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Sturm Type Theorems for Differential Equations of Arbitrary Order with Operator-Valued Coefficients

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Abstract. The operators under consideration are generated by differential expressions with locally bounded operator coefficients both on finite and infinite intervals. A relation between spectral and oscillation properties of such differential operators of arbitrary order (even and odd) is investigated. The operators in question may be both self-adjoint and non self-adjoint with block-triangular matrix coefficients. Several well-known Sturm type Theorems, which were obtained by topological methods, are obtained also by operator-theoretical methods. The results contained in this survey for infinite systems, being applied to either finite systems or scalar problems, are at least as precise as the results already known for those cases; and sometimes appear to be even more precise.

Key Words and Phrases: differential operator, Sturm type oscillation theory, block-triangular matrix potential.

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1. Introduction

The oscillation theory of differential equations originates in the famous works (formerly, memoirs) by Sturm [93] – [95], together with the joint paper by Liouville and Sturm [54]. Among those, one should mention above all the fundamental memoir by Sturm'of 1836 [94]. Since then, the theory of Sturm-Liouville type equations and the associated boundary problems permanently attracts an attention of a growing number of specialists in mathematics, physics and engineering throughout the world due to the increasing number of important applications in various fields. Namely, these are direct and inverse scattering problems (in many important cases the Sturm-Liouville equation is identical to the Schrödinger equation), oscillation theory, mathematical physics (both classical and modern), the theory of periodic, almost periodic, and stochastic (non-ordered) systems (e.g., crystal lattices with impurities), geometry (vector fields along geodesics), topology, and symplectic geometry. Together with its various applications, the theory of Sturm-Liouville problems is itself a nice subject for the new mathematical theories and methods to be applied to, a sort of touchstone for those to become more sophisticated, to bring

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them into a comparison, to stimulate their creation and development (in particular, that of computational methods of modern mathematics; see, e.g., [7]).

The bicentenary of Jacques Charles Francois Sturm (1803 – 1855) and the development of the Sturm-Liouville theory till the present day have been made subjects of a conference in Switzerland (Genève 2003). A report by B. Simon at this conference [92] was dedicated to a review and new results on oscillation and Sturm type comparison theorems and applications. The basic milestones of this theory have been described in a report by D. Hinton [35] and W. N. Everitt [20]. The Collection of invited papers by participants of The Bicentenary Colloquium in Geneva (published in 2005) [96] contains a number of interesting works. Among those, in addition to the works cited above, let us mention the report by J. Weidmann [99], M. M. Malamud [57], and also a Catalogue of Sturm-Liouville differential equations produced by W. N. Everitt [21]. That very year (2005) our monograph [89] has been published. That monograph was dedicated to a generalization of the Sturm Oscillation Theory for infinite self-adjoint systems of differential equations and the related topics.

In this survey we consider differential equations with bounded (for every value of a variable) operator-valued coefficients in a separable Hilbert space H. Such equations are equivalent to infinite systems of ordinary differential equations, and in the special case of finite dimensional H they are reduced to finite systems of ordinary equations.

The purpose of this work is to demonstrate the way of generalizing the Sturm Oscillation Theorem for self-adjoint differential equations of an arbitrary order with operator-valued coefficients on finite and infinite intervals. An important special case of this result is the Morse Index Theorem (see [65], [66], and also [61]).

An interesting topological interpretation of the Sturm theorems in the finite dimensional case and their link to symplectic geometry was considered by V. I. Arnold [3], see also [4]. Note that V. I. Arnold states in [3] that he has no claim on novelty of his results, because in a very classical field like the Sturm Theory, it is hard to keep track of all the predecessors. Some results of this [3] are deducible from the authors' works [79], [82]. Additionally, the final part of our survey contains a generalization of the Sturm Oscillation Theory to non self-adjoint systems of differential equations with block-triangular matrix coefficients. It should be noted here that V. A. Marchenko introduced a notion of generalized spectral function R for a Sturm-Liouville operator with arbitrary complex valued potential on the half-line [58], [59], which was transferred under his scientific supervision to the case of finite and infinite non self-adjoint systems [60], [71]. The distribution (the matrix in the case of systems) R acts on the topological space of test functions. The spectral distribution R determines formulas of expansion in eigenfunctions and also allows to solve the inverse problem of spectral analysis in the non self-adjoint case. In the case of self-adjoint problems, R is generated by a non-negative measure (either scalar of matricial in the case of systems). Here we use the specific form of spectrum and spectral matricial distribution R for some classes of self-adjoint and non self-adjoint systems.

It is interesting to note that a large number of partial linear equations can be successfully investigation, being treated as Sturm-Liouville type equations of a single variable, but with an operator-valued (unbounded) coefficient. This approach requires some special techniques to be applied, e.g., the theory of rigged spaces by Gel'fand and Kostyuchenko [22], Berezanskiy [8], theory of hyperfunctions, etc. Latter developments, starting from M. L. Gorbachuk [26], are contained in the monographs [27] – [28], [55], and review papers, e.g., [29].

The Sturm oscillation theory, along with its various generalizations to the ordinary differential equations and finite systems of those related to the spectral theory, hae been considered in the monographs of F. Atkinson [6], N. Danford and J. T. Schwartz [16], I. M. Glazman [24], A. L. Gol'denveiser, V. B. Lidskiy, and P. E. Tovstik [25], A. G. Kostyuchenko and I. S. Sargsyan [48], B. M. Levitan and I. S. Sargsyan [51], L. A. Pastur and A. L. Figotin [67], [68], C. A. Swanson [97], V. A. Yakubovich, V. M. Starzhinskiy [105], [106], P. Hartman [33], F. A. Berezin and M. A. Shubin [10], H. Weyl [100] (see also [101] – [103]). The sources listed above contain also an extended bibliography on the subjects in questions.

The well-known monographs by V. A. Marchenko [58], [59] and B. M. Levitan [50] are dedicated to direct and inverse problems of spectral theory for the Sturm-Liouville operator in various settings.

A relation between oscillation and spectral properties for scalar differential equations of an arbitrary even order were investigated in the works by M. G. Krein [49] and E. Heintz [34] (see also [24]). A generalization of the Sturm Oscillation Theorem for systems of differential equations of order 4, which arise in studying free oscillations of thin elastic shells, is contained in the monographs by A. G. Aslanyan and V. B. Lidskiy [5], [25]. A similar result for finite systems of even order in the case of simple spectrum can be found in [37]. It should be noted that the 'scalar' formulations and proofs of those theorems, which are based on the notion of determinant and compactness of the finite dimensional sphere, become inapplicable in the infinite dimensional case. Our considerations also cover a broad class of systems of odd order differential equations [80], [85], [89], in particular canonical systems, for example, the Dirac systems on axis with operator-valued coefficients [38]. Note that the results contained in this survey for infinite systems, being applied to either finite systems or scalar problems, are at least as precise as the results already known for those cases; and sometimes appear to be even more precise. Some of our results turn out to be new even in the scalar and other special cases.

The oscillation properties for linear canonical systems have been investigated in various contexts. In the real case, they have been studied by V. B. Lidskiy [52] and V. A. Yakubovich [104]. Some of their results have been transferred to the complex case by V. I. Khrabustovskiy [45]. Another reference here is the work by V. I. Arnold [3] cited above. The case of second order equations with operator valued coefficients has been studied in the works by G. Etgen and J. Pawlowski [19], G. Etgen and R. Lewis [18].

We supply a topological interpretation of the Sturm oscillation theorems for differential equations of an arbitrary order with locally bounded operator-valued coefficients; this interpretation is compared to the operator approach. A link to the symplectic geometry is considered.

A relation between spectral and oscillation properties of the problem on finite interval or half-line for Sturm-Liouville differential equations with block-triangular matrix coefficients is investigated. These results are new and have been published by the authors recently in [42]. In the case of equation with block-triangular matrix potential that rise at infinity, matrix solutions are constructed which either rise or decrease at infinity; a structure of the spectrum for differential operator with such coefficients is established; the Green function is produced together with its series expansion; the Parseval equality is proved [43], [44].

The present survey is based on the authors' works [79] – [84], [42], [43], [44] their monographs [85], [89], and the reports at the Conferences of the Ukrainian Mathematical Congress 2001 [86] – [88], [40], and at the Conference in Varna (1981) [75]. Additionally, some new results of the authors that extend the Sturm Oscillation Theory to non self-adjoint systems of differential equations with block-triangular matrix coefficients are included in [42] [43].

Note that non self-adjoint operators correspond to a description of non-closed physical systems, and undoubtedly make a mathematical interest. In this connection we mention a paper by Yu. L. Daletskii [14] and Section 7.5.3 'The Riemann-Hilbert Problem for Triangular Matrices' from the book by M. J. Ablowitz and A. S. Fokas [1]. Note also papers [13] (for the self-adjoint case see [2]) and [91].

2. On the Relation Between Spectral and Oscillation Properties of the Sturm-Liouville Matrix or Operator Problem

2.1. The Problem on a Finite Interval

Consider the following boundary-value problem for vector-valued functions with values in a finite-dimensional or infinite-dimensional separable Hilbert space H:

$$l[y] := -(P(x)y')' + Q(x)y = \lambda W(x)y \tag{2.1}$$

$$\cos A \cdot P(0)y'(0) - \sin A \cdot y(0) = 0 \tag{2.2}$$

$$\cos B \cdot P(b)y'(b) + \sin B \cdot y(b) = 0 \tag{2.3}$$

Here $P(x) > 0$, $W(x) > 0$, $Q(x)$, A, B, together with $P^{-1}(x)$, $P'(x)$, $W^{-1}(x)$ are all bounded self-adjoint operators in H; the dependence on $x \in [0, b]$ is uniformly continuous; and

$$-\frac{\pi}{2}I << A, B \leq \frac{\pi}{2}I. \tag{2.4}$$

(The point $-\frac{\pi}{2}$ belongs to the resolvent sets of A and B. In the case $\dim H < \infty$, this restriction is removed. Here I denotes the identity operator in H.)

The differential expression

$$l_W[y] := W^{-1}(x)l[y] \tag{2.5}$$

with boundary conditions (2.1), (2.3) gives rise to a self-adjoint operator L in the Hilbert space

$$H(0,b) = L_2\{H;(0,b);W(x)dx\} \tag{2.6}$$

with the inner product

$$< u, v >= \int_0^b (W(x)u(x), v(x))_H dx.$$

Under the given conditions (including (2.4)), the operator L is lower semi-bounded (see [71], p. 337 – 338, and [30]). If $\cos A = 0$ and $\cos B = 0$, the boundary conditions (2.2), (2.3) become

$$y(0) = 0, \tag{2.7}$$
$$y(b) = 0. \tag{2.8}$$

Let us denote by L^0 the operator corresponding to the boundary value problem (2.1), (2.2), (2.8). In the case $\dim H = m < \infty$, the problem (2.1) – (2.3) and the corresponding self-adjoint operator L have purely discrete spectrum

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \dots,$$

However, this is not the case when $\dim H = \infty$ (see [71]†, p. 340, and [27], p. 20). Let

$$\lambda_e = \inf \sigma_e(L), \qquad \lambda_e^\circ = \inf \sigma_e(L^\circ)$$

be the greatest lower bounds of the essential spectrum σ_e of operators L and L° (the eigenvalues of infinite multiplicity belong to σ_e; in the case $\dim H < \infty$ we have $\lambda_e = +\infty$).

If λ is an eigenvalue of the problem (2.1) – (2.3), we denote its multiplicity by $æ(\lambda)$. If λ is not an eigenvalue of (2.1) – (2.3), we set $æ(\lambda) = 0$. Let us denote by $N(\lambda)$ the number of eigenvalues $\lambda_k < \lambda$ of the problem (2.1) – (2.3), counting multiplicities. The corresponding quantities $N(\lambda)$, $æ(\lambda)$, λ_n for the operator L° will be denoted by $N^\circ(\lambda)$, $æ^\circ(\lambda)$, λ_n°, respectively.

Let $Y(x,\lambda)$ be the fundamental matrix or operator solution to (2.1) with the initial conditions

$$Y(0,\lambda) = \cos A, \qquad P(0)Y'(0,\lambda) = \sin A. \tag{2.9}$$

Set

$$\operatorname{nul} Y(x,\lambda) = \dim \operatorname{Ker} Y(x,\lambda), \qquad \operatorname{def} Y(x,\lambda) = \dim \operatorname{Coker} Y(x,\lambda).$$

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†It follows from the example constructed therein that the spectrum of the problem (2.1) – (2.3) may prove even absolutely continuous.

Theorem 1 ([79], [85]; see also our monograph [89], Theorem 1.1, p. 3)**.**
For $\lambda \le \lambda_e^\circ \le \infty$ we have

$$\sum_{x\in(0,b)} \operatorname{nul} Y(x,\lambda) = N^\circ(\lambda)(\le\infty), \tag{2.10}$$

$$\forall\lambda\in\mathbb{R} \quad \operatorname{nul} Y(x,\lambda) = \operatorname{def} Y(x,\lambda) \tag{2.11}$$

(the sum here is taken over all $x\in(0,b)$ such that $\operatorname{nul} Y(x,\lambda)\ne 0$).

Theorem 2 ([79], [85]; see also our monograph [89], Theorem 1.2, p. 3)**.**
For the problem (2.1) – (2.3)*, the following estimates hold. If $\dim H<\infty$, we have*

$$N(\lambda) - \min\{\operatorname{rank}\cos B, \dim H - æ(\lambda)\} \le N^0(\lambda) \le N(\lambda). \tag{2.12}$$

Assume that $\dim H = \infty$ and $\lambda \le \lambda_e$.
If $\operatorname{rank}\cos B < \infty$, then $\lambda_e = \lambda_e^\circ$ and

$$N(\lambda) - \operatorname{rank}\cos B \le N^\circ(\lambda) \le N(\lambda). \tag{2.13}$$

If $\operatorname{rank}\cos B = \infty$, then $\lambda_e \le \lambda_e^\circ$ and

$$0 \le N^\circ(\lambda) \le N(\lambda). \tag{2.14}$$

The equalities may take place in the leftmost inequalities of this Theorem if the left hand sides of these inequalities are non-negative.

Corollary 1.
1) In the scalar case ($\dim H = 1$), for the problem (2.1) – (2.3)*, by* (2.10) *and* (2.12) *we have*

$$\sum_{x\in(0,b)} \operatorname{nul} Y(x,\lambda_n) = n-1 \tag{2.15}$$

and this is equivalent to the classical Sturm Oscillation Theorem.
2) If $\dim H<\infty$, $\lambda_n > \lambda_{n-1}$, and $æ(\lambda_n) = \dim H$, then (2.15) *holds.*
3) If $\dim H \le \infty$ and $æ^\circ(\lambda_n^\circ) = 1$, then (2.15) *holds for the problem* (2.1)*,* (2.2)*,* (2.8) *(since in this case $N^\circ(\lambda_n^\circ) = n-1$).*

Note that if $\dim H > 1$ and if the general boundary condition (2.3) holds, then the equation (2.15) can fail, as it is shown in the following example.

Example 1. *For the diagonal system*

$$-y_k'' = \lambda y_k, \qquad y_k'(0) = 0, \qquad y_k'(1) + (-1)^k y_k(1) = 0$$

we have $Y(x,\lambda) = I_2\cos\left(x\sqrt{\lambda}\right)$. The eigenvalues are simple and are determined by the equation $\lambda\left(\tan\sqrt{\lambda}\right)^2 = 1$, ($\lambda_1 < 0$). Here

$$\sum_{x\in(0,1)} \operatorname{nul} Y(x,\lambda_{2n}) = 2n-2,$$

which contradicts to (2.15)*.*

Remark 1. *Let* $\dim H \leq \infty$. *Consider the problem* (2.1), (2.7). *In this case, in the terminology of Morse,* $\operatorname{nul} Y(x,\lambda)$ *coincides with the index of the point* x *with respect to the endpoint* $x = 0$ *and the equation* (2.1) *for the given* λ. *On the other hand, the number* $N^\circ(\lambda)$ *of eigenvalues of the problem* (2.1), (2.7), (2.8) *less than the given* λ, *is the maximal dimension of the subspace of* $H(0,b)$ *on which the corresponding Dirichlet integral*

$$D_\lambda[u] = \int_0^b \left\{(P(x)u', u') + ((Q(x) - \lambda W(x))u, u)\right\} dx$$

is negative (the Morse quadratic form (i.e. Hessian of the action function (see [61], § 13) or index form of the geodesic (see [31], 4.5)) can be reduced to $D_\lambda[u]$*). Thus, as it was noted in the Introduction, Theorem 1 contains the Morse Index Theorem.*

Morse Index Theorem ([61], Sec. 4.6 or [31], Sec. 15, see also our monograph [89], p. xxii, and Remark 1.1., p.5) *Index of Hessian for the action function*

$$E_{**} : T\Omega_\gamma \times T\Omega_\gamma \to \mathbb{R}$$

(i.e. maximal dimension of subspaces $T\Omega_\gamma$ *on which the form* E_{**} *is negative definite) is finite and equal to the number of points of the geodesic* $\gamma(t)$, $0 < t < 1$, *such that* $\gamma(t)$ *is conjugate to* $\gamma(0)$ *along* γ, *where every conjugate point is counted according to its multiplicity. (In Morse terminology, this multiplicity is also called the index of point with respect to the endpoint* $t = 0$ *and the equation* (2.1). *The sum of indices of the point* t *along a given geodesic segment is called "Morse index of the given geodesic segment.")*

The works [72] – [85], [89], contain a generalization of the Sturm Oscillation Theorem for the infinite system of second order differential equations

$$-\big(P(x)y' + R(x)y\big)' + R^*(x)y' + Q(x)y = \lambda W(x)y, \tag{2.16}$$

with the boundary conditions

$$\cos A \cdot y^{[1]}(a) - \sin A \cdot y(a) = 0, \tag{2.17}$$
$$\cos B \cdot y^{[1]}(b) + \sin B \cdot y(b) = 0, \tag{2.18}$$

where $y^{[1]}(x) := P(x)y'(x) + R(x)y$ is the quasi-derivative.

Set

$$Y[x,\lambda,B] := \cos B \cdot Y^{[1]}(x,\lambda) + \sin B \cdot Y(x,\lambda). \tag{2.19}$$

Theorem 3 ([78]; see also our monograph [89], Proposition 1.1, p. 4)**.**
For the problem (2.16) – (2.18) *with* $\lambda \leq \lambda_e(\leq \infty)$ *one has*

$$\sum_{x\in(a,b)} \operatorname{nul} Y[x,\lambda,B] = N(\lambda), \tag{2.20}$$

$$\operatorname{nul} Y[x,\lambda,B] = \operatorname{def} Y[x,\lambda,B],$$

provided that for $a \le x \le b$

$$H_{2,2}(x,\lambda,B) := \cos B \cdot \left\{-Q(x) + \lambda W(x) + R^*(x)P^{-1}(x)R(x)\right\}\cos B + \\ + \sin B \cdot P^{-1}(x)R(x)\cos B + \cos B \cdot R^*(x)P^{-1}(x)\sin B + \sin B \cdot P^{-1}(x)\sin B \gg 0, \\ a \le x \le b, \quad (2.21)$$

and, in addition to (2.4)*, the operator* h_a *is lower semi-bounded, self-adjoint in the closure of its domain* $D(h_a)$*, where*

$$h_a := -\left(\cos A \sin B + \sin A \cos B\right)^{-1}\left(\cos A \cos B - \sin A \sin B\right) \ge c \cdot I_{\overline{D(h_a)}}, \qquad c \in \mathbb{R}. \quad (2.22)$$

Remark 2. *The condition* (2.21) *is always true for* λ *big enough, and the condition* (2.22) *is always true if* $\dim H < \infty$*, i.e., for finite systems, when the operator* h_a *is self-adjoint.*

Note that with $\cos B = 0$, Theorem 3 transforms to Theorem 1 (i.e., Theorem 1.1 of our monograph [89]).

In the general case the proof of Theorem 3 goes similarly to that of Theorem 1.1 of [89], using analogs of Lemmas 1.1, 1.2 and that of Theorem 1.3 from [89]. Under assumptions of Theorem 3, those analogs are established in our case for $Y[x,\lambda,B]$ instead of $Y(x,\lambda)$ in a similar way, after one applies the condition (2.22) for the operator h_a. Theorem 3 allows one to prove the Arnold Alternation Theorem in the generalized form. Let us formulate this theorem using our definitions.

Theorem 4. (on the alternation; V. I. Arnold [3]; see also our monograph [89], Theorem 4.9, p. 120)
If the Hamiltonian function $H(t)$ *is positive definite on Lagrangian planes* α *and* β*, then the numbers* ν *of non-transversality instants with respect to the planes* α *and* β *of the Lagrangian plane that evolves in correspondence to the equation*

$$J\frac{d}{dt}\begin{pmatrix} y \\ y^{[1]} \end{pmatrix} = H(t,\lambda)\begin{pmatrix} y \\ y^{[1]} \end{pmatrix}, \quad (2.23)$$

can differ among themselves in an arbitrary interval at most by the number of degrees of freedom

$$|\nu_\alpha - \nu_\beta| \le \dim H (< \infty). \quad (2.24)$$

Note that in [3], a real finite dimensional space H is considered, $\lambda = 0$. (2.23) is derived by reducing the system

$$l[y] \equiv -(P(t)y' + R(t)y)' + R^*(t)y' + Q(t)y = \lambda W(t)y \quad (2.25)$$

to a first order system for the vector function $\mathrm{col}\{y, y^{[1]}\}$ in the double space $H \oplus H$, where $J = \begin{pmatrix} 0_H & -I_H \\ I_H & 0_H \end{pmatrix}$, 0_H and I_H are the zero and the identity operators in H, respectively,

$$H(t,\lambda) = \begin{pmatrix} -Q + \lambda W + R^*P^{-1}R & -R^*P^{-1} \\ -P^{-1}R & P^{-1} \end{pmatrix}. \quad (2.26)$$

Corollary 2 ([3])**.** *On an interval containing* $\dim H + 1$ *non-transversality instants with respect to* α*, there is a non-transversality instant with respect to* β*.*

Suppose that for given A and, correspondingly, B and B_1, the semi-boundedness condition (2.22) is satisfied (in the case $\dim H < \infty$, this condition is satisfied automatically). Since the Hamiltonian $H(t,\lambda)$ (2.26) is positive on both Lagrangian planes α and β, we have (2.21) and, analogously

$$H_{2,2}(t,\lambda,B_1) \gg 0, \qquad a \le t \le b, \tag{2.27}$$

whence we get (2.20) and, correspondingly,

$$\sum_{x\in(a,b)} \operatorname{nul} Y[x,\lambda,B_1] = N_1(\lambda), \tag{2.28}$$

where $Y[x,\lambda;B_1]$ is defined by (2.19) with B_1 instead of B, and $N_1(\lambda)$ is the analogue of $N(\lambda)$ for the case of condition (2.18) with B_1 instead of B. (We remind the reader that we assume that the conditions (2.4) for A, B, and B_1 are also satisfied.) Now by Lemma 1.4 of [89], p. 14, we have

$$|N(\lambda) - N_1(\lambda)| \le p := \operatorname{Def}\{\Lambda\} \le \dim H, \tag{2.29}$$

where Λ is a symmetric operator equal to the common part of two self-adjoint operators corresponding to the boundary-value problems (2.16) – (2.18) with B and, correspondingly, B_1 in (2.18). Hence by Theorem 4.2 of [89], p. 95, we have

$$\operatorname{Def}\{\Lambda\} = \operatorname{rank}\{\sin B_1 \cdot \cos B - \cos B_1 \cdot \sin B\}. \tag{2.30}$$

From this,by (2.29), (2.28), (2.20), we obtain the following generalization of the Alternation Theorem 4.

Theorem 5 ([78]; see also our monograph [89], Theorem 4.10, p. 122)**.**
Under the above conditions, we have:

$$\left|\sum_{t\in(a,b)} \operatorname{nul} Y[t,\lambda;B] - \sum_{t\in(a,b)} \operatorname{nul} Y[t,\lambda;B_1]\right| \le \operatorname{Def}\{\Lambda\} = \\ = \operatorname{rank}\{\sin B_1 \cdot \cos B - \cos B_1 \cdot \sin B\} \le \dim H \le \infty. \tag{2.31}$$

Corollary 3. *On an interval containing* $1 + \operatorname{Def}\{\Lambda\}$ *points where* $\operatorname{nul} Y[t,\lambda;B] > 0$*, there is a point where* $\operatorname{nul} Y[t,\lambda;B_1] > 0$ *(if* $\operatorname{Def}\{\Lambda\} < \infty$*).*

2.2. The Problem on a Half-Line

By L' we will denote the minimal with respect to $x = \infty$ operator in $H(0, \infty)$ (2.6) defined by the expression $l_W[y]$ (2.5) and the boundary condition at zero (2.2). We assume that L' is lower semi-bounded and denote by L_F the Friedrichs extension of L'. We also use the notation $\lambda_e^F = \inf \sigma_e(L_F) \leq \infty$, and denote the eigenvalues of L_F by

$$\lambda_1^F \leq \lambda_2^F \leq \ldots (< \lambda_e^F).$$

Theorem 6 ([80] – [85]; see also our monograph [89], Theorem 1.4, p. 20)**.**
Let L' denote the minimal with respect to $x = \infty$ operator in $H(0, \infty) = L_2\{H; (0, \infty); W(x)dx\}$ defined by the expression $W^{-1}(x)l[y]$ (2.1) and the boundary condition (2.2). Assume that L' is lower semi-bounded. Let $N_F(\lambda)$ be the number of eigenvalues $\lambda_n^F < \lambda$ corresponding to the Friedrichs extension L_F of L', and let $N_0(\lambda)$ be the number of jumps $\lambda_n^0 < \lambda$ (counting multiplicities) of the spectral operator-valued function $\rho(\lambda)$ obtained from spectral functions of the problems of type (2.1), (2.2), $y(b_j) = 0$ by passing to the limit as $b = b_j \to \infty$.

Then for $\lambda \leq \lambda_e^F$ we have

$$\sum_{x \in (0,\infty)} \operatorname{nul} Y(x, \lambda) = N_F(\lambda) = N_0(\lambda).$$

Theorem 7 ([80] – [85]; see also our monograph [89], Theorem 1.4, p. 21)**.**
In the assumptions of Theorem 6, let Λ denote a self-adjoint extension of L' in $H(0, \infty)$ such that on the left of a fixed $\mu < \lambda_e^F$

$$\lambda_n(\Lambda) = \lambda_n^F (< \mu), \qquad (n = 1, 2, \ldots)$$

Let L be an arbitrary semi-bounded self-adjoint extension of L' in $H(0, \infty)$, and let

$$p = \min_{\Lambda} \operatorname{Def} \left\{ L|_{D(L) \cap D(\Lambda)} \right\},$$

where Def T *denotes the deficiency number*‡ *of an operator T. Let $N_L(\lambda)$ be the function counting eigenvalues $< \lambda$ of the operator L. Then for all $\lambda < \mu$, the following holds:*

$$N_L(\lambda) - p \leq \sum_{x \in (0,\infty)} \operatorname{nul} Y(x, \lambda) = N_F(\lambda) = N_\Lambda(\lambda) \leq N_L(\lambda).$$

If Def $L' = d$ *and λ is not an eigenvalue of the closed operator L', then for all $\lambda \leq \mu$ we have*

$$N_L(\lambda) - \min\{p, d - æ_L(\lambda)\} \leq \sum_{x \in (0,\infty)} \operatorname{nul} Y(x, \lambda) = N_F(\lambda) \leq N_L(\lambda).$$

‡For a semi-bounded operator, the two deficiency numbers are equal.

Corollary 4. *If* $\dim H = 1$, *then*

$$\sum_{x\in(0,\infty)} \operatorname{nul} Y(x, \lambda_n^L) = n - 1,$$

which corresponds to [16], [51].

Here is a consequence of Theorem 6.

Corollary 5. *Under assumptions of Theorem 6, suppose that* $\operatorname{rank}\cos A$ *in* (2.2) *is finite and the restriction* L'' *of the operator* L' *by the condition* $y(0) = 0$ *is non-negative (the last condition is satisfied if, for example,* $Q(x)$ *is a non-negative operator for all* x*). Then*

$$\sum_{x\in(0,\infty)} \operatorname{nul} Y(x, 0) \le \operatorname{rank}\cos A. \tag{2.32}$$

3. Semi-Bounded Operators of Even Order

Consider the self-adjoint differential equation of order $r = 2n$ with operator coefficients from $B(H)$

$$l[y] = \sum_{k=0}^{r} i^k l_k[y] = \lambda W(x) y, \tag{3.1}$$

where

$$l_{2j} = D^j p_j(x) D^j, \qquad p_j^*(x) = p_j(x),$$

$$l_{2j-1} = \frac{1}{2} D^{j-1}\{D q_j(x) + q_j^*(x) D\} D^{j-1}, \qquad D = d/dx_j.$$

Here, the operator-valued coefficients $p_j(x)$, $q_j(x)$, together with their derivatives of order up to and including j, uniformly continuously depend on x, and the coefficient at the highest order derivative $p_n(x)$ in the equation (2.1) has a bounded inverse in H for all $x \in (a, b)$, including a and b if they are finite.

Denote by L the minimal differential operator in $H(a, b)$ determined by the differential expression

$$l_w[y] = W^{-1}(x) l[y]. \tag{3.2}$$

A linear condition of the form

$$U_a[y] = 0, \tag{3.3}$$

where U_a is a linear map from $H(a, c)$, $a < c \le b$, into a Hilbert space $\mathbf{H}_a$, will be called a boundary condition at a if every two vector-valued functions, which coincide in a neighbourhood of a, either both satisfy this condition or they both don't.

Denote by L_ξ the closure of the operator in $H(a, \xi)$, $a < \xi \le b$, determined by the differential expression (3.2) and the condition (3.3), on the smooth functions that vanish in a neighbourhood of ξ. We call L_ξ the minimal operator with respect to the end-point ξ corresponding to (3.2), (3.3).

We call the boundary condition $U_a[y] = 0$ self-adjoint at the point a, if the operator L_b is symmetric, and the functions $y \in D(L_b^*)$ satisfy this boundary condition. Then L_ξ is symmetric for all $\xi \in (a, b)$.

Similarly, we define the self-adjoint condition

$$U_b[y] = 0 \tag{3.4}$$

at b. The general form of self-adjoint conditions for equations of arbitrary order with bounded operator-valued coefficients is derived in [72], [73].

An operator L in $H(a, b)$ has a self-adjoint extension determined by separated boundary conditions if and only if there exist self-adjoint boundary conditions for this operator at the points a and b. The restriction of the operator L^* to a manifold satisfying those conditions is a self-adjoint extension of L.

A semi-bounded differential operator in $H(a, b)$ always has self-adjoint extensions with separated boundary conditions; for example, the Friedrichs extension. It is easy to demonstrate that a lower semi-bounded differential operator should be of an even order $r = 2n$, and its highest coefficient $p_n(x)$, which is assumed to have a bounded inverse at every x in the closure of (a, b), should be, in addition, strictly positive at every x, that is $p_n(x) \gg 0$.

On the other hand, the assumption of lower semi-boundedness for the essential spectrum is insufficient to guarantee that the highest coefficient $p_n(x)$ is strictly positive or at least non-negative, as one can see from the following example.

Example 2. *Suppose the differential operator L with operator-valued coefficients is an orthogonal sum of two operators: $L = L_1 \oplus L_2$ in such a way that L_1 acts in the space $H_1(a, b) = L_2\{H_1; (a, b); W_1(x)dx\}$ and is lower semi-bounded, hence its essential spectrum is a fortiori lower semi-bounded. Here $\dim H_1 \leq \infty$. Then we let L_2 act in the space $H_2(a, b) = L_2\{H_2; (a, b); W_2(x)dx\}$ with $\dim H_2 < \infty$, so that L_2 has a strictly negative highest coefficient and a purely discrete spectrum (certainly, lower unbounded). In this case the operator $L = L_1 \oplus L_2$ that acts in the space $H(a, b) = H_1(a, b) \oplus H_2(a, b) = L_2\{H_1 \oplus H_2; (a, b); (W_1 \oplus W_2)dx\}$ has the same essential spectrum as L_1: $\sigma_e(L) = \sigma_e(L_1) > -\infty$. In particular, the essential spectrum of L is lower semi-bounded, while its highest coefficient is not a positive operator-valued function, as at every $x \in (a, b)$ it is an orthogonal sum of a positive operator from $B(H_1)$ and a negative operator from $B(H_2)$.*

The even order operators on a finite interval always have the separated self-adjoint boundary conditions without semi-boundedness assumption [72], [73]. Meanwhile, an even order operator on the half-line $(0, \infty)$ may fail to have self-adjoint separated boundary conditions with $\dim H \leq \infty$. To see that sort of example, consider a separated infinite system of scalar second order equations in which the system formed by the initial two equations, generates a symmetric operator with deficiency indices (2;3) or (3;2) [36], and the rest of equations form a diagonal system with all the scalar operators having deficiency indices (1;1). Then L has deficiency indices (∞, ∞) and admits self-adjoint extensions; on the other hand, the extension of L determined by the self-adjoint boundary condition at $x = 0$ has the deficiency indices (0;1) or (1;0) and has no self-adjoint extensions.

Condition 1. In what follows we assume that for the minimal operator L in $H(\alpha, \beta)$ there exist self-adjoint separated boundary conditions on any interval $(\alpha, \beta) \subseteq (a, b)$.

For equations of arbitrary order with bounded coefficients on (a, b), self-adjoint separated boundary conditions have the following form (see [72] – [73]):

$$U_a[y] \equiv \cos A \cdot y^{\vee}(a) - \sin A \cdot y^{\wedge}(a) = 0, \tag{3.5}$$

$$U_b[y] \equiv \cos B \cdot y^{\vee}(b) + \sin B \cdot y^{\wedge}(b) = 0, \tag{3.6}$$

$$-\frac{\pi}{2} I_n \ll A, B \leq \frac{\pi}{2} I_n, \tag{3.7}$$

that is, the point $-\frac{\pi}{2}$ belongs to the resolvent set of the self-adjoint operators A, B in $H^n = H \oplus H \oplus \ldots \oplus H$ (I_n is the identity operator H^n),

$$y^{\vee}(x) = \operatorname{col}\left\{y(x), y'(x), \ldots, y^{(n-1)}(x)\right\},\ y^{\wedge}(x) = \operatorname{col}\left\{y^{[2n-1]}(x), y^{[2n-2]}(x), \ldots, y^{[n]}(x)\right\}, \tag{3.8}$$

$y^{[k]}$ are quasi-derivatives corresponding to the operation $l[y]$ (2.1) and defined as in [72] – [73].

For the problem on an unbounded interval (a, ∞), $a \geq -\infty$, in the absolutely indeterminate case, self-adjoint separated boundary conditions have been in [39]. An existence criterion and a description of all separated self-adjoint boundary conditions for an expression of even order on half-line are contained in [62]. It was demonstrated in [64] that for first order symmetric systems such conditions exist only for Hamiltonian systems.

As for the operator-valued Weyl-Titchmarsh characteristic function, the work [46], among other results, contains necessary and sufficient conditions providing that such characteristic operator-valued function corresponds to a boundary problem with separated (Sturm) boundary conditions, both in the case of finite and infinite intervals.

Let $Y(x, \lambda) \in B(\mathbf{H}, H)$ be the fundamental solution of the problem (3.1), (3.3), with $\mathbf{H}$ being a Hilbert space.

If $a > -\infty$, the fundamental solution $Y(x, \lambda)$ of the problem (3.1), (3.3) can be obtained as a solution of (3.1) with the Cauchi operator data

$$Y^{\wedge}(a, \lambda) = \cos A \cdot K, \qquad Y^{\vee}(a, \lambda) = \sin A \cdot K. \tag{3.9}$$

Here, as in (3.8), the operators $Y^{\wedge}(a, \lambda)$ and $Y^{\vee}(a, \lambda)$ are given by $Y^{\wedge}h = (Yh)^{\wedge}$, $Y^{\vee}h = (Yh)^{\vee}$, $\forall h \in \mathbf{H}$, $K \in B(\mathbf{H}, H^n)$, $K^{-1} \in B(\mathbf{H}, H^n)$. Since the initial data is independent of λ, $Y(x, \lambda)$ is an entire function of λ.

If $\mathbf{H} = H^n$, one can set $K = I_n$. In what follows, we will assume this setting wherever possible.

If $a = -\infty$ or a is a finite singular point, a construction of the fundamental solution of the problem (3.1), (3.3) is not reduced to the Cauchi problem. However, in several cases the fundamental solution can be constructed explicitly (see [53], [90], [75]).

Denote by $\sigma_d(T)$, $\sigma_e(T)$, the discrete and the essential spectra of an operator T, respectively, $\operatorname{nul} T = \dim \operatorname{Ker} T$, $\operatorname{Def} T = \dim \operatorname{Coker} T$.

The existence of fundamental solution $Y(x,\lambda) \in B(\mathbf{H}, H)$ for the equation (2.1) of arbitrary order, both even and odd, with the boundary condition (2.3) and with $\dim \mathbf{H} = \frac{r}{2} \dim H$, $\lambda \in \mathbb{C} \setminus \sigma_e(L_\xi)$, has been established in [81], [41]. The fundamental solution $Y(x,\lambda)$ could be made analytic in λ in a neighbourhood Λ_ρ of the subset $\Lambda := \mathbb{R} \setminus \sigma_e(L_\xi)$ of complex plane. Another facts proved in those papers are self-consistency of the fundamental solution $Y(x,\lambda)$ for $\lambda \in \Lambda$, the relation $\operatorname{nul} Y^\wedge(x,\lambda) = \operatorname{nul} Y^{\wedge *}(x,\lambda)$. Also it is demonstrated for $\lambda \in \mathbb{R} \setminus \sigma_e(L_x^0)$ that $Y(x,\lambda)$ is Fredholm (here L_x^0 is the extension of the operator L_x in $H(a,x)$ by the condition $y^\wedge(x) = 0$, i.e., it is Friedrichs for semi-bounded L_x, $\sigma_e(L_\xi)$ does not depend on ξ, and is void with the regular endpoint $a > -\infty$). In [63] the fundamental solution is constructed for a proper extension (not necessarily self-adjoint) of a symmetric differential operator of an even order with arbitrary deficiency indices.

Consider a minimal with respect to the endpoint $b \le \infty$ differential operator L_b of an even order $r = 2n$ with operator-valued coefficients. Assume it to be lower semi-bounded.

Denote by $N(\lambda)$ the number of eigenvalues $\lambda_k < \lambda$ of the operator $\widetilde{L}$, determined by the problem (3.1), (3.3), (3.4), with each eigenvalue counted according to its multiplicity (the latter is denoted by $\text{æ}(\lambda_k)$). In the case of regular endpoints a or b, the self-adjoint boundary conditions in them have the form (3.5), (3.6). If λ is not an eigenvalue of the problem in question, we set $\text{æ}(\lambda) = 0$. The quantities $N(\lambda)$, $\text{æ}(\lambda)$, λ_k for the Friedrichs extension L_b^F of the operator L_b will be denoted by $N^F(\lambda)$, $\text{æ}^F(\lambda)$, λ_k^F, respectively. With $b < \infty$ one has $L_b^F = L_b^0$, where L_b^0 is the operator corresponding to the problem (2.1), (2.3), $y^\wedge(b) = 0$. Thus the index 0 is about to be used along with the index F.

Theorem 8 ([80] – [85]; see also our monograph [89], Theorem 4.1, p. 91)**.**
When $\lambda < \lambda_e\left(\widetilde{L}\right) := \inf \sigma_e\left(\widetilde{L}\right)$ *one has*

$$N(\lambda) - p \le \sum_{x \in (a,b)} \operatorname{nul} Y^\wedge(x,\lambda) = N^F(\lambda) \le N(\lambda), \tag{3.10}$$

where $p = \operatorname{Def}\left\{\widetilde{L}\,\middle|\, D(L_b^F) \cap D(\widetilde{L})\right\}$. *If* λ *is not an eigenvalue of* L_b, *then with* $\lambda < \lambda_e\left(\widetilde{L}\right)$ *one has*

$$N(\lambda) - \min\{p, \operatorname{Def} L_b - \text{æ}(\lambda)\} \le \sum_{x \in (a,b)} \operatorname{nul} T^\wedge(x,\lambda) = N^F(\lambda) \le N(\lambda) \tag{3.11}$$

(if $b < \infty$ *is a regular endpoint,* $p = \operatorname{rank} \cos B$, *with* B *coming from* (2.32), $\operatorname{Def} L_b = n \cdot \dim H$*). If the following condition is satisfied*

$$\lambda_e\left(L_\xi^F\right) > \lambda_e\left(L_b^F\right) \text{ for } \xi \in (a,b), \tag{3.12}$$

in particular, if the endpoint $a > -\infty$ *is regular, then* (3.10) *is also true for* $\lambda = \lambda_e\left(\widetilde{L}\right)$, $p < \infty$. *(The sum here is taken over all* $x \in (a,b)$ *where* $\operatorname{nul} Y^\wedge(x,\lambda) \ne 0$. *The lower bound in* (3.10), (3.11), *if non-negative, can be attained.)*

Remark 3. *If $a > -\infty$ and for some $\lambda \in \mathbb{R}$ there exists a fundamental solution $Y(x, \lambda)$ of the problem* (3.1), (3.3) *such that the operator $Y^{\wedge}(x, \lambda)$ is Fredholm for all $x \in (a, b)$, when $\lambda \leq \lambda_e\left(L_b^F\right)$.*

Corollary 6. *If $\lambda_k > \lambda_{k-1}$ for L_b^F, and for this k one has*

$$\sum_{x \in (a,b)} \operatorname{nul} Y^{\wedge}(x, \lambda_k) = k - 1, \tag{3.13}$$

and if $æ(\lambda_k) = n \cdot \dim H < \infty$ and $\lambda_k\left(\widetilde{L}\right) > \lambda_{k-1}\left(\widetilde{L}\right)$, $b < \infty$, then (3.13) *is also true for $\widetilde{L}$.*

In particular, for a scalar second order equation one has $Y^{\wedge}(x, \lambda) = Y(x, \lambda)$, and (3.13) contains the classical Sturm Oscillation Theorem, together with its extensions to the case of non-real coefficients (these are allowed at the first order derivative, hence the solution $Y(x, \lambda)$ is non-real), and to the case of infinite interval, since for $r = 2$ (3.13) holds for $-\infty \leq a < b \leq \infty$ and for $\widetilde{L}$.

Definition 1. *A solution $Y(x, \lambda) \in B(\mathbf{H}, H)$ of* (3.1) *with $\lambda \in \mathbb{R}$ is said to be self-consistent if for $\xi \in (a, b)$ there exists a self-adjoint boundary condition*

$$U_{\xi,\lambda}[y] := \cos A_{\xi,\lambda} y^{\vee}(\xi) - \sin A_{\xi,\lambda} y^{\wedge}(\xi) = 0, \tag{3.14}$$

which is satisfied at $x = \xi$ by all the functions of the form $y(x, \lambda) = Y(x, \lambda)h$ (here $A_{\xi,\lambda}$ is a self-adjoint operator in H^n).

An interesting topological interpretation of the Sturm theorems in the finite dimensional case and their link to symplectic geometry was considered by V. I. Arnold [3], but with no claim on novelty of his results. The latter was substantiated in [3] by observing that in a very classical field like Sturm theory, it is hard to keep track of all the predecessors. Some results of this interesting paper is deducible from the author's works [79], [82].

Definition 2. *A Lagrange plane in $H \oplus H$ is said to be vertical if it contains a non-zero vector $\{y, z\}$ with $y = 0$.*

Theorem 8 announced in [80] and proved in [82], [83] (in particular, in the case of second order equation of the form

$$-(P(x)y')' + Q(x)y = \lambda W(x)y, \tag{3.15}$$

the Theorem is proved in [79]), implies

Corollary 7. *Assume that* $\operatorname{rank} \cos A$ *in* (2.2) *is finite, and the restriction L'' of the operator L' given by the additional condition $y^{\wedge}(0) = 0$ is non-negative. Then*

$$\sum_{x \in (0,\infty)} \operatorname{nul} Y^{\wedge}(x, 0) \leq \operatorname{rank} \cos A. \tag{3.16}$$

In fact, the Friedrichs extensions L_1 and L_0 of the operators L' and L'', respectively, are in addition finite dimensional extensions of their intersection Λ, whose deficiency indices are $(\operatorname{rank}\cos A, \operatorname{rank}\cos A)$. In view of this, it follows from Lemma 1.4 of [85], [89] that

$$N_1(0) - \operatorname{rank}\cos A \leq N_0(0).$$

Since the operator L_0 is non-negative, one has $N_0(0) = 0$, whence we deduce (3.16) in view of the equality in (3.10) with $\lambda = 0$.

The Corollary 7 contains the Non-Oscillation Theorem form [3], where it was formulated for a real finite dimensional matrix Sturm-Liouville equation with $W(x) = I$.

4. Theorems of Comparison, Alternation, and the Theorem on Zeros for Equations with Matrix and Operator-Valued Coefficients

Consider the operation $l[y]$ (3.1) of an even order $r = 2n$ and associate it with the form $l_\Delta[y, y]$, the Dirichlet integral over the interval Δ:

$$l_\Delta[y,y] = \int\limits_\Delta \left\{ \sum_{j=0}^{n} \left(p_j(x)y^{(j)}, y^{(j)}\right) + \frac{i}{2}\sum_{j=0}^{n-1}\left[\left(q_{j+1}^*(x)y^{(j+1)}, y^{(j)}\right) - \right.\right.$$
$$\left.\left. - \left(q_{j+1}(x)y^{(j)}, y^{(j+1)}\right)\right]\right\} dx.$$

Theorem 9 ([80] – [85]; see also our monograph [89], Theorem 4.2, p. 95)**.**
Let $-\infty < a < b < \infty$, $Y_1(x)$, $Y_2(x)$ be the fundamental solutions of problems of the form (3.1), (3.5) $l^{(k)}[Y_k] = 0$, $k = 1, 2$, *with $A = A_k$ in* (3.5), *respectively. The operators $L_b^{(k)0}$ are assumed to be lower semi-bounded and*

$$\inf \sigma_e(L_b^{(2)0}) > 0, \qquad W_k(x) \equiv I, \qquad r = 2n, \qquad l^{(1)}_{(a,b)}[y,y] \leq l^{(2)}_{(a,b)}[y,y] \tag{4.1}$$

for the vector-functions that vanish in a neighbourhood of b. If

$$\operatorname{rank}\left\{Y_2^{\vee *}Y_1^{\wedge} - Y_2^{\wedge *}Y_1^{\vee}\right\}_{x=a} = m < \infty,$$

which is equivalent to $\operatorname{rank}\{\sin A_2 \cdot \cos A_1 - \cos A_2 \cdot \sin A_1\} = m < \infty$, *then for any $\beta \in (a, b]$ one has*

$$\sum_{x\in(a,\beta]} \operatorname{nul} Y_1^{\wedge}(x) \geq \sum_{x\in(a,\beta]} \operatorname{nul} Y_2^{\wedge}(x) - m. \tag{4.2}$$

The latter of the conditions (4.1) *is satisfied, in particular, if for $j = 0, 1, \ldots, n$*

$$p_j^{(1)}(x) \leq p_j^{(2)}(x), \qquad q_j^{(1)}(x) = q_j^{(2)}(x), \qquad x \in (a, b). \tag{4.3}$$

If $l^{(1)} = l^{(2)}$, the summing in (4.2) *can be done in $x \in [\alpha, \beta] \subset (a, b]$. If $\dim H < \infty$, one can sum in* (4.2) *in $x \in [\alpha, \beta]$. Also, under the condition* (4.3), *after replacement of m by $n \cdot \dim H$, one has*

$$\sum_{x\in[\alpha,\beta]} \operatorname{nul} Y_1^{\wedge}(x) \geq \sum_{x\in[\alpha,\beta]} \operatorname{nul} Y_2^{\wedge}(x) - n \cdot \dim H. \tag{4.4}$$

Corollary 8. *If $l^{(1)} = l^{(2)}$, then for any $[\alpha,\beta] \subset (a,b]$ one has*

$$\left| \sum_{x\in[\alpha,\beta]} \operatorname{nul} Y_1^{\wedge}(x) - \sum_{x\in[\alpha,\beta]} \operatorname{nul} Y_2^{\wedge}(x) \right| \leq m, \tag{4.5}$$

and if $\sum_{x\in[\alpha,\beta]} \operatorname{nul} Y_1^{\wedge}(x) \geq m+1$, *then* $[\alpha,\beta]$ *contains at least one point where* $\operatorname{nul} Y_2^{\wedge}(x) \geq 1$.

In particular, with $\dim H < \infty$ one has $m \leq n \cdot \dim H$, hence Theorem 9 implies the Heinz-Rellich Comparison and Alternation Theorems [34], [24], sec. 44. The above Alternation Theorem is a kind of refinement of the well-known Sturm Alternation Theorems in the finite dimensional case, which are due to M. Morse [65], G. B. Birkhoff [11], W. Reid [70]. This theorem also implies a comparison theorem, which is close in its formulation to that of V. I. Arnold [3]. Corollary 5 contains the Theorem on Zeros [3] in a refined and generalized form.

We reproduce here the Theorem on Zeros from [3]. Let H be real, $\dim H < \infty$, $\lambda = 0$, the matrices $P(x)$ and $Q(x)$ in (3.15) are real and symmetric, $P(x)$ is positive definite for all $x \in [a,b]$, $-\infty < a < b < \infty$.

The Lagrange plane that evolves according to (3.15), under our assumptions is given by the equation

$$Y^*(x,0)z - Y'^*(x,0)P(x)y = 0, \qquad \{y,z\} \in H \oplus H, \tag{4.6}$$

for some $A = A^*$ in (2.9), and its verticality moments are those $x \in (a,b)$ where $\det Y(x,0) = 0$.

Theorem 10. (Theorem on Zeros [3], see also our monograph [89], Theorem 4.7, p. 115)
On a segment containing $1 + \dim H$ moments of verticality of one Lagrangian plane, any other Lagrangian plane (4.6) (i.e., the plane related to the solution $Y_1(x,0)$ instead of $Y(x,0)$ with A_1 instead of A in (2.9)) becomes vertical at least once. Moreover, the difference between the moments of verticality of two arbitrary Lagrangian planes, evolving under the same system, on any segment of the time axis does not exceed $\dim H$.

Now consider the Comparison Theorem from [3]. The subjects of that work are systems with $2n\times 2n$ matrices $H_1(t)$ and $H_2(t)$, so that both matrices at each t are positive definite on a fixed Lagrange plane α. Those systems are of the form

$$J\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = H_j(t)\begin{pmatrix} u \\ v \end{pmatrix}, \tag{4.7}$$

where $j = 1,2$, the vectors u and v are in the same space H, $\dim H = n$, $J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}$, with I_n and 0_n being the identity and the zero $n \times n$ matrices. Denote by $N(H_j)$ the number of non-transversality instants to α of a Lagrange plane that evolves under (4.7) and the corresponding $j = 1,2$.

Theorem 11. (Comparison Theorem [3], see also our monograph [89], Theorem 4.8, p. 115)
If $H_1(t) \geq H_2(t)$, then $N(H_1) \geq N(H_2) - n$.

We are about to demonstrate that a close version of this Comparison Theorem follows from our Theorem 9 related to an even order r equation in an infinite dimensional space H, as a special case with $\dim H < \infty$ and $r = 2$. Suppose that the canonical systems (4.7) are given in an infinite dimensional 'double' Hilbert space, containing the vectors $u(t)$ and $v(t)$ at every t.

Every Lagrange plane α in $H \oplus H$ can be defined by the equation

$$\cos A \cdot u - \sin A \cdot v = 0, \tag{4.8}$$

where $A = A^*$ is an arbitrary self-adjoint operator in H. Conversely, given $A = A^*$, the equation (4.8) determines a Lagrange plane in $H \oplus H$ with respect to a skew Hermitian scalar product $\left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} s \\ w \end{pmatrix}\right] = (v, s) - (u, w)$ (see [72], [73]).

Let us substitute

$$\begin{pmatrix} y \\ z \end{pmatrix} = \Phi_A \begin{pmatrix} u \\ v \end{pmatrix}, \tag{4.9}$$

where Φ_A is an operator in $H \oplus H$ with block 2×2 matrix

$$\Phi_A = \begin{pmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{pmatrix},$$

and the operator A is chosen according to (4.8) corresponding to a given Lagrange plane α, on which the matrices $H_j(t)$ are positive definite. Then the non-transversality instants to α of a Lagrange plane that evolves under (4.7), are in a correspondence to the instants t, where for a matrix (operator) solution $\begin{pmatrix} Y \\ Z \end{pmatrix}$ of the equation, determining the evolution, one has $\operatorname{nul} Y(t) > 0$. Here $\operatorname{nul} Y(t) := \dim \operatorname{Ker} Y(t)$, so that $\operatorname{nul} Y^*(t) = \operatorname{nul} Y(t)$,

$$Y^*(t)Z(t) - Z^*(t)Y(t) = 0, \qquad Y^*(t)Y(t) + Z^*(t)Z(t) > 0,$$

and the evolution of a Lagrange plane, managed by (4.7), is given by

$$Z^*(t)y - Y^*(t)z = 0, \tag{4.10}$$

where the evolving Lagrange plane at every t is formed by all the vectors $\begin{pmatrix} y \\ z \end{pmatrix}$ that satisfy (4.10).

By virtue of our substitution (4.9), the equation (4.7) transforms to

$$J \frac{d}{dt} \begin{pmatrix} y \\ z \end{pmatrix} = H_j^A \begin{pmatrix} y \\ z \end{pmatrix}, \tag{4.11}$$

where $H_j^A(t) = \Phi_A H_j(t) \Phi_A^*$, and the Lagrange plane α is now given by $y = 0$. Due to the positivity condition on α for the forms corresponding to the operators $H_j(t)$, we have also the positivity in H of the operators $H_j^A(t)$:

$$H_j^A(t) = \begin{pmatrix} H_{11}^{jA}(t) & H_{12}^{jA}(t) \\ H_{21}^{jA}(t) & H_{22}^{jA}(t) \end{pmatrix}. \tag{4.12}$$

Here $(H_j^A(t))^* = H_j^A(t)$. In view of the above observations, the equations (4.11) with $j = 1, 2$ are equivalent to the Sturm-Liouville systems of the form

$$l^j[Y] = -(P_j Y' + R_j Y)' + R_j * Y' + Q_j Y = 0, \tag{4.13}$$

where $P_j = P_j^* = (H_{22}^{jA})^{-1} > 0$ and $Q_j = Q_j^*$. Conversely, the system (4.11) – (4.12) corresponding to (4.13), is derived when one sets there $z = P_j y' + R_j y$ and

$$H_j^A(t) = \begin{pmatrix} -Q_j + R_j^* P_j^{-1} & -R_j^* P_j^{-1} \\ -P_j^{-1} R_j & P_j^{-1} \end{pmatrix}. \tag{4.14}$$

The quantities $P_j = \left(H_{22}^{jA}\right)^{-1}$, Q_j, R_j are in one-to-one correspondence with $H_{11}^{jA}, H_{22}^{jA} > 0$, $H_{12}^{jA} = \left(H_{21}^{jA}\right)^*$, and these are also linked to the operator blocks of the matrices $H_j(t)$ in view of the relation $H_j^A(t) = \Phi_A H_j(t) \Phi_A^*$.

Now observe that by virtue of (4.14), the quadratic form $H_j^A(t)$ can be written in the form

$$\left(H_j^A(t)\begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix}\right) = (P_j^{-1}(z - R_j y), z - R_j y) - (Q_j y, y). \tag{4.15}$$

On the other hand, the Dirichlet integral for the equation (4.13) is an integral of the quadratic form

$$D_j \left[\begin{pmatrix} y \\ y' \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix}\right] := (P_j(t) y', y') + (R_j y, y') + (R_j^* y', y) + (Q_j y, y), \tag{4.16}$$

which has a form of 2×2 block matrix

$$D_j(t) = \begin{pmatrix} Q_j & R_j^* \\ R_j & P_j \end{pmatrix}. \tag{4.17}$$

The assumptions of Comparison Theorem [3] require the inequality

$$H_1^A(t) \geq H_2^A(t) \tag{4.18}$$

for the matrices (4.14) as well as for the forms (4.15). If one requires additionally that $R_1(t) = R_2(t)$, the above inequalities become equivalent to

$$D_1(t) \leq D_2(t), \tag{4.19}$$

which is sufficient to make the condition (4.1) of our Theorem 9 satisfied (with $r = 2$), and, consequently, to make true our claim (4.2). Thus, under the additional assumption $R_1(t) = R_2(t)$, the Comparison Theorem of [3] follows from our Theorem 9 as the special case with $r = 2$, $\dim H < \infty$, because in this case one has $m < \dim H$. However, if one withdraws the requirement $R_1 = R_2$, the inequalities (4.18) and (4.19) fail to be equivalent, so the Corollary 8 of our Theorem 9 does not cover the Comparison Theorem of [3] and is not covered by the latter. As for the general case of our Theorem 9 with $r > 2$, it covers the Hamiltonian systems with matrices, which may happen to be strictly positive on no Lagrange plane, unlike the assumptions of [3].

5. Estimating the Number of Eigenvalues in a Gap of the Essential Spectrum

Consider the self-adjoint differential equation (3.1) of an arbitrary order $r \geq 1$, either even or odd, with operator-valued coefficients from $B(H)$. As in Sec. 3 above, we assume the Condition 1 to be satisfied, hence either $r \cdot \dim H = \infty$ or $r \cdot \dim H$ is an even number (see [85], [89]).

Without assuming semi-boundedness for the minimal operator $L \subseteq \widetilde{L}$, where $\widetilde{L}$ is some self-adjoint extension of L, we suppose that C^{2r} is the smoothness class of the coefficients of $l_W[y]$ (3.2). Then the operation

$$l_W^2[y] = l_W[l_W[y]] \tag{5.1}$$

can be treated as an ordinary differential one.

Denote by M the closed minimal differential operator generated in $H(a,b)$ by the operation (5.1). Since the operators M and $(\widetilde{L})^2$ coincide on C_0^∞, they also coincide on the domain of M. Therefore $M \subseteq (\widetilde{L})^2$, hence $(\widetilde{L})^2$ is a self-adjoint extension $\widetilde{M}$ of the positive symmetric operator M.

Denote by M_b the restriction of the operator $\widetilde{M}$ by the minimality condition with respect to b. Let also $p = \operatorname{Def}\left\{\widetilde{M}|D(\widetilde{M}) \bigcap D(M_b^F)\right\}$, $N(\lambda,\mu)$ be the number of eigenvalues $\lambda_k \in (\lambda,\mu)$ of the operator $\widetilde{L}$, counting their multiplicities æ(λ_k).

Let $Y(x,\lambda) \in B(\mathbf{H}, H)$, be the fundamental solution of the problem (3.1), (3.5), where $\mathbf{H}$ is a Hilbert space. Set

$$Y(x,\lambda,\mu) = \{Y(x,\lambda); Y(x,\mu)\} : \mathbf{H}^2 \to H,$$

$$Y^{\triangle}(x,\lambda,\mu) = \operatorname{col}\left\{Y(x,\lambda,\mu); Y'(x,\lambda,\mu); \ldots; Y^{(r-1)}(x,\lambda,\mu)\right\}. \tag{5.2}$$

Theorem 12 ([80], [85], [89], see our monograph [89], Theorem 4.6, p. 108)**.**
Let (α,β) be a gap in the essential spectrum of the operator $\widetilde{L}$, $\alpha < \lambda < \mu < \beta$. Then one has

$$N(\lambda,\mu) - p \leq \sum_{x\in(a,b)} \operatorname{nul} Y^{\triangle}(x,\lambda,\mu) \leq N(\lambda,\mu), \tag{5.3}$$

so that $\operatorname{nul} Y^{\triangle}(x,\lambda,\mu) = \operatorname{nul} Y^{\triangle *}(x,\lambda,\mu)$. *If* $æ_{L_b}(\lambda) = æ_{L_b}(\mu) = 0$, *where* $L_b \subseteq \widetilde{L}$ *is a minimal operator with respect to the endpoint* b, *then one can replace* p *in* (5.3) *with*

$$\min\{p, \operatorname{Def} M_b - æ(\lambda) - æ(\mu)\}.$$

If $a > -\infty$, *then the Theorem is also true for* $\lambda = \alpha$, $\mu = \beta$.

Remark 4. *If* $b < \infty$, *then* $0 < p = \frac{r}{2} \dim H$, $\operatorname{Def} M_b = r \cdot \dim H$. *(It is clear from the inequality* $p > 0$ *that with* $b < \infty$ *the Friedrichs extension* M_b^F *can not be represented as a square of self-adjoint differential operator* $\widetilde{L}$, *that is,* $(M_b^F)^{1/2}$ *is not a differential operator, although* $M_b^F \geq 0$*).*

Another approach to studying the discrete levels in spectral gaps, based on the phase function method, has been developed in [76], Theorem 4, for the scalar Schrödinger equation. A result similar to Theorem 12 for the scalar Sturm-Liouville equation is contained in the paper by F. Gesztesy, B. Simon, and G. Teschl [23].

Remark 5. *Consider the self-adjoint operator* $\widetilde{L}$, *generated in* $H(-\infty,\infty)$ *by the symmetric first order system*

$$l[y] = Jy' + H(x)y = \lambda y, \tag{5.4}$$

where $J^* = J^{-1} = -J$, $H(x)$ *is a uniformly continuously differentiable operator-valued function from* $B(H^2)$. *In the case when*

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad H(x) = \begin{pmatrix} P(x) & Q(x) \\ Q(x) & -P(x) \end{pmatrix}, \qquad P = P^* \in B(H), \quad Q = Q^* \in B(H),$$

we get a self-adjoint Dirac operator on the axis.

Suppose that the essential spectrum of the operator $\widetilde{L}$ does not cover the entire axis, and let (α,β) be a gap in its essential spectrum. With $\lambda \in (\alpha,\beta)$, denote by $Y(x,\lambda) \in B(H,H^2)$ the fundamental solution of (5.4) and set

$$Y(x,\lambda,\mu) = \{Y(x,\lambda); Y(x,\mu)\} : \mathbf{H}^2 \to H^2.$$

Denote by $N(\lambda,\mu)$ the number of eigenvalues $\lambda_k \in (\lambda,\mu)$ of the operator $\widetilde{L}$, counting their multiplicities $æ(\lambda_k)$.

Theorem 13. [§] *Let* (α,β) *be a gap in the essential spectrum of the operator* $\widetilde{L}$. *With* $\alpha < \lambda < \mu < \beta$ *one has*

$$\sum_{x \in (-\infty,\infty)} \operatorname{nul} Y(x,\lambda,\mu) = N(\lambda,\mu).$$

[§] In the finite dimensional case this Theorem has been proved in [38].

6. The Sturm Type Oscillation Theorems for Equations with Block-Triangular Matrix Coefficients

Consider the differential equation with matrix coefficients

$$l[y] = -(P(x)y')' + \frac{i}{2}((Q(x)y)' + Q(x)y') + V(x)y = \lambda W(x)y, \tag{6.1}$$

so that the coefficients $P(x)$, $Q(x)$, together with their derivatives, as well as the coefficients $V(x)$, $W(x)$, depend continuously on $x \in [0, \infty)$.

Suppose that the coefficients $P(x)$, $Q(x)$, $V(x)$ of the equation (6.1) have a block-triangular form, in particular, the potential $V(x)$ is of the form

$$V(x) = \begin{pmatrix} V_{11}(x) & V_{12}(x) & \dots & V_{1r}(x) \\ 0 & V_{22}(x) & \dots & V_{2r}(x) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & V_{rr}(x) \end{pmatrix} \tag{6.2}$$

and the weight $W(x)$ is a block-diagonal matrix.

The diagonal blocks $P_{kk}(x)$, $V_{kk}(x)$, $Q_{kk}(x)$, $W_{kk}(x)$, $k = \overline{1,r}$, are Hermitian $m_k \times m_k$ matrices with $m_k \geq 1$ (in particular, with $m_k = 1$ one has real scalar functions). Suppose that $\sum\limits_{k=1}^{r} m_k = m$.

Denote by H_m the m-dimensional Hilbert space. A vector $h \in H_m$ will be written in the form $h = \text{col}(h_1, h_2, \dots, h_r)$, with h_k, $k = \overline{1,r}$, being a vector from H_{m_k}.

6.1. The Problem on a Finite Interval

Suppose that at the ends of the interval $(0, b)$, $b < \infty$, one has the boundary conditions:

$$A \cdot y'(0) - B \cdot y(0) = 0, \tag{6.3}$$

$$C \cdot y'(b) - D \cdot y(b) = 0, \tag{6.4}$$

where A and B, C and D are commuting block-triangular matrices of the same structure as the coefficients of the differential equation, subject to the conditions

$$\det\left(A^2 + B^2\right) \neq 0, \qquad \det\left(C^2 + D^2\right) \neq 0. \tag{6.5}$$

Denote by $Y(x, \lambda)$ the matrix solution of the equation (6.1), satisfying the initial conditions

$$Y(0, \lambda) = A, \qquad Y'(0, \lambda) = B.$$

This solution also has a block-triangular structure

$$Y(x, \lambda) = \begin{pmatrix} Y_{11}(x,\lambda) & Y_{12}(x,\lambda) & \dots & Y_{1r}(x,\lambda) \\ 0 & Y_{22}(x,\lambda) & \dots & Y_{2r}(x,\lambda) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & Y_{rr}(x,\lambda) \end{pmatrix},$$

with $Y_{kk}(x,\lambda)$, $k=\overline{1,r}$, being $m_k\times m_k$ matrices, $m_k\geq 1$.

Consider the system

$$l_k[y_k] = -\left(P_{kk}(x)y_k'\right)'+\frac{i}{2}\left((Q_{kk}(x)y_k)'+Q_{kk}(x)y_k'\right)+V_{kk}(x)y_k=\lambda W_{kk}(x)y_k, \qquad k=\overline{1,r}, \tag{6.6}$$

with the boundary conditions

$$A_{kk}\cdot y_k'(0)-B_{kk}\cdot y_k(0)=0, \tag{6.7}$$

where $y_k(x)$ is a vector function with values in H_{m_k}.

Denote by L the differential operator generated by the differential expression $l_W[y]=W^{-1}(x)l[y]$ and the boundary conditions (6.3) and (6.4). Let L_k be the self-adjoint operator generated by the differential expression $l_{k,w}[z]=W_{kk}^{-1}(x)l_k[z]$ and the boundary conditions (6.7) and

$$C_{kk}\cdot y_k'(b)-D_{kk}\cdot y_k(b)=0, \tag{6.8}$$

where A_{kk}, B_{kk}, C_{kk}, D_{kk} are Hermitian $m_k\times m_k$ matrices that satisfy conditions similar to (4.19).

If the matrix C in the boundary condition (6.4) has the form

$$C=\begin{pmatrix} 0 & C_{12} & \dots & C_{1r}\\ 0 & 0 & \dots & C_{2r}\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 0\end{pmatrix}, \tag{6.9}$$

then we denote the operator L by L^0. If the boundary condition (6.8) acquires the form

$$y_k(b)=0, \tag{6.10}$$

then we denote the operator L_k by L_k^0.

Denote by $\sigma_k=\bigcup_s\{\lambda_{sk}\}_s$, $k=\overline{1,r}$, the set of eigenvalues for the self-adjoint operator L_k, by $N_k(\lambda)$ the set of eigenvalues $\lambda_{sk}<\lambda$ with a fixed k, counting their multiplicities. The quantities λ_{sk}, $N_k(\lambda)$ for the operator L_k^0 are denoted by λ_{sk}^0, $N_k^0(\lambda)$, respectively.

Lemma 1 ([42]).
The spectrum of L is discrete, real, and it coincides to the union of spectra of the self-adjoint operators L_k, i.e.,

$$\sigma(L)=\sigma_d(L)=\bigcup_{k=1}^{r}\sigma_k. \tag{6.11}$$

Let us enumerate the eigenvalues of L^0 in an increasing order

$$\lambda_1^0\leq\lambda_2^0\leq\dots\leq\lambda_n^0\leq\dots$$

and denote by $N_a^0(\lambda)$ the number of eigenvalues $\lambda_n^0<\lambda$ of L^0, counting their algebraic multiplicities.

Suppose $Y_{kk}(x,\lambda)$ is a matrix solution of the differential equation (6.6) with Hermitian coefficients. Although this matrix, in general, can fail to be Hermitian, it was proved in [79], [80], [85], [89] that

$$\operatorname{nul} Y_{kk}(x,\lambda) = \operatorname{Def} Y_{kk}(x,\lambda), \tag{6.12}$$

where, given an arbitrary matrix T, we use the conventional notation $\operatorname{nul} T = \dim \operatorname{Ker} T$, $\operatorname{Def} T = \dim \operatorname{Coker} T$.

With $m \geq 1$, denote by $\operatorname{nul}_a Y(x,\lambda)$ the algebraic multiplicity of zero as an eigenvalue of the matrix $Y(x,\lambda)$ under fixed x and λ. In particular, with $m = 1$ we have $\operatorname{nul}_a Y(x,\lambda) = 1$ if x is a root of the scalar equation $Y(x,\lambda) = 0$, and $\operatorname{nul}_a Y(x,\lambda) = 0$ if x is not a root of this equation.

Theorem 14 ([42]).
Suppose that the operator L^0 is generated by the differential expression $l_W[y]$ with matrix block-triangular coefficients, which satisfy the conditions listed above, and by the boundary conditions (6.3), (6.4), *with the matrix C of the form* (6.9). *Assume that the blocks $P_{kk}(x)$ of the coefficient at highest derivative $P(x)$ and the blocks $W_{kk}(x)$ of the matrix weight $W(x)$ are simultaneously either Hermitian positive or negative at every $x \in [0,b]$, and the blocks $V_{kk}(x)$ are Hermitian. Then with $\lambda \in \mathbb{R}$ one has*

$$\sum_{x \in (0,b)} \operatorname{nul}_a Y(x,\lambda) = N_a^0(\lambda) \tag{6.13}$$

(here the sum is through all $x \in (0,b)$ where $\operatorname{nul}_a Y(x,\lambda) \neq 0$*).*

Corollary 9. *With $\lambda \in \mathbb{R}$ one has*

$$\sum_{k=1}^{r} \sum_{x \in (0,b)} \operatorname{nul} Y_{kk}(x,\lambda) = N_a^0(\lambda).$$

Theorem 15 ([42]).
For the problem (6.1), (6.3), (6.4) *with $\lambda \in \mathbb{R}$ one has*

$$N_a(\lambda) - \sum_{k=1}^{r} \min\{\operatorname{rank} C_{kk}, m_k - æ_k(\lambda)\} \leq \sum_{x \in (0;b)} \operatorname{nul}_a Y(x,\lambda) \leq N_a(\lambda).$$

6.2. The problem on half-line. Denote by L' the minimal with respect to $x = \infty$ differential operator, generated in $L_2(H_m,(0,\infty),W(x)dx)$ (here H_m is an m-dimensional Hilbert space) by the differential expression $l_W[y]$ and the boundary condition at zero (6.3), and by L'_k, $k = \overline{1,r}$, the symmetric operator, generated in $L_2(H_{m_k},(0,\infty),W_{kk}(x)dx)$ by the differential expression $l_{k,w}[z]$ and the boundary condition (6.7). Assume that every symmetric operator L'_k is lower semi-bounded. (If $P(x) = I_m$, then the minimal symmetric semi-bounded operators L'_k, $k = \overline{1,r}$, are essentially self-adjoint (see [9]), and their self-adjoint extensions are derived by closing the domain of minimal operator).

Suppose that for the symmetric operator L'_k one has a self-adjoint boundary condition at infinity

$$U_k[y_k] = 0, \qquad k = \overline{1, r}, \tag{6.14}$$

where U_k is a linear map from $L_2(H_{m_k}, (0, \infty), W_{kk}(x)dx)$ to H_{m_k} such that $U_k[y_k] = U_k[z_k]$ if $y_k(x) = z_k(x)$ for x big enough.

Suppose that for L' one has some boundary condition at infinity

$$U[y] = 0 \tag{6.15}$$

such that

$$(U[y])_1 = U_1[y_1, y_2, \ldots, y_r]$$

$$(U[y])_2 = U_2[y_2, y_3, \ldots, y_r]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(U[y])_{r-1} = U_{r-1}[y_{r-1}, y_r]$$

$$(U[y])_r = U_r[y_r].$$

Denote by L an extension of the operator L' determined by the boundary condition (6.15) and such that it satisfies the conditions

$$\begin{gathered} U_1[y_1, 0, \ldots, 0] = U_1[y_1] \\ U_2[y_2, 0, \ldots, 0] = U_2[y_2] \\ U_{r-1}[y_{r-1}, 0] = U_{r-1}[y_{r-1}]. \end{gathered} \tag{6.16}$$

If the conditions (6.14) determine the Friedrichs extension L_k^0 of the semi-bounded symmetric operator L'_k, then the corresponding extension of the operator L' is denoted by L^0. It is demonstrated in [74], [75] that the spectral function $\rho(\lambda)$ of L_k^0 is produced by passage to a limit as $b \to \infty$ from the spectral function $\rho_b(\lambda)$ of the problem (6.6), (6.7), (6.10) on $[0, b]$.

Denote by $\sigma_k = \bigcup\limits_s \{\lambda_{sk}\}_s$, $k = \overline{1, r}$, the set of eigenvalues $\lambda_{sk} < \lambda_e(L_k)$ with fixed k of the self-adjoint operator L_k, and by $N_k(\lambda)$ the number of eigenvalues $\lambda_{sk} < \lambda < \lambda_e(L_k)$ counting their multiplicities. The quantities λ_{sk}, $N_k(\lambda)$ for the operator L_k^0 are denoted by λ_{sk}^0, $N_k^0(\lambda)$, respectively.

Lemma 2 ([43]).
The discrete spectrum of L is real and is contained in the union of discrete spectra of L_k, i.e.,

$$\sigma_d(L) \subseteq \bigcup_{k=1}^{r} \sigma_k. \tag{6.17}$$

It was noted above that with $b < \infty$, the smoothness of coefficients implies Lemma 1, i.e., (6.11) holds. However, this condition can fail for the problem on half-line. Sufficient conditions that provide the coincidence of the spectrum $\sigma_d(L)$ of L with the union of the discrete spectra of L_k, $k = \overline{1,r}$, is presented in [13]. In that work, the validity of (6.11) is proved for the differential equation of the form

$$-y'' + V(x)y = \lambda y, \tag{6.18}$$

with triangular matrix $m \times m$ potential $V(x)$, which has a bounded first moment, i.e.,

$$\int_0^\infty x \cdot |V(x)| dx < \infty.$$

Another sufficient condition for the equation (6.18), whose potential grows on its diagonal, has been established by the authors and is presented in [43].

Consider the equation with a block-triangular matrix potential

$$l[\overline{y}] = -\overline{y}'' + V(x)\overline{y} = \lambda \overline{y}, \qquad 0 \le x < \infty, \tag{6.19}$$

where

$$V(x) = w(x) \cdot I_m + U(x), \qquad U(x) = \begin{pmatrix} U_{11}(x) & U_{12}(x) & \dots & U_{1r}(x) \\ 0 & U_{22}(x) & \dots & U_{2r}(x) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & U_{rr}(x) \end{pmatrix}, \tag{6.20}$$

$w(x)$ is a real scalar function, $0 < w(x) \to \infty$ monotonically as $x \to \infty$, and has a monotonic absolutely continuous derivative. The diagonal blocks $U_{kk}(x)$, $k = \overline{1,r}$, are Hermitian $m_k \times m_k$ matrices, $m_k \ge 1$ (in particular, with $m_k = 1$ these are real scalar functions). Let $\sum\limits_{k=1}^{r} m_k = m$, and I_m being the identity $m \times m$ matrix.

In the case when

$$w(x) \ge C x^{2\alpha}, \qquad C > 0, \quad \alpha > 1, \tag{6.21}$$

we assume that the coefficients of equation (6.19) satisfy the conditions

$$\int_a^\infty |U(t)| \cdot w^{-\frac{1}{2}}(t) dt < \infty, \tag{6.22}$$

$$\int_a^\infty w'^2(t) \cdot w^{-\frac{5}{2}}(t) dt < \infty, \qquad \int_a^\infty w''(t) \cdot w^{-\frac{3}{2}}(t) dt < \infty \qquad a > 0. \tag{6.23}$$

(A studying the spectral properties of the one dimensional Schrödinger operator with a polynomial potential is the subject of [17]).

In the case when $w(x) = x^{2\alpha}$, $0 < \alpha \le 1$, let us assume that the coefficients of equation (6.19) satisfy the condition

$$\int_a^\infty |U(t)| \cdot t^{-\alpha} dt < \infty, \qquad a > 0. \tag{6.24}$$

The following Theorem has been proved in [43].

Theorem 16 ([43]). [¶]
Consider the equation (6.19)*. If either conditions* (6.21)*,* (6.22)*,* (6.23) *with* $\alpha > 1$ *or the condition* (6.24) *with* $0 < \alpha \le 1$ *are satisfied, then the spectrum of* L *is discrete, real and coincides with the union of spectra of the self-adjoint operators* L_k*,* $k = \overline{1, r}$*, i.e.,* $\sigma(L) = \bigcup\limits_{k=1}^{r} \sigma(L_k)$*.*

Condition 2. In what follows we assume that the coefficients of the differential equation (6.19) for the problem on half-line are such that the discrete spectrum of L coincides with the union of discrete spectra of the self-adjoint operators L_k, $k = \overline{1, r}$, i.e., that (6.11) holds.

Let us enumerate the eigenvalues of L^0 in the increasing order

$$\lambda_1^0 \le \lambda_2^0 \le \ldots \le \lambda_n^0 \le \ldots < \lambda_e(L^0).$$

Denote by $N_a^0(\lambda)$ the number of eigenvalues $\lambda_n^0 < \lambda < \lambda_e(L^0)$ of L^0, counting their algebraic multiplicities.

Theorem 17 ([42]).
Assume that the Condition 2 is satisfied. Suppose that the operator L^0 *is generated by the differential expression* $l_W[y]$ *with matrix block-triangular coefficients, the boundary condition at zero* (6.3)*, and such boundary conditions at infinity, that for semi-bounded symmetric operators* L_k' *one gets this way the Friedrichs extensions. Suppose that the diagonal blocks* $P_{kk}(x)$ *of the coefficient at the highest derivative* $P(x)$ *and the diagonal blocks* $W_{kk}(x)$ *of the weight* $W(x)$ *are either both Hermitian positive or both negative at every* $x \in [0; \infty)$*, and the blocks* $V_{kk}(x)$ *are Hermitian. Then with* $\lambda < \lambda_e(L^0)$ *one has*

$$\sum_{x \in (0,\infty)} \operatorname{nul}_a Y(x, \lambda) = N_a^0(\lambda).$$

(here the sum is taken over all $x \in (0, \infty)$ *where* $\operatorname{nul}_a Y(x, \lambda) \ne 0$*).*

Let L_k be an arbitrary self-adjoint extension of the semi-bounded symmetric operator L_k' in $L_2(H_{m_k}, (0, \infty), W_{kk}(x)dx)$, determined by the condition at infinity (6.14). A description of self-adjoint extensions for symmetric differential operators of an arbitrary

[¶] Lemma 3 in [42] is a special case of this Theorem. It was promised in [42] to present a proof in a subsequent paper, and this has been done in [43].

order (both even and odd) with operator coefficients on an infinite interval (axis, half-line) in the absolutely indefinite case is obtained in [39] (see also [85], [89]). In the case of intermediate indices, these problems have been investigated in [15], [32], [56], [62], [64].

Denote by L the extension of the operator L' by the boundary condition at infinity (6.15), with the conditions (6.16) being satisfied.

Theorem 18 ([42]).
Suppose the Condition 2 is satisfied. With $\lambda < \lambda_e(L)$ one has for the operator L

$$N_a(\lambda) - \sum_{k=1}^{r} p_k \le \sum_{x\in(0,\infty)} \operatorname{nul}_a Y(x,\lambda) = N_a^0(\lambda) \le N_a(\lambda),$$

where $p_k = \operatorname{Def}\left\{L_k \,\middle|\, D\left(L_k^0\right) \cap D\left(L_k\right)\right\}$. If λ is not an eigenvalue of L', then with $\lambda < \lambda_e(L)$ one has

$$N_a(\lambda) - \sum_{k=1}^{r} \min\left\{p_k, d_k - æ_k(\lambda)\right\} \le \sum_{x\in(0,\infty)} \operatorname{nul}_a Y(x,\lambda) = N_a^0(\lambda) \le N_a(\lambda),$$

$d_k = \operatorname{Def} L_k'$, $æ_k(\lambda)$ *the multiplicity of λ as an eigenvalue of the self-adjoint operator L_k.*

Remark 6. *With a regular endpoint $b < \infty$, one has*

$$p_k = \operatorname{rank} C_{kk}, \qquad d_k = m_k.$$

7. The Green Function, Resolvent, Parseval Equality for a Differential Operator with Block-Triangular Matrix Coefficients

Consider the equation (6.19) with a block-triangular matrix potential. With $\alpha > 1$ and under the conditions (6.21), (6.22), (6.23), we define the functions $\gamma_0(x,\lambda)$ and $\gamma_\infty(x,\lambda)$ by setting

$$\gamma_0(x,\lambda) = \frac{1}{\sqrt[4]{4w(x)}} \cdot \exp\left(-\int_0^x \sqrt{w(u)}du\right),$$

$$\gamma_\infty(x,\lambda) = \frac{1}{\sqrt[4]{4w(x)}} \cdot \exp\left(\int_0^x \sqrt{w(u)}du\right),$$

and with $0 < \alpha \le 1$, $w(x) = x^{2\alpha}$ by

$$\gamma_0(x,\lambda) = \frac{1}{\sqrt[4]{4\left(x^{2\alpha} - \lambda\right)}} \cdot \exp\left(-\int_a^x \sqrt{u^{2\alpha} - \lambda}du\right),$$

$$\gamma_\infty(x,\lambda) = \frac{1}{\sqrt[4]{4\left(x^{2\alpha}-\lambda\right)}} \cdot \exp\left(\int\limits_a^x \sqrt{u^{2\alpha}-\lambda}du\right).$$

The asymptotics of these functions as $x \to \infty$ has been established in [43].

Theorem 19 ([43])**.** *With $\alpha > 1$ and under the conditions (6.21), (6.22), (6.23), as well as with $0 < \alpha \le 1$ under the condition (6.24), the equation (6.19) has a unique decreasing at infinity matrix solution $\Phi(x,\lambda)$, which satisfies the condition*

$$\lim_{x\to\infty} \frac{\Phi(x,\lambda)}{\gamma_0(x,\lambda)} = I_m$$

and is such that

$$\lim_{x\to\infty} \frac{\Phi'(x,\lambda)}{\gamma_0'(x,\lambda)} = I_m,$$

and also a unique increasing at infinity matrix solution $\Psi(x,\lambda)$ which satisfies the condition

$$\lim_{x\to\infty} \frac{\Psi(x,\lambda)}{\gamma_\infty(x,\lambda)} = I_m,$$

and is such that

$$\lim_{x\to\infty} \frac{\Psi'(x,\lambda)}{\gamma_\infty'(x,\lambda)} = I_m.$$

The asymptotics of $\Phi(x,\lambda)$ and $\Psi(x,\lambda)$ were used in [43] to prove Theorem 16 formulated above.

Suppose we are given a boundary condition at $x = 0$

$$B \cdot y'(0) - C \cdot y(0) = 0, \tag{7.1}$$

where B and C are commuting block-triangular matrices of the same structure as coefficients of the differential equation, subject to the condition

$$\det\left(B^2 + C^2\right) = \prod_{k=1}^{r} \det\left(B_{kk}^2 + C_{kk}^2\right) \neq 0.$$

Lemma 3 ([43])**.** *The boundary condition (7.1) can be written down in the equivalent form*

$$\cos A \cdot \overline{y}'(0) - \sin A \cdot \overline{y}(0) = 0, \tag{7.2}$$

where A is a block-triangular matrix of the same structure as B and C.

Along with the problem (6.19), (7.2), let us consider the split system

$$l_k\left[\overline{y}_k\right] = -\overline{y}_k'' + \left(w(x)I_{m_k} + U_{kk}(x)\right)\overline{y}_k = \lambda \overline{y}_k, \qquad k = \overline{1,r}, \tag{7.3}$$

with the boundary conditions

$$\cos A_{kk} \cdot \overline{y}_k'(0) - \sin A_{kk} \cdot \overline{y}_k(0) = 0, \qquad k = \overline{1,r}. \tag{7.4}$$

Denote by L_0 the minimal differential operator generated by the differential expression $l\,[\overline{y}]$ (6.19) and the boundary condition (7.2). Also, denote by L_k, $k = \overline{1,r}$, the minimal symmetric operators in $L_2\left(H_{m_k}, (0,\infty)\right)$, generated by the differential expressions $l_k\,[\overline{y}_k]$ (7.3) and the boundary conditions (7.4). In view of the assumptions on coefficients for each symmetric operator L_k, $k = \overline{1,r}$, we encounter the case of limit point at infinity. It follows that the self-adjoint extensions $\widetilde{L}_k$ of those operators are given by closures of L_k. The operators $\widetilde{L}_k$ are semi-bounded, and their spectra are discrete.

Denote by L the extension of the operator L_0 determined by requiring that the functions of the domain of L are in $L_2\left(H_{m_k}, (0,\infty)\right)$. Enumerate the eigenvalues of L in the increasing order, counting their multiplicities:

$$\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \le \ldots$$

Together with the equation (6.19), we consider the left equation

$$\widetilde{l}\,[\widetilde{y}] = -\widetilde{y}'' + \widetilde{y}V(x) = \lambda \widetilde{y}, \qquad \widetilde{y} = (\widetilde{y}_1 \;\; \widetilde{y}_2 \ldots \widetilde{y}_r)\,. \tag{7.5}$$

Let us denote by $\widetilde{\Phi}(x,\lambda)$ (respectively, $\widetilde{\Psi}(x,\lambda)$) the increasing (respectively, the decreasing) at infinity solution of (7.5).

Denote by $Y(x,\lambda)$ and $\widetilde{Y}(x,\lambda)$ the solutions of (6.19) and (7.5), respectively, which satisfy the initial conditions

$$Y(0,\lambda) = \cos A, \quad Y'(0,\lambda) = \sin A, \quad \widetilde{Y}(0,\lambda) = \cos A, \quad \widetilde{Y}'(0,\lambda) = \sin A, \qquad \lambda \in \mathbb{C}.$$

Set

$$G(x,t,\lambda) = \begin{cases} Y(x,\lambda)\left(W\left(\widetilde{\Phi}, Y\right)\right)^{-1}\widetilde{\Phi}(t,\lambda), & 0 \le x \le t \\ -\Phi(x,\lambda)\left(W\left(\widetilde{Y}, \Phi\right)\right)^{-1}\widetilde{Y}(t,\lambda), & x \ge t \end{cases}. \tag{7.6}$$

Theorem 20 ([44])**.** *The matrix function $G(x,t,\lambda)$ is the Green function of the differential operator L, i.e.,*

1. *The function is continuous at all $x, t \in [0,\infty)$.*
2. *At every fixed t, the function $G(x,t,\lambda)$ has a continuous derivative in x on each of the intervals $[0,t)$ and (t,∞); also at $x = t$ it has a jump*
$$G_x'(x+0,x,\lambda) - G_x'(x-0,x,\lambda) = -I_m.$$
3. *With t fixed, the function $G(x,t,\lambda)$ of the variable x is a matrix solution of (6.19) on each of the intervals $[0,t)$ and (t,∞); it satisfies the boundary condition (7.2). Also, with x fixed, the function $G(x,t,\lambda)$ of the variable t is a matrix solution of (7.5) on each of the intervals $[0,x)$ and (x,∞); it satisfies the boundary condition $\widetilde{y}'(0) \cdot \cos A - \widetilde{y}(0) \cdot \sin A = 0$.*

In view of its definition (7.6), the function $G(x,t,\lambda)$ is meromorphic in λ, whose poles are just eigenvalues of L.

Consider the operator R_λ in $L_2\left(H_m,(0,\infty)\right)$ given by

$$\left(R_\lambda \overline{f}\right)(x) = \int\limits_0^\infty G(x,t,\lambda)\overline{f}(t)dt =$$

$$= -\int\limits_0^x \Phi(x,\lambda)\left(W\left(\widetilde{Y},\Phi\right)\right)^{-1}\widetilde{Y}(t,\lambda)\overline{f}(t)dt + \int\limits_x^\infty Y(x,\lambda)\left(W\left(\widetilde{\Phi},Y\right)\right)^{-1}\widetilde{\Phi}(t,\lambda)\overline{f}(t)dt.$$

Theorem 21 ([44])**.** *R_λ is a resolvent of the operator L.*

As above, denote by $Y(x,\lambda)$ the matrix solution of (6.19), which satisfies the initial conditions $Y(0,\lambda)=\cos A$, $Y'(0,\lambda)=\sin A$, and by $Z(x,\lambda)$ the matrix solution of (6.19), which satisfies the initial conditions $Z(0,\lambda)=-\sin A$, $Z'(0,\lambda)=\cos A$. Then the solutions $\Phi(x,\lambda)$, $\widetilde{\Phi}(x,\lambda)$ admit representations in the form

$$\Phi(x,\lambda) = Z(x,\lambda)W\left\{\widetilde{Y},\Phi\right\} - Y(x,\lambda)W\left\{\widetilde{Z},\Phi\right\},$$
$$\widetilde{\Phi}(x,\lambda) = W\left\{\widetilde{\Phi},Z\right\}\widetilde{Y}(x,\lambda) - W\left\{\widetilde{\Phi},Y\right\}\widetilde{Z}(x,\lambda).$$

Now the Green function (7.6) can be rewritten in the form

$$G(x,t,\lambda) = Y(x,\lambda)\left(W\left\{\widetilde{\Phi},Y\right\}\right)^{-1}W\left\{\widetilde{\Phi},Z\right\}\widetilde{Y}(t,\lambda)+\ldots =$$
$$= Y(x,\lambda)W\left\{\widetilde{Z},\Phi\right\}\left(W\left\{\widetilde{Y},\Phi\right\}\right)^{-1}\widetilde{Y}(t,\lambda)+\ldots$$

The ellipsis here stands for an entire function of λ. Consider a circle on the complex plane bounded by a circumference C_{R_n} with radius R_n, centered at the origin and such that for n big enough one has $|\lambda_n| < R_n$ and $\lambda_{n+1} > R_n$. Integrate the function $\dfrac{G(x,t,\lambda)}{\lambda - z}$ along the above contour to get

$$\frac{1}{2\pi i}\int\limits_{C_{R_n}} \frac{G(x,t,\lambda)}{\lambda - z}d\lambda =$$
$$= G(x,t,z) + \sum_{j=1}^{n} \operatorname{Re}s_{\lambda_j}\left\{\frac{1}{\lambda - z}Y(x,\lambda)W\left\{\widetilde{Z},\Phi\right\}\left(W\left\{\widetilde{Y},\Phi\right\}\right)^{-1}\widetilde{Y}(t,\lambda)\right\}.$$

Sending n to the infinity, one obtains

$$G(x,t,z) = -\sum_{j=1}^{\infty} \operatorname{Re}s_{\lambda_j}\left\{\frac{1}{\lambda - z}Y(x,\lambda)W\left\{\widetilde{Z},\Phi\right\}\left(W\left\{\widetilde{Y},\Phi\right\}\right)^{-1}\widetilde{Y}(t,\lambda)\right\}.$$

Similarly to [2], [13], [12], we define the normalizing polynomials by either

$$N_j(t) = e^{-\lambda_j t} \operatorname{Re} s_{\lambda_j} \left\{ e^{\lambda t} \left(W \left\{ \widetilde{Y}, \Phi \right\} \right)^{-1} W \left\{ \widetilde{Y}, \Psi \right\} \right\}$$

or

$$N_j(t) = \sum_{k=0}^{r_j-1} \left(\sum_{l=0}^{r_j-(k+1)} \operatorname{Re} s_{\lambda_j} \left\{ \left(W \left\{ \widetilde{Y}, \Phi \right\} \right)^{-1} (\lambda - \lambda_j)^{l+k} \right\} \frac{1}{l!} \frac{d^l}{d\lambda^l} W \left\{ \widetilde{Y}, \Psi \right\} \Big|_{\lambda=\lambda_j} \right) \frac{t^k}{k!}.$$

Lemma 4 ([44]). ‖ *For all $k = 0, 1, \dots, r_j - 1$ one has*

$$\sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \Big|_{t=0} \frac{d^l}{d\lambda^l} W \left\{ \widetilde{\Phi}, Z \right\} \Big|_{\lambda=\lambda_j} = \operatorname{Re} s_{\lambda_j} \left\{ \left(W \left\{ \widetilde{Y}, \Phi \right\} \right)^{-1} (\lambda - \lambda_j)^k \right\}, \tag{7.7}$$

$$\sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \Big|_{t=0} \frac{d^l}{d\lambda^l} W \left\{ \widetilde{\Phi}, Y \right\} \Big|_{\lambda=\lambda_j} = 0, \tag{7.8}$$

$$\sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^l}{d\lambda^l} W \left\{ \widetilde{Y}, \Phi \right\} \Big|_{\lambda=\lambda_j} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \Big|_{t=0} = 0. \tag{7.9}$$

Theorem 22 ([44]). *Let L_0 be the minimal differential operator generated by the differential expression* (6.19)*, whose coefficients are subject either to* (6.21)*,* (6.22)*,* (6.23) *with $\alpha > 1$, or to* (6.24) *with $0 < \alpha \le 1$, and the boundary condition* (7.2)*. Let L be the extension of L_0 given by requiring that the functions in the domain of L are in $L_2(H_m, (0, \infty))$. Then the Green function of L has the form*

$$G(x,t,z) = \\ = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} \left(\frac{1}{\lambda - z} \Phi(x,\lambda) \right) \Bigg|_{\lambda=\lambda_j} \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \Big|_{t=0} \frac{d^l}{d\lambda^l} \left(\widetilde{\Phi}(t,\lambda) \right) \Big|_{\lambda=\lambda_j}. \tag{7.10}$$

If all the eigenvalues λ_j of L are simple, i.e., the poles λ_j of the matrix $\left(W \left\{ \widetilde{Y}, \Phi \right\} \right)^{-1}$ are simple, then the matrix $N_j(t)$ is given by

$$N_j = \operatorname{Re} s_{\lambda_j} \left\{ \left(W \left\{ \widetilde{Y}, \Phi \right\} \right)^{-1} \right\} W \left\{ \widetilde{Y}, \Psi \right\} \Big|_{\lambda=\lambda_j}.$$

In this case one has

$$\operatorname{Re} s_{\lambda_j} \left\{ \left(W \left\{ \widetilde{Y}, \Phi \right\} \right)^{-1} \right\} = N_j \, W \left\{ \widetilde{\Phi}, Z \right\} \Big|_{\lambda=\lambda_j},$$

‖Our formulas (7.8), (7.9) are similar to (25) of [13], but are supplied here with a different proof.

Therefore, under the assumption that all the eigenvalues of L are simple, the formula (7.10) becomes less cumbersome:

$$G(x,t,z) = -\sum_{j=1}^{\infty} \frac{1}{\lambda_j - z} Y(x,\lambda_j)\, W\left\{\widetilde{Z},\Phi\right\}\Big|_{\lambda=\lambda_j} N_j\, W\left\{\widetilde{\Phi},Z\right\}\Big|_{\lambda=\lambda_j} \widetilde{Y}(t,\lambda_j) =$$

$$= \sum_{j=1}^{\infty} \frac{1}{\lambda_j - z} \Phi(x,\lambda_j)\, N_j \widetilde{\Phi}(t,\lambda_j).$$

This formula is equivalent to (57) of [12], where the formula for the Green function is obtained in the case of potential decreasing at infinity, which has a bounded first moment.

Let $U(x)$, $V(x)$ be arbitrary matrix functions from $L_2(H_m,(0,\infty))$. Set

$$E(U,\lambda) = \int_0^{\infty} U(t)\Phi(t,\lambda)dt, \qquad \widetilde{E}(U,\lambda) = \int_0^{\infty} \widetilde{\Phi}(t,\lambda)U(t)dt.$$

Theorem 23 ([44]). *Suppose that the coefficients of the problem* (6.19), (7.2) *are subject either to* (6.21), (6.22), (6.23) *with* $\alpha > 1$, *or to* (6.24) *with* $0 < \alpha \le 1$. *Then the arbitrary matrix functions* $U(x), V(x) \in L_2(H_m,(0,\infty))$ *admit an expansion in the solutions* $\Phi(x,\lambda)$, $\widetilde{\Phi}(x,\lambda)$ *of equations* (6.19), (7.5), *respectively:*

$$U(x) = \sum_{j=1}^{\infty}\sum_{k=0}^{r_j-1} \frac{1}{k!}\frac{d^k}{d\lambda^k}\left(E(U,\lambda)\right)\Big|_{\lambda=\lambda_j} \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!}\frac{d^{k+l}}{dt^{k+l}} N_j(t)\Big|_{t=0} \frac{d^l}{d\lambda^l}\left(\widetilde{\Phi}(x,\lambda)\right)\Big|_{\lambda=\lambda_j},$$

$$U(x) = \sum_{j=1}^{\infty}\sum_{k=0}^{r_j-1} \frac{1}{k!}\frac{d^k}{d\lambda^k}\left(\Phi(x,\lambda)\right)\Big|_{\lambda=\lambda_j} \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!}\frac{d^{k+l}}{dt^{k+l}} N_j(t)\Big|_{t=0} \frac{d^l}{d\lambda^l}\left(\widetilde{E}(U,\lambda)\right)\Big|_{\lambda=\lambda_j}.$$

Also, the Parseval equation is valid:

$$\int_0^{\infty} U(x)V(x)dx = \sum_{j=1}^{\infty}\sum_{k=0}^{r_j-1} \frac{1}{k!}\frac{d^k}{d\lambda^k} E(U,\lambda)|_{\lambda=\lambda_j} \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!}\frac{d^{k+l}}{dt^{k+l}} N_j(t)\Big|_{t=0} \frac{d^l}{d\lambda^l}\widetilde{E}(V,\lambda)\Big|_{\lambda=\lambda_j}.$$

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