Azerbaijan Journal of Mathematics
V. 8, No 2, 2018, July
ISSN 2218-6816

Quantum Cones and Quantum Balls

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Abstract. The present note is dedicated to description of quantum systems among the quantum spaces. In the normed case we obtain a complete solution to the problem when an operator space turns out to be an operator system. The min and max quantizations of a local order are described in terms of the min and max envelopes of the related state spaces.

Key Words and Phrases: quantum cone, quantum ball, operator systems, quantum systems.

2010 Mathematics Subject Classifications: 46L07, 46B40, 47L25

1. Introduction

A subspace of the operator space $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H is called a concrete operator space. The known result of Ruan [8, 2.3] asserts that every abstract operator space can be realized as a concrete operator space up to a matrix isometry. A unital self-adjoint subspace of $\mathcal{B}(H)$ is known as an operator system. Operator spaces can be regarded as (Paulsen's) corners of operator systems. Abstract characterization of operator systems was proposed by Choi and Effros in [1] (see also [14, Ch. 13]), which is described in terms of the matrix-ordered $*$-vector spaces with their Archimedian matrix order units or closed, separated, unital quantum cones. The operator systems provide an alternative background to quantum functional analysis [8, 17, 11, 2]. Ruan's result on abstract characterization of operator spaces mentioned above can be driven from the duality of quantum cones (see [6]). An operator system version of Effros-Webster-Winkler operator bipolar theorem (see [9, 10, 18]) was obtained in [5, 6] (see also [7]) within the framework of duality of quantum cones. A pioneering work on operator system structures of ordered spaces was carried out in [16] and [15] by Paulsen, Todorov and Tomforde. It is all about quantizations of unital cones in a $*$-vector space. For the sake of completeness of this theory one needs to have parallel developments of operator space constructions for operator systems. Tensor products of operator systems were considered in [12]. For the quotients, exactness and nuclearity in the operator system category see [13]. In that concern the following problem is of great interest. Which operator space structures on a unital $*$-vector space are indeed operator systems? A solution to this problem would lead to the right outlines in the theory of operator systems being operator spaces automatically.

The present note is dedicated to the solution of the above stated problem. Basically, a possible operator system structure of an operator space generates a matrix norm which is equivalent to the original matrix norm, that is, the related quantum topologies coincide. A similar situation takes place in the general case of quantum spaces. The quantum bornology (see [4] and [19]) in the topological dual plays a central technical role in our approach and solves the problem for quantum spaces.

2. Involution and quantum cones

By *an involution* on a vector space X we mean a $*$-linear mapping $x \mapsto x^*$ on X such that $x^{**} = x$ for all $x \in X$. A vector space equipped with an involution is called *a $*$-vector space*. The set of all hermitian elements (that is, $x^* = x$) is denoted by X_h, which is a real linear subspace in X. Now assume that X is a $*$-vector space and (X, Y) is a dual pair such that the involution on X is $\sigma(X, Y)$-continuous. Then Y possesses the canonical involution $y \mapsto y^*$, $\langle x, y^* \rangle = \langle x^*, y \rangle^*$. In this case (X, Y) is called *a dual $*$-pair.* The given pairing $\langle \cdot, \cdot \rangle$ defines a *matrix pairing* $\langle\langle \cdot, \cdot \rangle\rangle : M(X) \times M(Y) \to M$, $\langle\langle v, w \rangle\rangle = [\langle v_{ij}, w_{st} \rangle]_{(i,s),(j,t)}$, and the weak topology $\sigma(X, Y)$ admits only one quantization $\mathfrak{s}(X, Y)$ called *the weak quantum topology of the dual pair* (X, Y) [3]. The involutions on X and Y are naturally extended to involutions over the related matrix spaces such that $\langle\langle x, y^* \rangle\rangle = \langle\langle x^*, y \rangle\rangle^*$ for all $x \in M(X)$ and $y \in M(Y)$ [6]. Put $M(X)_h = \{x \in M(X) : x^* = x\}$, and we have a well defined real linear mapping $h_X : M(X) \to M(X)_h$, $h(x) = \begin{bmatrix} 0 & x \\ x^* & 0 \end{bmatrix}$ called (*Paulsen's*) *hermitifier* [6]. Note that $h_X(x) \in M_{2n}(X)_h$ whenever $x \in M_n(X)$. A quantum set $\mathfrak{C} \subseteq M(X)_h$ is said to be *a quantum cone on* X if $\mathfrak{C} + \mathfrak{C} \subseteq \mathfrak{C}$ and $a^*\mathfrak{C}a \subseteq \mathfrak{C}$ for all $a \in M$. A quantum cone $\mathfrak{C}$ on X is called *a separated quantum cone on* X if $\mathfrak{C} \cap -\mathfrak{C} = \{0\}$. If $\mathfrak{K}$ is a quantum set on X then its *quantum polar* $\mathfrak{K}^{\boxdot}$ *in* $M(Y)$ is defined as the quantum set $\mathfrak{K}^{\boxdot} = \{y \in M(Y)_h : \langle\langle \mathfrak{K}, y \rangle\rangle \geq 0\}$. The latter is $\mathfrak{s}(Y, X)$-closed quantum cone on Y. If $\mathfrak{C}$ is a $\mathfrak{s}(X, Y)$-closed, quantum cone on X, then $\mathfrak{C} = \mathfrak{C}^{\boxdot\boxdot}$ [6]. Now let V be a vector space with its conjugate space $\overline{V}$. Put $\mathcal{P}_V = \left\{ \begin{bmatrix} \lambda & u \\ v^* & \mu \end{bmatrix} : \lambda, \mu \in \mathbb{C}, u, v \in V \right\}$ to be a $*$-vector space with the involution $\begin{bmatrix} \lambda & u \\ v^* & \mu \end{bmatrix}^* = \begin{bmatrix} \lambda^* & v \\ u^* & \mu^* \end{bmatrix}$ called Paulsen power of V. If (V, W) is a dual pair, then so is $(\mathcal{P}_V, \mathcal{P}_W)$ with the related canonical duality. Actually, $(\mathcal{P}_V, \mathcal{P}_W)$ is a dual $*$-pair and V is included into $\mathcal{P}_V$ by means of the canonical linear mapping $\iota_V : V \to \mathcal{P}_V$, $\iota_V(x) = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$. For the hermitian elements from $M(\mathcal{P}_V)$ we use the following notations: $v_b^a = \begin{bmatrix} a & v \\ v^* & b \end{bmatrix}$ and $v_{a,\varepsilon,b} = (a + \varepsilon I)^{-1/2} v (b + \varepsilon I)^{-1/2} \in M(V)$ whenever $a, b \in M_+$ and $\varepsilon > 0$. If $\mathfrak{B} \subseteq M(V)$ is a quantum set, then we define a new hermitian quantum set $\mathfrak{C}_{\mathfrak{B}}$ on $\mathcal{P}_V$ in the following way: $\mathfrak{C}_{\mathfrak{B}} = \{v_b^a \in M(\mathcal{P}_V)_h : v_{a,\varepsilon,b} \in \mathfrak{B}, a, b \in M_+, \varepsilon > 0, v \in M(V)\}$. If $\mathfrak{B}$ is an absolutely matrix convex set, then $\mathfrak{C}_{\mathfrak{B}}$ is a quantum cone on $\mathcal{P}_V$ and $\mathfrak{C}_{\mathfrak{B}}^{\boxdot} = \mathfrak{C}_{\mathfrak{B}^{\odot}}$, where $\mathfrak{B}^{\odot} \subseteq M(W)$ is the absolute quantum polar of $\mathfrak{B}$ (see [9]) with respect to the matrix duality obtained from (V, W). Moreover, $\mathfrak{C}_{\mathfrak{B}^-} = \mathfrak{C}_{\mathfrak{B}}^-$, where $\mathfrak{B}^-$ is $\mathfrak{s}(V, W)$-closure of $\mathfrak{B}$ and $\mathfrak{C}_{\mathfrak{B}}^-$ is $\mathfrak{s}(\mathcal{P}_V, \mathcal{P}_W)$-closure of $\mathfrak{C}_{\mathfrak{B}}$. Finally, $\mathfrak{C}_{\mathfrak{B}}$ is a separated quantum cone iff so is $\mathfrak{B}$ in the sense of $\cap_{\varepsilon > 0} \varepsilon \mathfrak{B} = \{0\}$.

3. Unital quantum cones and unital hulls

Let X be a $*$-vector space with its fixed hermitian element e. The quantum set $(\{e_n\})$ on X is denoted by $\mathfrak{e}$, where $e_n = e^{\oplus n} \in M_n(X)_h$. The $*$-vector space $X = \mathcal{P}_V$ is unital with its unit $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. A quantum cone $\mathfrak{C}$ on the unital space (X, e) is said to be *a unital quantum cone* if $\mathfrak{C} - \mathfrak{e}$ is absorbent in $M(X)_h$ (in this case, $\mathfrak{e} \subseteq \mathfrak{C}$ (see [6])). The quantum set $\mathfrak{C}^- = \cap_{r > 0} r(\mathfrak{C} - \mathfrak{e})$ is called the algebraic closure of $\mathfrak{C}$. Note that $\mathfrak{C} \subseteq \mathfrak{C}^-$ if $\mathfrak{e} \subseteq \mathfrak{C}$. We say that $\mathfrak{C}$ is *a closed* (*or an Archimedian*) *quantum cone* if $\mathfrak{C} = \mathfrak{C}^-$. Note that $\mathfrak{C}^-$ is smaller than any topological closure of $\mathfrak{C}$ with respect to any polynormed topology in $M(X)$ whenever $\mathfrak{e} \subseteq \mathfrak{C}$. For a unital quantum cone $\mathfrak{C}$ we set $\widehat{\mathfrak{C}} = h_X^{-1}(\mathfrak{C} - \mathfrak{e})$. It is an absorbent absolutely matrix convex set in $M(X)$, which is separated whenever so is $\mathfrak{C}$ [6].

Now let (X, e) be a unital $*$-vector space and let (X, Y) be a dual $*$-pair. Put $M(Y)_e = \{y \in M(Y) : \langle\langle e, y \rangle\rangle = I\}$, which is $\mathfrak{s}(Y, X)$-closed and matrix additive set in $M(Y)$. We also put $M(Y)_{he} = M(Y)_h \cap M(Y)_e$. The unital bipolar theorem proven in [7] asserts that $\mathfrak{C} = \mathcal{S}(\mathfrak{C})^{\boxdot}$ for a $\mathfrak{s}(X, Y)$-closed, unital quantum cone $\mathfrak{C}$ on X, where $\mathcal{S}(\mathfrak{C}) = \mathfrak{C}^{\boxdot} \cap M(Y)_e$ is *the matricial state space of* $\mathfrak{C}$. If $\mathfrak{C} = \mathcal{S}^{\boxdot}$ for a certain quantum subset $\mathcal{S} \subseteq \mathcal{S}(\mathfrak{C})$, then $\mathcal{S}$ is called *a prematricial* *state space of* $\mathfrak{C}$. Now let $\mathfrak{B}$ be an absorbent, $\mathfrak{s}(X,Y)$-closed, absolutely matrix convex set on X. Then $\mathfrak{C}_{\mathfrak{B}}$ is $\mathfrak{s}(\mathcal{P}_X,\mathcal{P}_Y)$-closed, unital quantum cone on $\mathcal{P}_X$, which can be treated as a cone on X. The $\mathfrak{s}(X,Y)$-closed, quantum cone on X generated by $\mathfrak{C}_{\mathfrak{B}}$ is denoted by $\widetilde{\mathfrak{C}_{\mathfrak{B}}}$. Actually, $\widetilde{\mathfrak{C}_{\mathfrak{B}}}=\mathfrak{C}_{\mathfrak{B}}^{\boxdot\boxdot}$, where $\mathfrak{C}_{\mathfrak{B}}^{\boxdot}$ is the quantum polar of the cone $\mathfrak{C}_{\mathfrak{B}}$ with respect to the dual $*$-pair (X,Y).

Now let $\mathfrak{B}$ be a quantum set on X. Put $\widetilde{\mathfrak{B}}=\left(\mathfrak{B}^{\odot}\cap M(Y)_{he}\right)^{\odot}$ called the *unital hull of* $\mathfrak{B}$. Confirm that $\widetilde{\mathfrak{B}}$ is $\mathfrak{s}(X,Y)$-closed, absolutely matrix convex sets on X, $\widetilde{\widetilde{\mathfrak{B}}}=\widetilde{\mathfrak{B}}$ and $\mathfrak{B}\subseteq\mathfrak{B}^{\odot\odot}\subseteq\widetilde{\mathfrak{B}}$. A quantum set $\mathfrak{B}$ on X is said to be *unital* if $\widetilde{\mathfrak{B}}=\mathfrak{B}$.

Theorem 1. *Let (X,Y) be a dual $*$-pair. The correspondence $\mathfrak{B}\longmapsto\widetilde{\mathfrak{C}_{\mathfrak{B}}}$ is a bijection between the set of all $\mathfrak{s}(X,Y)$-closed, unital, absorbent, absolutely matrix convex sets on X, and the set of all $\mathfrak{s}(X,Y)$-closed, unital, quantum cones on X. Moreover, $\widetilde{\mathfrak{C}_{\mathfrak{B}}}$ is a separated quantum cone iff $\mathfrak{B}$ is a separated absolutely matrix convex set.*

Corollary 1. *If $\mathfrak{C}$ is a $\mathfrak{s}(X,Y)$-closed, unital, quantum cone on X, then $\widehat{\mathfrak{C}}=\mathcal{S}(\mathfrak{C})^{\odot}$. If $\mathfrak{B}$ is an absorbent, $\mathfrak{s}(X,Y)$-closed, absolutely matrix convex set on X, then $\mathcal{S}\left(\widetilde{\mathfrak{C}_{\mathfrak{B}}}\right)=\mathfrak{B}^{\odot}\cap M(Y)_{he}$.*

Corollary 2. *Let $\mathfrak{C}$ be a $\mathfrak{s}(X,Y)$-closed, unital, quantum cone on X and let $\mathcal{S}$ be a quantum subset of $\mathcal{S}(\mathfrak{C})$. Then $\mathcal{S}$ is a prematricial state space of $\mathfrak{C}$ iff $\widehat{\mathfrak{C}}=\mathcal{S}^{\odot}$.*

4. Quantum order and quantum $*$-topology compatible with a $*$-duality

Let X be a unital $*$-vector space with its unit e, and let $\mathcal{F}$ be a filter base of quantum cones in $M(X)_h$. We say that $\mathcal{F}$ is *a quantum order* if it consists of closed, unital, quantum cones such that $\cap\mathcal{F}$ is a separated quantum cone. In this case, $(X,\mathcal{F})$ is called a quantum system [7]. Any separated, closed, unital, quantum cone $\mathfrak{C}$ on a unital $*$-vector space X defines the quantum order $\{\mathfrak{C}\}$, and $(X,\mathfrak{C})$ is *an* (*abstract*) *operator system.* A linear mapping $\varphi:(X,\mathcal{F})\to(X',\mathcal{F}')$ between quantum systems is called *a quantum* (or *local matrix*) *positive* if $\mathcal{F}'\preceq\varphi^{(\infty)}(\mathcal{F})$ for the filter bases in $M(X')$, where $\varphi^{(\infty)}:M(X)\to M(X')$ is the canonical extension of φ. If $(X,\mathcal{F})$ is a quantum system, then $\widehat{\mathcal{F}}=\left\{\widehat{\mathfrak{C}}:\mathfrak{C}\in\mathcal{F}\right\}$ is a filter base of absorbent absolutely matrix convex sets in $M(X)$, which in turn defines a Hausdorff quantum topology in $M(X)$ whose neighborhood filter base is $\left\{r\widehat{\mathfrak{C}}:\mathfrak{C}\in\mathcal{F},r>0\right\}$. We use the same notation $\widehat{\mathcal{F}}$ for the latter quantum topology. In particular, we have the dual pair (X,Y) with the topological dual

$Y = X'$. Actually, it is a $*$-dual pair, and each $\mathfrak{C} \in \mathcal{F}$ turns out to be $\widehat{\mathcal{F}}$-closed quantum cone [7], which is $\mathfrak{s}(X,Y)$-closed by Mazur's theorem. Thus $\mathcal{F}$ is a quantum order of $\mathfrak{s}(X,Y)$-closed, unital quantum cones called $\mathfrak{s}(X,Y)$-*quantum order on* X. In the case of a given dual $*$-pair (X,Y) with the unital $*$-vector space (X,e), and a $\mathfrak{s}(X,Y)$-quantum order $\mathcal{F}$ on X, we say that $\mathcal{F}$ is *compatible with the duality* (X,Y) if the relevant quantum topology $\widehat{\mathcal{F}}$ is compatible with the duality (X,Y). Recall that a quantum topology $\mathfrak{t}$ on X is said to be compatible with the duality (X,Y) if $(X,\mathfrak{t}|X)' = Y$. In this case, $\mathfrak{t}$ has a neighborhood filter base of the origin, which consists of $\mathfrak{s}(X,Y)$-closed, absorbent, absolutely matrix convex sets in $M(X)$. Moreover, $\mathfrak{s}(X,Y) \preceq \mathfrak{t} \preceq \mathfrak{r}(X,Y)$ [2, Lemma 5.1], where $\mathfrak{r}(X,Y) = \max \varkappa(X,Y)$ and $\varkappa(X,Y)$ is the Mackey topology of the dual pair (X,Y).

If $\mathfrak{t}$ is a filter base defining a quantum topology on X compatible with the duality (X,Y), then we define its *unitization* to be a quantum topology determined by the filter base $\widetilde{\mathfrak{t}} = \left\{\widetilde{\mathfrak{B}} \colon \mathfrak{B} \in \mathfrak{t}\right\}$. Since $\langle\langle e, M(Y)_{he}\rangle\rangle = I$, it follows that $e \in (\mathfrak{B}^{\odot} \cap M(Y)_{he})^{\odot} = \widetilde{\mathfrak{B}}$ for all $\mathfrak{B} \in \mathfrak{t}$. Moreover, $\widetilde{\mathfrak{t}}$ is a filter base of $\mathfrak{s}(X,Y)$-closed, absorbent, absolutely matrix convex sets on X, which in turn defines a (weaker) quantum topology $\widetilde{\mathfrak{t}}$. If $\mathfrak{t} = \|\cdot\|$ is a normed quantum topology determined by a matrix norm $\|\cdot\|$, then $\mathfrak{t} = \{\mathfrak{B}\}$ and $\widetilde{\mathfrak{t}} = \left\{\widetilde{\mathfrak{B}}\right\}$, where $\mathfrak{B} = \operatorname{ball}\|\cdot\|$. It follows that $\widetilde{\mathfrak{t}} = \|\cdot\|_e$ is a seminormed quantum topology determined by a new matrix seminorm $\|\cdot\|_e$. Actually it is a matrix norm as follows from the following result.

Lemma 1. *If (X,Y) is a dual $*$-pair and $\mathfrak{t}$ is a quantum topology on X compatible with the duality (X,Y), then $\widetilde{\mathfrak{t}}$ is a Hausdorff quantum topology on X compatible with the duality (X,Y). In particular, $\widetilde{\mathfrak{s}}(X,Y) = \mathfrak{s}(X,Y)$ and $\widetilde{\mathfrak{r}}(X,Y) \preceq \mathfrak{r}(X,Y)$.*

If $\widetilde{\mathfrak{t}} = \mathfrak{t}$ (as the filter bases), then we say that $\mathfrak{t}$ is *a unital quantum topology.* A quantum topology $\mathfrak{t}$ on X compatible with the duality (X,Y) is said to be *a quantum $*$-topology* if its neighborhood filter base consists of *hermitian quantum sets*, that is, $\mathfrak{B}^* = \mathfrak{B}$ and $e \in \mathfrak{B}$ for all $\mathfrak{B} \in \mathfrak{t}$.

Theorem 2. *Let (X,Y) be a dual $*$-pair, and let $\mathfrak{t}$ be a quantum topology on X compatible with the duality (X,Y). Then $\mathfrak{t}$ is a quantum $*$-topology on X if and only if $\mathfrak{t}$ is unital. In particular, $\widetilde{\mathfrak{r}}(X,Y)$ is a quantum $*$-topology, and all unital quantum topologies compatible with the duality (X,Y) are exactly arranged into the $*$ -scale $\mathfrak{s}(X,Y) \preceq \mathfrak{t} \preceq \widetilde{\mathfrak{r}}(X,Y)$ of all quantum $*$-topologies.*

Using Theorem 1, we conclude that a quantum $*$-topology $\mathfrak{t}$ on X being a unital one generates a quantum order $\widetilde{\mathfrak{C}}_{\mathfrak{t}} = \left\{\widetilde{\mathfrak{C}_{\mathfrak{B}}} : \mathfrak{B} \in \mathfrak{t}\right\}$ on X. For brevity we put $\mathfrak{C}_{\mathfrak{t}} = \widetilde{\mathfrak{C}}_{\mathfrak{t}}$.

Theorem 3. *Let (X, Y) be a dual $*$-pair. The correspondence $\mathfrak{t} \longmapsto \mathfrak{C}_{\mathfrak{t}}$ is a bijection from the $*$ -scale $\mathfrak{s}(X, Y) \preceq \mathfrak{t} \preceq \widetilde{\mathfrak{r}}(X, Y)$ of quantum $*$-topologies onto the scale of all $\mathfrak{s}(X, Y)$-quantum orders on X compatible with the duality (X, Y).*

5. The matrix normed case

A normed quantum $*$-topology $\mathfrak{t}$ on X is given by the Minkowski functional $\|\cdot\|$ such that $\|x^*\| = \|x\|$ and $\|e\| = 1$ for all $x \in M(X)$.

Proposition 1. *Let (X, Y) be a dual $*$-pair, $\mathfrak{t} = \|\cdot\|$ a normed quantum $*$-topology on X compatible with the duality (X, Y), and let $\mathfrak{B}$ be the related unit ball, which is an hermitian quantum set in $M(X)$. Then X is an operator system such that $M(X)_+ = \mathcal{S}^{\boxdot}$, where $\mathcal{S} = (3\mathfrak{B}^{\odot}) \cap M(Y)_{he}$ is a prematricial state space. Moreover, $\mathfrak{t} = \|\cdot\|_e$ with $\|x\|_e = \sup \|\langle\langle x, \mathcal{S}\rangle\rangle\|$, and $3^{-1}\|x\|_e \leq \|x\| \leq 10\|x\|_e$ for all $x \in M(X)$.*

Confirm that the prematricial state space $\mathcal{S}$ is far from being unique, it depends on a particular choice of the matrix norm on X. It can be much smaller than $(3\mathfrak{B}^{\odot}) \cap M(Y)_{he}$. For example, it can be $\mathfrak{B}^{\odot} \cap M(Y)_{he}$.

6. Quantizations of unital cones

Let (X, Y) be a dual $*$-pair with the unital space (X, e), and let t be a polynormed topology in X compatible with the duality. If t has a neighborhood filter base of hermitian sets (that is, $e \in \mathfrak{b}$ and $\mathfrak{b}^* = \mathfrak{b}$ for all $\mathfrak{b} \in t$), then we say that t is *a $*$-topology.*

Theorem 4. *Let (X, Y) be a dual $*$-pair and let t be $*$-topology in X compatible with the duality (X, Y). Then $\min t$ and $\max t$ are quantum $*$-topologies on X compatible with (X, Y). In particular, there are unique $\mathfrak{s}(X, Y)$-quantum orders $\mathfrak{C}_{\min t} = \left\{(\mathfrak{b}^{\circ} \cap Y_{he})^{\boxdot} : \mathfrak{b} \in t\right\}$ and $\mathfrak{C}_{\max t} = \left\{\left(((\mathfrak{b}^{\circ} \cap Y_{he})^{\circ})^{\odot} \cap M(Y)_{he}\right)^{\boxdot} : \mathfrak{b} \in t\right\}$ on X compatible with (X, Y) that correspond to $\min t$ and $\max t$, respectively.*

The couples $(X, \mathfrak{C}_{\min t})$ and $(X, \mathfrak{C}_{\max t})$ are called *the minimal and maximal quantum systems associated with the $*$-topology t in X.*

7. Quantizations of a local order within Paulsen-Todorov-Tomforde framework

Let (X, e) be a unital $*$-vector space. By *a local order in* X we mean a filter base $\mathfrak{f}$ of closed, unital cones in X such that $\cap\mathfrak{f}$ is a separated cone in X. For every $\mathfrak{c} \in \mathfrak{f}$ we have the state space $S(\mathfrak{c})$ to be the set of all unital $\mathfrak{c}$-positive functionals $y : X \to \mathbb{C}$. The filter base $\widehat{\mathfrak{f}} = \{S(\mathfrak{c})^{\circ} : \mathfrak{c} \in \mathfrak{f}\}$ of absolutely convex subsets in X generates the related polynormed Hausdorff topology. The set of all $\widehat{\mathfrak{f}}$-continuous functionals on X is denoted by Y. Thus (X, Y) is a dual $*$-pair, and $\cup\{S(\mathfrak{c}) : \mathfrak{c} \in \mathfrak{f}\} \subseteq Y_{he}$. By *a quantization of* $\mathfrak{f}$ we mean a quantum order $\mathcal{F}$ on X such that $\mathcal{F}|X = \mathfrak{f}$. For every $\mathfrak{c} \in \mathfrak{f}$ we define its *maximal envelope* $\overline{\mathfrak{c}}$ to be the quantum polar $S(\mathfrak{c})^{\boxdot}$ on X, and put $\min\mathfrak{f} = \{\overline{\mathfrak{c}} : \mathfrak{c} \in \mathfrak{f}\}$, which is a filter base of $\mathfrak{s}(X, Y)$-closed, quantum cones on X.

Proposition 2. *The filter base* $\min\mathfrak{f}$ *is a* $\mathfrak{s}(X, Y)$*-quantum order on* X*, which is a quantization of* $\mathfrak{f}$*. Moreover,* $\widehat{\min\mathfrak{f}}|X = \widehat{\mathfrak{f}}$ *and the quantum topology* $\widehat{\min\mathfrak{f}}$ *is given by the family of matrix seminorms* $p_{\overline{\mathfrak{c}}}(x) = \sup\|\langle\langle x, S(\mathfrak{c})\rangle\rangle\|$*,* $x \in M(X)$*.*

Fix again $\mathfrak{c} \in \mathfrak{f}$ and define its minimal envelope $\underline{\mathfrak{c}}$ to be the algebraic closure of the quantum cone in $M(X)_h$ generated by $\mathfrak{c}$ (see [15]). Put $\max\mathfrak{f} = \{\underline{\mathfrak{c}} : \mathfrak{c} \in \mathfrak{f}\}$, which is a filter base of closed, unital, quantum cones on X. Since $\overline{\mathfrak{c}}$ is closed (being topologically closed), it follows that $\underline{\mathfrak{c}} \subseteq \overline{\mathfrak{c}}$. In particular, $\cap\max\mathfrak{f} \subseteq \cap\min\mathfrak{f}$ and $\cap\max\mathfrak{f}$ is a closed, separated, unital, quantum cone by Proposition 2. Hence $\max\mathfrak{f}$ is a quantum order on X. Using [7, Theorem 5.1], we conclude that $\widehat{\max\mathfrak{f}}$ is a Hausdorff quantum $*$-topology on X such that $\max\mathfrak{f}$ consists of topologically closed quantum cones. For a while we put $Y_{\max}$ to be the topological dual of $\left(X, \widehat{\max\mathfrak{f}}\right)$. Since $\widehat{\max\mathfrak{f}}$ is stronger than $\widehat{\min\mathfrak{f}}$ in the quantum topology scale, it follows that $Y \subseteq Y_{\max}$.

Lemma 2. *Let* $\mathfrak{f}$ *be a local order in* X*. Then* $\widehat{\max\mathfrak{f}}|X = \widehat{\min\mathfrak{f}}|X = \widehat{\mathfrak{f}}$ *and* $Y = Y_{\max}$*.*

Based on Lemma 2 and Proposition 2, one can prove the following result.

Theorem 5. *Let* $\mathfrak{f}$ *be a local order in* X*. Then* $\max\mathfrak{f}$ *is a* $\mathfrak{s}(X, Y)$*-quantum order on* X*, which is a quantization of* $\mathfrak{f}$*. Thus* $\underline{\mathfrak{c}} = \mathfrak{c}^{\boxdot\boxdot}$ *for every* $\mathfrak{c} \in \mathfrak{f}$ *with respect to the dual* $*$*-pair* (X, Y)*. Moreover, for every quantization* $\mathcal{F}$ *of* $\mathfrak{f}$ *on* X *we have* $\min\mathfrak{f} \preceq \mathcal{F} \preceq \max\mathfrak{f}$*. In particular,* $\widehat{\mathcal{F}}|X = \widehat{\mathfrak{f}}$*,* $\widehat{\min\mathfrak{f}} = \min\widehat{\mathfrak{f}}$ *and* $\widehat{\max\mathfrak{f}} = \max\widehat{\mathfrak{f}}$*.*

8. The entanglement breaking maps

Let (X, e) be a unital $*$-vector space with a local order $\mathfrak{f}$ in X. Based on Theorem 5, we can assume $\mathfrak{f}$ is $\sigma(X, Y)$-local order and all quantizations $\mathcal{F}$

of $\mathfrak{f}$ are $\mathfrak{s}(X,Y)$-quantizations for a certain Y $(= Y_{\max})$ such that (X,Y) is a dual $*$-pair. Let (X',Y') be another dual $*$-pair equipped with a $\mathfrak{s}(X',Y')$-quantum order $\mathcal{F}'$. If $\varphi:(X,\mathcal{F})\rightarrow(X',\mathcal{F}')$ is a quantum positive mapping, then $\varphi:\left(X,\widehat{\mathcal{F}}\right)\rightarrow\left(X',\widehat{\mathcal{F}'}\right)$ is a quantum continuous mapping [7, Corollary 6.2], which in turn implies that $\varphi^{*}(Y')\subseteq Y$, where φ^{*} is the algebraic dual mapping to φ. Moreover, for every $\mathfrak{K}\in\mathcal{F}'$ there corresponds $\mathfrak{C}\in\mathcal{F}$ such that $(\varphi^{*})^{(\infty)}(S(\mathfrak{K}))\subseteq\mathbb{R}_{+}S(\mathfrak{C})$, where $S(\mathfrak{K})$ is the state space of the quantum cone $\mathfrak{K}$ and $\mathbb{R}_{+}S(\mathfrak{C})$ indicates to the quantum set of all $\mathfrak{C}$-positive functionals on the matrix spaces.

A linear mapping $\varphi:(X,\mathfrak{f})\rightarrow(X',\mathcal{F}')$ of a local ordered system $(X,\mathfrak{f})$ to a quantum system $(X',\mathcal{F}')$ is called *an entanglement breaking mapping* if for each $\mathfrak{K}\in\mathcal{F}'$ there corresponds $\mathfrak{c}\in\mathfrak{f}$ such that $(\varphi^{*})^{(\infty)}(S(\mathfrak{K}))\subseteq\mathbb{R}_{+}S(\mathfrak{C})$ for every quantization $\mathfrak{C}$ of $\mathfrak{c}$. An entanglement breaking mapping $\varphi:(X,\mathfrak{f})\rightarrow(X',\mathcal{F}')$ is continuous automatically being a mapping from $\left(X,\widehat{\mathfrak{f}}\right)$ to $\left(X',\widehat{\mathcal{F}'}|X'\right)$. Thus φ^{*} in the definition of an entanglement breaking mapping is the topological dual.

Proposition 3. *Let $(X,\mathfrak{f})$ be a local ordered space, $(X',\mathcal{F}')$ a quantum system, and let $\varphi:(X,\mathfrak{f})\rightarrow(X',\mathcal{F}')$ be a linear mapping. The following conditions are equivalent: (i) φ is an entanglement breaking mapping; (ii) for each $\mathfrak{K}\in\mathcal{F}'$ there corresponds $\mathfrak{c}\in\mathfrak{f}$ such that $(\varphi^{*})^{(\infty)}(S(\mathfrak{K}))\subseteq\mathbb{R}_{+}S\overline{\otimes}S(\mathfrak{c})$; (iii) $\varphi:(X,\min\mathfrak{f})\rightarrow(X',\mathcal{F}')$ is quantum positive, where $S\overline{\otimes}S(\mathfrak{c})$ is the space of all w^{*}-limits of separable $\mathfrak{c}$-positive functionals.*

Concluding remarks. Based on the central result Theorem 3, we derive that there is no operator column and row Hilbert systems as well as Haagerup tensor product of operator systems in their direct proper senses. Nonetheless the operator Hilbert space of Pisier turns out to be an operator system, and the projective tensor product of operator systems admits interesting quantizations that have not been seen before in [12].

References

[1] M.D. Choi, E.G. Effros, *Injectivity and operator spaces*, J. Functional Anal., **24**, 1977, 156-209.

[2] A.A. Dosi, *Quantum duality, unbounded operators and inductive limits*, J. Math. Physics, **51(6)**, 2010, 1-43.

[3] A.A. Dosi, *Noncommutative Mackey theorem*, Inter. J. Math., **22(3)**, 2011, 1-7.

[4] A.A. Dosi, *Quotients of quantum bornological space*, Taiwanese J. Math., **15(3)**, 2011, 1287-1303.

[5] A.A. Dosi, *Bipolar theorem for quantum cones*, Funct. Anal. and its Appl., **46(3)**, 2012, 228-231.

[6] A.A. Dosi, *Quantum cones and their duality*, Houston J. Math., **39(3)**, 2013, 853-887.

[7] A.A. Dosi, *Quantum systems and representation theorem*, Positivity, **17(3)**, 2013, 841-861.

[8] E.G. Effros, Z-.J. Ruan, *Operator Spaces*, Clarendon Press, Oxford, 2000.

[9] E.G. Effros, C. Webster, *Operator analogues of locally convex spaces*, Operator Algebras and Applications, (Samos 1996), (Adv. Sci. Inst. Ser. C Math. Phys. Sci., 495), Kluwer (1997).

[10] E.G. Effros, S. Winkler, *Matrix convexity: operator analogues of the Bilpolar and Hanh-Banach theorems*, J. Funct. Anal., **144**, 1997, 117-152.

[11] A.Ya. Helemskii, *Quantum functional analysis*, MCCME, Moscow, 2009.

[12] A. Kavruk, V.I. Paulsen, I.G. Todorov, M. Tomford, *Tensor products of operator systems*, J. Func. Anal., **261**, 2011, 267-299.

[13] A. Kavruk, V.I. Paulsen, I.G. Todorov, M. Tomford, *Quotients, exactness and nuclearity in the operator system category*, Adv. Math., **235**, 2013, 321-360.

[14] V.I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Stud. Advan. Math., **78**, 2002.

[15] V.I. Paulsen, V.I. Todorov, M. Tomford, *Operator system structures on ordered spaces*, Proc. London Math. Soc., **102(3)**, 2011, 25-49.

[16] V.I. Paulsen, M. Tomford, *Vector spaces with an order unit*, Indiana Univ. Math. J., **58**, 2009, 1319-1359.

[17] G. Pisier, *Introduction to operator space theory*, Cambridge Univ. Press, 2003.

[18] C. Webster, *Local operator spaces and applications*, Ph.D. University of California, Los Angeles, 1997.

[19] A.A. Dosi, *Multinormed semifinite von Neumann algebras, unbounded operators and conditional expectations*, J. Math. Anal. Appl., 2018, 1-36 (in press).

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Received 14 February 2018
Accepted 14 May 2018