2}} y(t) = \frac{e^{-t}}{(11 + e^t)} \left[\frac{y_t}{1 + y_t} - \frac{D^{\frac{3}{2}} y(t)}{1 + D^{\frac{3}{2}} y(t)} \right], \ \textit{for each}, \ t \in [1, e], \tag{12}$$

$$y(t) = \chi(t), \ t \in [1 - r, 1], \ r > 0, \tag{13}$$

$$y(t) = \psi(t),\ t \in [e, e+h],\ h > 0, \tag{14}$$

where $\chi \in C([1-r, 1], \mathbb{R})$ *with* $\chi(1) = 0$ *and* $\psi \in C([e, e+h], \mathbb{R})$ *with* $\psi(e) = 0$.
Set

$$f(t, u, v) = \frac{e^{-t}}{(11 + e^t)} \left[\frac{u}{1+u} - \frac{v}{1+v} \right],\ t \in [1, e],\ u \in C\left([-r, h], \mathbb{R}\right)\ \textit{and}\ v \in \mathbb{R}.$$

Clearly, the function f *is jointly continuous.*

For each $u, \bar{u} \in C\left([-r, h], \mathbb{R}\right), v, \bar{v} \in \mathbb{R}$ *and* $t \in [1, e]$,

$$\begin{aligned} |f(t, u, v) - f(t, \bar{u}, \bar{v})| &\leq \frac{e^{-t}}{(11 + e^t)} (\|u - \bar{u}\|_{[-r,h]} + |v - \bar{v}|) \\ &\leq \frac{1}{12} \|u - \bar{u}\|_{[-r,h]} + \frac{1}{12} |v - \bar{v}| . \end{aligned}$$

Hence condition (H2) is satisfied with $K = L = \dfrac{1}{12}$.
Thus condition

$$\frac{2K}{(1-L)\Gamma(\alpha+1)} = \frac{\frac{1}{6}}{(1 - \frac{1}{12})\Gamma(\frac{5}{2})} = \frac{8}{33\sqrt{\pi}} < 1,$$

is satisfied with $\alpha = \dfrac{3}{2}$. *It follows from Theorem 3 that the problem (12)–(14) has a unique solution.*

Example 2. *Consider the following problem*

$$D^{\frac{3}{2}} y(t) = \frac{2 + |y_t| + |D^{\frac{3}{2}} y(t)|}{108 e^{t+3} (1 + |y_t| + |D^{\frac{3}{2}} y(t)|)},\ \textit{for each}\ t \in [1, e], \tag{15}$$

$$y(t) = \chi(t),\ t \in [1-r, 1],\ r > 0, \tag{16}$$

$$y(t) = \psi(t),\ t \in [e, e+h],\ h > 0, \tag{17}$$

where $\chi \in C([1-r, 1], \mathbb{R})$ *with* $\chi(1) = 0$ *and* $\psi \in C([e, e+h], \mathbb{R})$ *with* $\psi(e) = 0$.
Set

$$f(t, u, v) = \frac{2 + |u| + |v|}{108 e^{t+3} (1 + |u| + |v|)},\quad t \in [1, e],\ u \in C([-r, h], \mathbb{R}), v \in \mathbb{R}.$$

Clearly, the function f *is jointly continuous.*

For any $u, \bar{u} \in C([-r, h], \mathbb{R}), v, \bar{v} \in \mathbb{R}$ *and* $t \in [1, e]$:

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq \frac{1}{108e^3}(\|u - \bar{u}\|_{[-r,h]} + |v - \bar{v}|).$$

Hence condition (H2) is satisfied with $K = L = \frac{1}{108e^3}$.

We have, for each $t \in [1, e]$:

$$|f(t, u, v)| \leq \frac{1}{108e^{t+3}}(2 + \|u\|_{[-r,h]} + |v|).$$

Thus condition (H3) is satisfied with

$$p(t) = \frac{1}{54e^{t+3}}, \quad q(t) = r(t) = \frac{1}{108e^{t+3}}.$$

Clearly

$$p^* = \frac{1}{54e^4}, \ q^* = \frac{1}{108e^4}, \ r^* = \frac{1}{108e^4} < 1.$$

Thus condition

$$\frac{2q^*}{(1 - r^*)\Gamma(\alpha + 1)} = \frac{144}{54\sqrt{\pi}(108e^4 - 1)} < 1,$$

is satisfied with $\alpha = \dfrac{3}{2}$. *It follows from Theorem 4 that the problem (15)–(17) has at least one solution.*

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Received 09 December 2016
Accepted 05 May 2017