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Approximate Construction of the Jost Function by the Collocation Method for Sturm-Liouville Boundary Value Problem

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Abstract. This work presents an approximate construction of the Jost function for Sturm-Liouville boundary value problem by means of collocation method.

Key Words and Phrases: Sturm-Liouville problem, Jost function, collocation method, Lagrange polynomial.

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1. Introduction

Assume that $q(x)$ is a real continuous function defined on the semi-axis $[0,\infty)$, and the condition

$$\int_0^\infty x\,|q(x)|\,dx < \infty \tag{1}$$

is satisfied.

Consider the following boundary value problem in the space $L_2[0,\infty)$:

$$\left.\begin{array}{c} -y'' + q(x)\,y = \lambda^2 y, \\ y(0) = 0. \end{array}\right\} \tag{2}$$

It is known that this problem is a self-adjoint problem. It has a finite number of negative eigenvalues $\lambda_1^2, ..., \lambda_k^2$ (see, [1]), the positive real axis $\lambda^2 > 0$ is its continuous spectrum. It is known that the numbers $\lambda_1, \lambda_2, ..., \lambda_k$ are the roots of the Jost function on the upper half-plane (see [2]). Therefore, it is required to find the Jost function.

If in problem (2) $q(x) \equiv 0$, then it is easy to find the Jost function. We consider the case $q(x) \neq 0$. Since in this case it is very difficult to find the Jost function precisely, it is important to construct it approximately.

The present paper is dedicated to the solution of this problem.

It is known that (see [2]) there is a function $K(x,t)$ such that

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$$f(\lambda, x) = e^{i\lambda x} + \int_x^{\infty} K(x,t)\, e^{i\lambda t} dt, \tag{3}$$

where the function $f(\lambda, x)$ is the solution of problem (2). Then the numbers $\lambda_1, \lambda_2, ..., \lambda_k$ are the roots of the function $f(\lambda, 0)$, i.e. the solution of the equation $f(\lambda, 0) = 0$.

To construct the Jost function approximatly we proceed as follows.

1) It is known that (according to condition (1)) $\forall \varepsilon > 0$ there is a number $N > 0$ such that

$$\int_N^{\infty} x\,|q(x)|\, dx < \varepsilon.$$

Then the Jost functions of the following new problem and of the previous problem (2), and also their λ_k's differ from each other by an infinitesimal quantity (see [3])

$$\left. \begin{array}{c} -y'' + q_N(x)\, y = \lambda^2 y, \\ y(0) = 0, \end{array} \right\} \tag{4}$$

where

$$q_N(x) = \begin{cases} q(x), & x \le N, \\ 0, & x > N. \end{cases}$$

So, we have a boundary value problem (4) with a finite coefficient. It is known that, by substitution, the segment $[0, N]$ may be reduced to $[0, 1]$. Therefore, for simplicity we can take $N = 1$. Then instead of problem (4) we have the following boundary value problem:

$$\left. \begin{array}{c} -y'' + q(x)\, y = \lambda^2 y, \\ y(0) = 0, \end{array} \right\} \ 0 \le x \le 1. \tag{5}$$

2) By the above mentioned relation (3), the Jost function of the problem (5) is as follows:

$$f(\lambda, x) = e^{i\lambda x} + \int_x^{2-x} K(x,t)\, e^{i\lambda t} dt, \tag{6}$$

and since the function $K(x,t)$ is unknown, the function $f(\lambda, x)$ is also unknown. And the main problem in the construction of the Jost function is to find $K(x,t)$. The function $K(x,t)$ is found from some Volterra type integral equation by means of the collocation method (see [4]).

To find the function $K(x,t)$, the scheme of G.M.Vainikko's paper on the solution of multi-dimensional integral equation by the collocation method (see [5]) is used. In his work G.M.Vainikko has given the solution of Fredholm type integral equations.

The present paper consists of introduction and four sections. The problem statement and the work plan of the paper are given in introduction. The known properties of the function $K(x,t)$ is recalled in §2. In §3, lemmas to guarantee the convergence of the Lagrange interpolation process in the space $L_2(D)$ are proved. A theorem on approximate finding of the function $K(x,t)$ and the convergence is proved in §4. In §5 the Jost function $f(\lambda, 0)$ is constructed.

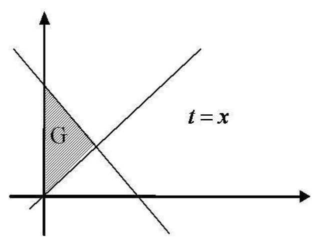
2. The properties of the function $K(x,t)$

It is known that the function $K(x,t)$ in expression (6) is the solution of the following problem:

$$\frac{\partial^2 K(x,t)}{\partial x^2} - q(x) K(x,t) = \frac{\partial^2 K(x,t)}{\partial t^2}, \tag{7}$$

$$\left.\begin{array}{c} K(x, 2-x) = 0, \\ K(x,x) = \frac{1}{2}\int_x^1 q(s)\, ds, \end{array}\right\} \tag{8}$$

where $(x,t) \in G$, $G = \left\{\ (x,t) |\quad 0 \le x \le 1;\quad x \le t \le 2-x\ \right\}$. The latter is known as the Goursat problem. The existence and uniqueness of the solution of the Goursat problem are known (see [6]).



It is obvious that to find the numbers $\lambda_1, \lambda_2, ..., \lambda_k$ we have to find the solution $K(0,t)$ $0 \le t \le 2$ of the Goursat problem (7), (8).

Write the integral equation to which the Goursat problem (7), (8) is equivalent (see [2]):

$$K(x,t) = \frac{1}{2}\int_{\frac{x+t}{2}}^1 q(s)\, ds + \int_{\frac{x+t}{2}}^1 d\alpha \int_0^{\frac{t-x}{2}} q(\alpha-\beta) K(\alpha-\beta, \alpha+\beta)\, d\beta, \tag{9}$$

where $K(x,t) = 0$ for $x > t$.

To reduce this integral equation we accept the following notation:

$$\left.\begin{array}{r} H(\alpha,\beta) = K(\alpha-\beta, \alpha+\beta), \\ x+t = 2u, \\ t-x = 2v. \end{array}\right\} \tag{10}$$

Then equation (9) takes the form

$$H(u,v) = \frac{1}{2}\int_u^1 q(s)\, ds + \int_u^1 d\alpha \int_0^v q(\alpha-\beta) H(\alpha,\beta)\, d\beta, \tag{11}$$

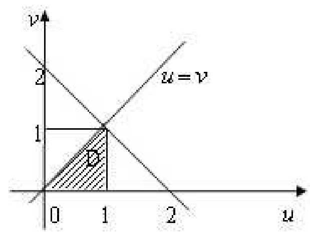
since $D = \{\ (u,v);\quad 0 \le v \le u \le 1\ \}$ and $H(u,v) = 0,\ v > u$. We can write equation (11) as follows:

$$H(u,v) = \frac{1}{2}\int_u^1 q(s)\,ds + \iint_{D_{u,v}} q(\alpha-\beta)\,H(\alpha,\beta)\,d\alpha d\beta. \tag{12}$$

It is clear that $D_{u,v} = \{\ \alpha,\beta:\quad \alpha \in [u,1],\beta \in [0,v];\quad v \le u\ \};\ (u,v) \in D,\ D_{u,v} \subset D$.

Take the following auxiliary function $\Gamma(u,v;\alpha,\beta)$:

$$\Gamma(u,v;\alpha,\beta) = \begin{cases} 1, & (\alpha,\beta) \in D_{u,v}, \\ 0, & (\alpha,\beta) \notin D_{u,v}. \end{cases} \tag{13}$$



Using this auxiliary function $\Gamma(u,v;\alpha,\beta)$, we can replace integral equation (12) by the following Fredholm type integral equation:

$$H(u,v) = \frac{1}{2}\int_u^1 \Gamma(u,v;\alpha,\beta)\,q(\alpha)\,d\alpha + \iint_{D_{u,v}} \Gamma(u,v;\alpha,\beta)\,q(\alpha-\beta)\,H(\alpha,\beta)\,d\alpha d\beta. \tag{14}$$

So, we reduced the finding of the solution of Goursat problem (7), (8) to the finding of solution of the Fredholm type integral equation (14). Now we find the approximate solution of integral equation (14) by using the collocation method.

For that we prove some lemmas on convergence of Lagrange interpolation process in two-dimensional space.

3. Notation and auxiliary results

We will use the following notation throughout work.

Denote by D a bounded closed domain in R^2 and by $\chi_D(u,v)$ the characteristic function, (R) integrated in this domain (in Riemannian sense), i.e.

$$\chi_D(u,v) = \begin{cases} 1, & (u,v) \in D, \\ 0, & (u,v) \notin D. \end{cases}$$

Ω denotes the square

$$\Omega : \{\ 0 \le u \le 1;\quad 0 \le v \le 1\ \}$$

containing the domain D.

Let the nodal points

$$u_{0n}, \quad u_{1n}, ..., \quad u_{nn}, \tag{15}$$

$$v_{0\nu}, \quad v_{1\nu}, ..., \quad v_{\nu\nu} \tag{16}$$

be the roots of polynomials $r_{n+1}(u)$ and $p_{\nu+1}(v)$, orthogonal with respect to non-negative, summable weights $g(u)$ and $p(v)$, respectively, satisfying the conditions

$$\int_0^1 \frac{du}{g(u)} < \infty, \int_0^1 \frac{dv}{p(v)} < \infty. \tag{17}$$

The polynomials $\{r_n(u)\}_{n=0}^{\infty}$ and $\{p_\nu(v)\}_{\nu=0}^{\infty}$ are of degrees n and ν, respectively. It is known that (see [7]) nodal points (15), (16) are simple, real and lie in the interval (0,1) (along the axes u and v, respectively). Write the Lagrange fundamental interpolation polynomials corresponding to these roots:

$$l_{kn}(u) = \frac{l_n(u)}{l_n'(u_{kn})(u - u_{kn})} \ \ (k = 0,\, 1,\, ...\, ,\, n),$$

$$\omega_{\phi\nu}(v) = \frac{\omega_\nu(v)}{\omega_\nu'(v_{\phi\nu})(v - v_{\phi\nu})} \ \ (\phi = 0,\, 1,\, ...\, ,\, \nu),$$

where

$$l_n(u) = (u - u_{0n})(u - u_{1n}) \cdot\ ...\ \cdot (u - u_{nn}),$$

$$\omega_\nu(v) = (v - v_{0\nu})(v - v_{1\nu}) \cdot\ ...\ \cdot (v - v_{\nu\nu}).$$

Recall some properties of fundamental polynomials:

$$l_{kn}(u_{in}) = \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases} \ (i,\, k = 0,\, 1,\, ...\, ,\, n), \tag{18}$$

$$\int_0^1 g(u)\, l_{in}(u)\, l_{kn}(u)\, du = 0, \ \ i \neq k \ \ (i,\, k = 0,\, 1,\, ...\, ,\, n), \tag{19}$$

$$\sum_{k=0}^{n} \int_0^1 g(u)\, l_{kn}^2(u)\, du = \int_0^1 g(u)\, du. \tag{20}$$

Write these properties for the polynomial $\omega_{\phi\nu}(v)$:

$$\omega_{\phi\nu}(v_{j\nu}) = \begin{cases} 0, & j \neq \phi \\ 1, & j = \phi \end{cases} \ (j,\, \phi = 0,\, 1,\, ...\, ,\, \nu), \tag{21}$$

$$\int_0^1 p(v)\, \omega_{j\nu}(v)\, \omega_{\phi\nu}(v)\, dv = 0, \ \ j \neq \phi \ \ (j,\, \phi = 0,\, 1,\, ...\, ,\, \nu), \tag{22}$$

$$\sum_{\phi=0}^{\nu}\int_{0}^{1} p\,(v)\,\omega_{\phi\nu}^{2}\,(v)\,dv=\int_{0}^{1} p\,(v)\,dv. \tag{23}$$

Now construct the Lagrange interpolation polynomial for any function $Z\,(u,v)$ defined in the domain D:

$$\mathrm{P}_{n,\nu}Z\,(u,v)=\sum_{k=0}^{n}\sum_{\phi=0}^{\nu} Z\,(u_{kn},\,v_{\phi\nu})\,l_{kn}\,(u)\,\omega_{\phi\nu}\,(v)\,,\ \ (u_{kn},\,v_{\phi\nu})\in D \tag{24}$$

(summation is taken over the indices k and ϕ, which correspond to the points $(u_{kn},\,v_{\phi\nu})$ in the domain D).

Consider the space $L^2_{gp}\,(D)$ (Banach space) of the functions square-summable with respect to the weight $g\,(u)\,p\,(v)$ in the domain D:

$$\|Z\|_{L^2_{gp}(D)}=\left[\iint_{D} g\,(u)\,p\,(v)\,|z\,(u,v)|^2\,dudv\right]^{\frac{1}{2}}. \tag{25}$$

Prove the following lemmas.

Lemma 1. *The relation*

$$\|Z-\mathrm{P}_{n,\nu}Z\|_{L^2_{gp}(D)}\to 0,\ \ n,\,\nu\to\infty \tag{26}$$

is true for each function $Z\,(u,v)$ (R)-integrable in the domain D.

Proof. Extend the function $Z\,(u,v)$ to the square Ω in the following way:

$$\Upsilon\,(u,\,v)=\begin{cases} Z\,(u,\,v)\,, & (u,\,v)\in D,\\ 0\,, & (u,\,v)\in \Omega\backslash D.\end{cases}$$

Then we get

$$\mathrm{P}_{n,\nu}Z=\sum_{k=0}^{n}\sum_{\phi=0}^{\nu}\Upsilon\,(u_{kn},\,v_{\phi\nu})\,l_{kn}\,(u)\,\omega_{\phi\nu}\,(v)=Q_{n,\nu}\Upsilon.$$

Indeed, $\mathrm{P}_{n,\nu}Z$ and the polynomial $Q_{n,\nu}\Upsilon$, being the interpolation polynomial of the function Υ, coincide along all the constructed nodal points, i.e. at the points $(u_{kn},\,v_{\phi\nu})$ $(k=0,1,\ldots,n;\ \phi=0,1,\ldots,\nu)$. As the functions $Z\,(u,v)$ and $\chi_D\,(u,v)$ are (R) – integrable, the function $\Upsilon\,(u,v)$ also is (R)-integrable in the square Ω.

Now prove that for the function $\Upsilon\,(u,v)$ (R) – integrable in the square,

$$\|\Upsilon-Q_{n,\nu}\Upsilon\|_{L^2_{gp}(\Omega)}=\left[\iint_{\Omega} g\,(u)\,p\,(v)\,|Q_{n,\nu}\Upsilon-\Upsilon|^2\,dudv\right]^{\frac{1}{2}}\to 0$$

as $n,\,\nu\to\infty$ (as in single variable function).

Denote by $\Upsilon_{n\nu}\,(u,v)$ best approximating to $\Upsilon\,(u,v)$ algebraic polynomial of order $\le n$ with respect to u, and of order $\le\nu$ with respect to v. Then it is known that the relations

$Q_{n,\nu}\Upsilon_{n\nu}(u,v)=\Upsilon_{n\nu}(u,v)$ and $(A+B)^2\leq 2\left(A^2+B^2\right)$ are true. The best approximation is $E_{n\nu}(\Upsilon)$. It is clear that

$$\left|\Upsilon(u,v)-\Upsilon_{n\nu}(u,v)\right|\leq E_{n\nu}(\Upsilon).$$

Taking into account what has been said, we have:

$$\|\Upsilon-Q_{n,\nu}\Upsilon\|_{L^2_{gp}(\Omega)}=\left[\iint_{\Omega}g(u)\,p(v)\left|Q_{n,\nu}\Upsilon-\Upsilon\right|^2dudv\right]^{\frac{1}{2}}=$$

$$=\left\{\iint_{\Omega}g(u)\,p(v)\left|\left[(\Upsilon-\Upsilon_{n\nu})+(Q_{n,\nu}\Upsilon_{n\nu}-Q_{n,\nu}\Upsilon)\right]\right|^2dudv\right\}^{\frac{1}{2}}\leq$$

$$\leq\left\{2\iint_{\Omega}g(u)\,p(v)\left[\left|\Upsilon-\Upsilon_{n\nu}\right|^2+\left|Q_{n,\nu}\Upsilon_{n\nu}-Q_{n,\nu}\Upsilon\right|^2\right]dudv\right\}^{\frac{1}{2}}\leq$$

$$\leq\left\{2\iint_{\Omega}g(u)\,p(v)\left[E^2_{n\nu}(\Upsilon)+\right.\right.$$

$$\left.\left.+\left(\sum_{k=0}^{n}\sum_{\phi=0}^{\nu}\left(\Upsilon_{n\nu}(u_{kn},\,v_{\phi\nu})-\Upsilon(u_{kn},\,v_{\phi\nu})\right)l_{kn}(u)\,\omega_{\phi\nu}(v)\right)^2dudv\right\}^{\frac{1}{2}}\leq$$

$$\leq\left\{2\iint_{\Omega}g(u)\,p(v)\left[E^2_{n\nu}(\Upsilon)+\left(\sum_{k=0}^{n}\sum_{\phi=0}^{\nu}E_{n\nu}(\Upsilon)\,l_{kn}(u)\,\omega_{\phi\nu}(v)\right)^2\right]dudv\right\}^{\frac{1}{2}}=$$

$$=\left\{2E^2_{n\nu}(\Upsilon)\iint_{\Omega}g(u)\,p(v)\left[1+\left(\sum_{k=0}^{n}\sum_{\phi=0}^{\nu}l_{kn}(u)\,\omega_{\phi\nu}(v)\right)^2\right]dudv\right\}^{\frac{1}{2}}=$$

$$=\left\{2E^2_{n\nu}(\Upsilon)\iint_{\Omega}g(u)\,p(v)\,(1+1)\,dudv\right\}^{\frac{1}{2}}=$$

$$=2E_{n\nu}(\Upsilon)\left[\int_0^1g(u)\,du\cdot\int_0^1p(v)\,dv\right]^{\frac{1}{2}}=2E_{n\nu}(\Upsilon)\cdot M,$$

where

$$M=\left[\int_0^1g(u)\,du\cdot\int_0^1p(v)\,dv\right]^{\frac{1}{2}}.$$

It is clear that $E_{n\nu}(\Upsilon) \to 0$ as $n,\ \nu \to \infty$. Therefore, $\|\Upsilon - Q_{n,\nu}\Upsilon\|_{L^2_{gp}(\Omega)} \to 0$ as $n,\ \nu \to \infty$. At the points $(u,v) \in D$, $Z(u,v) = \Upsilon(u,v)$ and $\mathrm{P}_{n,\nu}Z(u,v) = Q_{n,\nu}\Upsilon(u,v)$. Therefore, $\|Z - \mathrm{P}_{n,\nu}Z\|_{L^2_{gp}(D)} \le \|\Upsilon - Q_{n,\nu}\Upsilon\|_{L^2_{gp}(\Omega)} \to 0$ as $n,\ \nu \to \infty$. Thus, we get that $\|Z - \mathrm{P}_{n,\nu}Z\|_{L^2_{gp}(D)} \to 0$ if $n,\ \nu \to \infty$. The lemma is proved.◄

This lemma is the analog of the Erdesh-Turan theorem on mean quadratic convergence of Lagrange interpolation process for one-variable functions (see [8]).

Lemma 2. *For any polynomial* $Z_{n\nu}(u,v)$ *of the form*

$$Z_{n\nu}(u,v) = \sum_{k=0}^{n}\sum_{\phi=0}^{\nu} Y_{k\phi} l_{kn}(u)\,\omega_{\phi\nu}(v)\,,\quad (u_{kn},\ v_{\phi\nu}) \in D,$$

the estimation

$$\|Z_{n\nu}\|_{L^2_{gp}(D)} \le \|Z_{n\nu}\|_{L^2_{gp}(\Omega)} \le C_{gp} \max_{\substack{0\le k\le n;\ 0\le \phi\le \nu \\ (u_{kn},\ v_{\phi\nu}) \in D}} |Y_{k\phi}|$$

is true, where

$$C_{gp} = \left[\int_0^1 g(u)\,du \cdot \int_0^1 p(v)\,dv\right]^{\frac{1}{2}}. \tag{27}$$

Proof. We have

$$\|Z_{n\nu}\|_{L^2_{gp}(\Omega)} = \left[\iint_\Omega g(u)\,p(v)\,|Z_{n\nu}(u,v)|^2\,dudv\right]^{\frac{1}{2}} =$$

$$= \left[\iint_\Omega g(u)\,p(v)\left|\sum_{k=0}^{n}\sum_{\phi=0}^{\nu} Y_{k\phi} l_{kn}(u)\,\omega_{\phi\nu}(v)\right|^2 dudv\right]^{\frac{1}{2}} \le$$

$$\le \left[\iint_\Omega g(u)\,p(v)\left(\sum_{k=0}^{n}\sum_{\phi=0}^{\nu} |Y_{k\phi} l_{kn}(u)\,\omega_{\phi\nu}(v)|\right)^2 dudv\right]^{\frac{1}{2}} \le$$

$$\le \left[\iint_\Omega g(u)\,p(v)\left(\sum_{k=0}^{n}\sum_{\phi=0}^{\nu} \max_{k,\,\phi}|Y_{k\phi}| \cdot |l_{kn}(u)| \cdot |\omega_{\phi\nu}(v)|\right)^2 dudv\right]^{\frac{1}{2}} \le$$

$$\le \left[\left(\max_{\substack{k,\ \phi \\ (u_{kn},\ v_{\phi\nu}) \in D}} |Y_{k\phi}|\right)^2 \iint_\Omega g(u)\,p(v)\sum_{k=0}^{n}|l_{kn}(u)|^2\sum_{\phi=0}^{\nu}|\omega_{\phi\nu}(v)|^2\,dudv\right]^{\frac{1}{2}} \le$$

$$\leq \max_{\substack{k,\, \phi \\ (u_{kn},\, v_{\phi\nu}) \in D}} |Y_{k\phi}| \cdot \left[\int_0^1 g(u)\, du \cdot \int_0^1 p(v)\, dv \right]^{\frac{1}{2}} =$$

$$= C_{gp} \max_{\substack{k,\, \phi \\ (u_{kn},\, v_{\phi\nu}) \in D}} |Y_{k\phi}| .$$

It is clear that $\|Z_{n\nu}\|_{L^2_{gp}(D)} \leq \|Z_{n\nu}\|_{L^2_{gp}(\Omega)}$. Lemma 2 is proved.◀

(24) and Lemma 2 yield

$$\|\mathrm{P}_{n,\nu}\|_{C(D) \to L^2_{gp}(D)} \leq C_{gp}, \tag{28}$$

where $C(D)$ is a Banach space of continuous functions on D with the norm

$$\|Z\|_{C(D)} = \max_{(u,v) \in D} |Z(u,\, v)| .$$

Lemma 3. *If D is a quadratic domain (i.e. if $D = \Omega$), then for any function $Z(u,\, v)$ continuous on D the estimation*

$$\|Z - \mathrm{P}_{n,\nu} Z\|_{L^2_{gp}(D)} \leq 2 C_{gp} E_{n\nu}(Z) \tag{29}$$

is true, where

$$C_{gp} = \left[\int_0^1 g(u)\, du \cdot \int_0^1 p(v)\, dv \right]^{\frac{1}{2}},$$

$$E_{n\nu}(Z) = \inf_{Y_{k\phi}} \max_{(u,\, v) \in D} \left| Z(u, v) - \sum_{k=0}^{n} \sum_{\phi=0}^{\nu} Y_{k\phi} u^k v^\phi \right|$$

is the best approximation of polynomials $Z_{n\nu}(u,\, v)$ to $Z(u,\, v)$ of degree $\leq n$ with respect to u, and of degree $\leq \nu$ with respect to v.

Proof. First we note that the continuous function posseses the best approximation (by the Haar theorem).

From $D = \Omega$ it follows that for any polynomial of the given form $\mathrm{P}_{n,\nu} Z_{n\nu}(u, v) = Z_{n\nu}(u, v)$. Therefore,

$$Z - \mathrm{P}_{n,\nu} Z = (Z - Z_{n\nu}) - \mathrm{P}_{n,\nu}(Z - Z_{n\nu})$$

is true and then

$$\|Z - \mathrm{P}_{n,\nu} Z\|_{L^2_{gp}(D)} \leq \left\{ \left[\int_0^1 g(u)\, du \cdot \int_0^1 p(v)\, dv \right]^{\frac{1}{2}} + \right.$$

$$+\left\|\mathrm{P}_{n,\nu}\right\|_{C(D)\to L^2_{gp}(D)}\Big\}\cdot\max_{(u,v)\in D}\left|Z(u,v)-Z_{n\nu}(u,v)\right|.$$

Using (28), and taking into account the arbitrariness of the polynomial $Z_{n\nu}$ we get the validity of the lemma. ◄

As far as we know, there is no propriate estimate for $\|Z-\mathrm{P}_{n,\nu}Z\|_{L^2_{gp}(D)}$ in case when the domain D is of arbitrary form.

4. Finding the function $K(x,t)$ by the collocation method

Denote

$$f(u)=\frac{1}{2}\int_0^1\Gamma(u,v;\alpha,\beta)\,q(\alpha)\,d\alpha \tag{30}$$

and

$$\mathrm{T}(u,v;\alpha,\beta)=\Gamma(u,v;\alpha,\beta)\,q(\alpha-\beta). \tag{31}$$

Then we can write the integral equation (14) in the following form:

$$H(u,v)=f(u)+\iint_D\mathrm{T}(u,v;\alpha,\beta)\,H(\alpha,\beta)\,d\alpha d\beta. \tag{32}$$

We solve integral equation (32) by the collocation method. For that, we take the function $l_{kn}(u)\cdot\omega_{\phi\nu}(v)$ as a linearly independent system and look for the solution of equation (32) in the following form:

$$H_{n\nu}(u,v)=\sum_{k=0}^{n}\sum_{\phi=0}^{\nu}c_{k\phi}l_{kn}(u)\,\omega_{\phi\nu}(v),\ \ (u_{kn},v_{\phi\nu})\in D. \tag{33}$$

If we substitute expression (33) into (32) and assume that the equation is satisfied, then we have:

$$H_{n\nu}(u,v)-f(u)-\iint_D\mathrm{T}(u,v;\alpha,\beta)\,H_{n\nu}(\alpha,\beta)\,d\alpha d\beta=0. \tag{34}$$

To find the unknown coefficients $c_{k\phi}$, we take $\{(u_{in},v_{j\nu})\}\in D$ $(i=0,1,\ldots,n;\ j=0,1,\ldots,\nu)$ as collocation (nodal) points. Then

$$\left\{H_{n\nu}(u,v)-\iint_D\mathrm{T}(u,v;\alpha,\beta)\,H_{n\nu}(\alpha,\beta)\,d\alpha d\beta-f(u)\right\}\Bigg|_{\substack{u=u_{in}\\ v=v_{j\nu}}}=0, \tag{35}$$

where, $i=0,1,\ldots,n$; $j=0,1,\ldots,\nu$, $(u_{in},v_{j\nu})\in D$. From this system of linear algebraic equations we can determine $c_{k\phi}$. By (33), from system (35) we get the following linear algebraic equation for the vector $c^{(n,\nu)}=\{c_{k\phi}\}$:

$$c^{(n,\,\nu)} = A_{n\nu} c^{(n,\,\nu)} + \mathrm{P}_n f, \tag{36}$$

where

$$\mathrm{P}_n f = \{f\,(u_{in})\} \tag{37}$$

is the vector of $f\,(u)$ at the nodal points u_{in} and $A_{n\nu}$ is the matrix

$$A_{n\nu} = \{a_{i,j;k,\phi}\} \tag{38}$$

with

$$a_{i,j;k,\phi} = \iint_D \mathrm{T}\,(u_{in},\, v_{j\nu};\, \alpha,\, \beta)\, l_{kn}\,(\alpha) \cdot \omega_{\phi\nu}\,(\beta)\, d\alpha d\beta \tag{39}$$

$$(i,\, k = 0,\, 1,\, \dots,\, n; j,\, \phi = 0,\, 1,\, \dots,\, \nu)\;\; ((u_{kn},\, v_{\phi\nu}) \in D, (u_{in},\, v_{j\nu}) \in D).$$

Show that $c^{(n,\,\nu)} = \{c_{k\phi}\}$ are uniquely found from linear algebraic equation (36).

Consider the integral equation (32) as a linear operator equation in $L^2_{gp}\,(D)$, i.e.

$$H = AH + f.$$

As the operator $\mathrm{P}_{n,\nu}$ is an operator projecting the function of two variables in domain D to the Lagrange polynomial, we can write the system of linear algebraic equations (35) as follows:

$$\mathrm{P}_{n,\nu}\,(H_{n\nu} - AH_{n\nu} - f) = 0, \tag{40}$$

where

$$AH = \iint_D \mathrm{T}\,(u,\, v;\, \alpha,\, \beta)\, H\,(\alpha,\, \beta)\, d\alpha d\beta.$$

The polynomial $H_{n\nu}$ is an algebraic polynomial of degree $\leq n$ with respect to u and of degree $\leq \nu$ with respect to v. Therefore, $\mathrm{P}_{n,\nu} H_{n\nu} = H_{n\nu}$.

So, we can write the system (40) as follows:

$$H_{n\nu} - \mathrm{P}_{n,\nu} A H_{n\nu} = \mathrm{P}_{n,\nu} f. \tag{41}$$

By lemma 1, the Lagrange polynomial of any continuous function of two variables converges to that function in the Banach space $L^2_{gp}\,(D)$

$$\|\mathrm{P}_{n,\nu} - J\|_{C(D)\to L^2_{gp}(D)} \to 0,\;\; as\;\; n,\, \nu \to \infty.$$

Then, by the Banach-Steinhaus theorem, the norm of the sequence of operators $\mathrm{P}_{n,\nu}$ is bounded, i.e.

$$\|\mathrm{P}_{n,\nu}\|_{C(D)\to L^2_{gp}(D)} \leq M.$$

It is known that the operator A is a completely continuous operator from the space $L^2_{gp}(D)$ to the space $C(D)$. Indeed, two conditions of compactness, i.e. the conditions of regular boundedness and equicontinuity, are satisfied in. It is known that

$$AH = \iint_D \mathrm{T}(u,\, v;\, \alpha,\, \beta)\, H(\alpha,\, \beta)\, d\alpha d\beta = \iint_{D_{u,v}} q(\alpha - \beta)\, H(\alpha,\, \beta)\, d\alpha d\beta.$$

1. Show the regular boundedness:

$$\|AH\|_{C(D)} = \left\| \iint_{D_{u,v}} q(\alpha - \beta)\, H(\alpha,\, \beta)\, d\alpha d\beta \right\|_{C(D)} =$$

$$= \left\| \iint_{D_{u,v}} \frac{[g(\alpha)\, p(\beta)]^{\frac{1}{2}}}{[g(\alpha)\, p(\beta)]^{\frac{1}{2}}} q(\alpha - \beta)\, H(\alpha,\, \beta)\, d\alpha d\beta \right\|_{C(D)} \leq$$

$$\leq \left\| \frac{1}{2} \iint_{D_{u,v}} \left\{ [g(\alpha)\, p(\beta)]^{\frac{1}{2}}\, H(\alpha,\, \beta) \right\}^2 d\alpha d\beta \right\|_{C(D)} +$$

$$+ \left\| \frac{1}{2} \iint_{D_{u,v}} \left\{ \frac{q(\alpha - \beta)}{[g(\alpha)\, p(\beta)]^{\frac{1}{2}}} \right\}^2 d\alpha d\beta \right\|_{C(D)} =$$

$$= \left\| \frac{1}{2} \iint_{D_{u,v}} g(\alpha)\, p(\beta)\, |H(\alpha,\, \beta)|^2\, d\alpha d\beta \right\|_{C(D)} +$$

$$+ \left\| \frac{1}{2} \iint_{D_{u,v}} \frac{|q(\alpha - \beta)|^2}{g(\alpha)\, p(\beta)} d\alpha d\beta \right\|_{C(D)} \leq$$

$$\leq \left| \frac{1}{2} M_1 \right| + \frac{1}{2} \left\| \iint_{D_{u,v}} \frac{|q(\alpha - \beta)|^2}{g(\alpha)\, p(\beta)} d\alpha d\beta \right\|_{C(D)} =$$

$$= \frac{1}{2} |M_1| + \frac{1}{2} \sup_{(u,\, v) \in D} \left| \iint_{D_{u,v}} \frac{|q(\alpha - \beta)|^2}{g(\alpha)\, p(\beta)} d\alpha d\beta \right|,$$

where

$$M_1 = \left\{ \|H(\alpha,\, \beta)\|_{L^2_{gp}(D)} \right\}^2 = \iint_{D_{u,v}} g(\alpha)\, p(\beta)\, |H(\alpha,\, \beta)|^2\, d\alpha d\beta.$$

Assume that the conditions $\int_0^1 g(\alpha)\, d\alpha < \infty$, $\int_0^1 p(\beta)\, d\beta < \infty$ are satisfied. Then for $\forall\ (u,\, v) \in D$

$$\sup \left| \iint_{D_{u,v}} \frac{|q(\alpha-\beta)|^2}{g(\alpha)\, p(\beta)} d\alpha d\beta \right| \le M_2,$$

i.e. for $\forall\ (u,\, v) \in D$

$$\|AH\|_{C(D)} \le M.$$

2. Now we show the equicontinuity. Take $\forall\, \varepsilon > 0$. Find $\exists\, \delta$ for $\forall\ (u_1,\, v_1) \in D$ and $\forall\ (u_2,\, v_2) \in D$ for that for $|u_1 - u_2| < \delta$, $|v_1 - v_2| < \delta$

$$\left\| (AH)_{(u_1,\, v_1)} - (AH)_{(u_2,\, v_2)} \right\|_{C(D)} \le \varepsilon.$$

For simplicity we take $u_1 < u_2$, $v_1 < v_2$.

$$\left| (AH)_{(u_1,\, v_1)} - (AH)_{(u_2,\, v_2)} \right| =$$

$$= \left| \iint_{D_{u_1,v_1}} q(\alpha-\beta)\, H(\alpha,\, \beta)\, d\alpha d\beta - \iint_{D_{u_2,v_2}} q(\alpha-\beta)\, H(\alpha,\, \beta)\, d\alpha d\beta \right| =$$

$$= \left| \int_{u_1}^{u_2} d\alpha \int_0^{v_1} q(\alpha-\beta)\, H(\alpha,\, \beta)\, d\beta - \int_{u_2}^{1} d\alpha \int_{v_1}^{v_2} q(\alpha-\beta)\, H(\alpha,\, \beta)\, d\beta \right| =$$

$$= \left| \left[\int_{u_1}^{u_2} d\alpha \int_0^{v_1} g(\alpha)\, p(\beta)\, |H(\alpha,\, \beta)|^2\, d\beta \right]^{\frac{1}{2}} \cdot \left[\int_{u_1}^{u_2} d\alpha \int_0^{v_1} \frac{|q(\alpha-\beta)|^2}{g(\alpha)\, p(\beta)} d\beta \right]^{\frac{1}{2}} - \right.$$

$$\left. - \left[\int_{u_2}^{1} d\alpha \int_{v_1}^{v_2} g(\alpha)\, p(\beta)\, |H(\alpha,\, \beta)|^2\, d\beta \right]^{\frac{1}{2}} \cdot \left[\int_{u_2}^{1} d\alpha \int_{v_1}^{v_2} \frac{|q(\alpha-\beta)|^2}{g(\alpha)\, p(\beta)} d\beta \right]^{\frac{1}{2}} \right| \le$$

$$\le \left| \left[\int_{u_1}^{u_2} v_1 N_1 d\alpha \right]^{\frac{1}{2}} \cdot \left[\int_{u_1}^{u_2} v_1 N_2 d\alpha \right]^{\frac{1}{2}} - \right.$$

$$\left. - \left[\int_{u_2}^{1} (v_2 - v_1)\, N_1 d\alpha \right]^{\frac{1}{2}} \cdot \left[\int_{u_2}^{1} (v_2 - v_1)\, N_2 d\alpha \right]^{\frac{1}{2}} \right| \le$$

$$\leq \left| \left[v_1^2 \left(u_2 - u_1\right)^2 N_1 N_2 \right]^{\frac{1}{2}} - \left[\left(1 - u_2\right)^2 \left(v_2 - v_1\right)^2 N_1 N_2 \right]^{\frac{1}{2}} \right| \leq$$

$$\leq \left| \left[N_1 N_2 v_1^2 \delta^2 \right]^{\frac{1}{2}} - \left[N_1 N_2 \left(1 - u_2\right)^2 \delta^2 \right]^{\frac{1}{2}} \right| \leq$$

$$\leq \left| \sqrt{N_1 N_2} v_1 \delta - \sqrt{N_1 N_2} \left(1 - u_2\right) \delta \right| \leq$$

$$\leq \left| \sqrt{N_1 N_2} \left(v_1 + u_2 - 1\right) \right| |\delta| \leq \varepsilon,$$

where

$$N_1 = \sup_{\alpha,\beta} g(\alpha) p(\beta) \left| H(\alpha, \beta) \right|^2, \; N_2 = \sup_{\alpha,\beta} \frac{\left| q(\alpha - \beta) \right|^2}{g(\alpha) p(\beta)}.$$

Thus, choosing

$$\delta = \frac{\varepsilon}{\sqrt{N_1 N_2} \left(v_1 + u_2 - 1\right)},$$

we get the validity of the above relation so, the operator A maps any bounded set to the compact set, i.e. the operator $A : L_{gp}^2(D) \to C(D)$ is completely continuous.

Now if we multiply the operators $\mathrm{P}_{n,\nu}$ from the right by the continuous operator, we obtain the sequence of operators $\mathrm{P}_{n,\nu} A \in \left[L_{gp}^2(D) \to C(D) \right]$ converging in norm to the operator $JA = A$ (J is a unique operator), i.e. we get the validity of the relation

$$\left\| \mathrm{P}_{n,\nu} A - A \right\|_{C(D) \to L_{gp}^2(D)} \to 0 \;\; as \;\; n, 0 \to \infty$$

If

$$A, \; \mathrm{P}_{n,\nu} A \in \left[L_{gp}^2(D) \to C(D) \right],$$

then for sufficiently large values of $n \geq n_0$ and $\nu \geq \nu_0$

$$\left\| A - \mathrm{P}_{n,\nu} A \right\| \left\| (J - A)^{-1} \right\| \leq \phi < 1.$$

Here the operator $(J - A)^{-1}$ does exist, because the integral equation has a solution and the operator A in the space $L_{gp}^2(D)$ is a completely continuous operator (also is an inversely continuous operator). According to the known theorem (see [9]), we get that there exists the operator $(J - \mathrm{P}_{n,\nu} A)^{-1}$. In other words, equation (35) (or (36)) has a unique solution for sufficiently large values of n and ν, $n \geq n_0$ and $\nu \geq \nu_0$.

Now estimate the error. Denote by H_0 the exact solution. As the operator $(J - \mathrm{P}_{n,\nu} A)^{-1}$ has the inverse, we have

$$H_{n\nu} = (J - \mathrm{P}_{n,\nu} A)^{-1} f = (J - \mathrm{P}_{n,\nu} A)^{-1} (J - A) H_0 =$$

$$= (J - \mathrm{P}_{n,\nu} A)^{-1} \left[(J - \mathrm{P}_{n,\nu} A) + (\mathrm{P}_{n,\nu} A - A) \right] H_0 =$$

$$= (J - \mathrm{P}_{n,\nu} A)^{-1} (J - \mathrm{P}_{n,\nu} A) H_0 + (J - \mathrm{P}_{n,\nu} A)^{-1} (\mathrm{P}_{n,\nu} A - A) H_0 =$$

$$= H_0 + (J - \mathrm{P}_{n,\nu} A)^{-1} (\mathrm{P}_{n,\nu} A - A) H_0.$$

Hence

$$H_{n\nu} - H_0 = (J - \mathrm{P}_{n,\nu} A)^{-1} (\mathrm{P}_{n,\nu} A - A) H_0.$$

Then

$$\| H_{n\nu} - H_0 \|_{L^2_{gp}(D)} \le \| J - \mathrm{P}_{n,\nu} A \|^{-1}_{L^2_{gp}(D) \to C(D)} \| A \| \left\| \mathrm{P}^{(n,\nu)} H_0 \right\|_{L^2_{gp}(D)},$$

where

$$\mathrm{P}^{(n,\nu)} = J - \mathrm{P}_{n,\nu}.$$

Thus we get

$$\| H_{n\nu} - H_0 \|_{L^2_{gp}(D)} = O \left(\| H_0 - \mathrm{P}_{n,\nu} H_0 \| \right).$$

Using Lemma 3, we get

$$\| H_{n\nu} - H_0 \|_{L^2_{gp}(D)} \le c \, \| H_0 - \mathrm{P}_{n,\nu} H_0 \|_{L^2_{gp}(D)} \le 2cC_{gp} E_{n\nu} (H_0)$$

where c is a constant.

So, we completed the proof of the following theorem.

Theorem. *Assume that the kernel $q(\alpha - \beta)$ is a continuous function in domain D, the conditions $\int_0^1 \frac{d\alpha}{g(\alpha)} < \infty$, $\int_0^1 \frac{d\beta}{p(\beta)} < \infty$ are satisfied, and $H_0(\alpha, \beta)$ is a unique exact solution of integral equation* (32)*. Then for sufficiently large values of n and ν, $n, \nu > N$ the system of linear algebraic equations* (35) *(or* (36)*) possesses a unique solution and this approximate solution $H_{n\nu}(\alpha, \beta)$ converges to the exact solution $H_0(\alpha, \beta)$ in the mean square sense with respect to $g(\alpha)\, p(\beta)$. Thus,*

$$\| H_{n\nu} - H_0 \|_{L^2_{gp}(D)} \le c \, \| H_0 - \mathrm{P}_{n,\nu} H_0 \|_{L^2_{gp}(D)} \le 2cC_{gp} E_{n\nu} (H_0),$$

where c is a constant.

Thus, we have found the approximate solution of integral equation (32) in the form

$$H_{n\nu}(u, v) = \sum_{k=0}^{n} \sum_{\phi=0}^{\nu} c_{k\phi} l_{kn}(u)\, \omega_{\phi\nu}(v), \quad (u_{kn}, v_{\phi\nu}) \in D.$$

According to substitution (10),

$$H\left(u,v\right)=K\left(u-v,\,u+v\right)=K\left(x,\,t\right).$$

So,

$$H\left(\frac{x+t}{2},\,\frac{t-x}{2}\right)=K\left(x,\,t\right).$$

Then

$$\tilde{K}\left(x,\,t\right)=\sum_{k=0}^{n}\sum_{\phi=0}^{\nu}c_{k\phi}l_{kn}\left(\frac{x+t}{2}\right)\omega_{\phi\nu}\left(\frac{t-x}{2}\right)=H_{n\nu}\left(\frac{x+t}{2},\,\frac{t-x}{2}\right),\;\;(x_{kn},\,t_{\phi\nu})\in G.$$

This is the approximate solution of integral equation (9) or Goursat problem (7), (8).

Now we can construct the Jost function. But, to find the numbers $\lambda_1,\,\lambda_2,\,...\,,\,\lambda_k$, we must construct $f\left(\lambda,\,x\right)$ in the upper half-plain. In other words, we must construct the Jost function $f\left(\lambda,\,0\right)$.

5. Construction of the function $f\left(\lambda,\,0\right)$

We showed above that the function $\tilde{K}\left(x,\,t\right)$ is of the form

$$\tilde{K}\left(x,\,t\right)=\sum_{k=0}^{n}\sum_{\phi=0}^{\nu}c_{k\phi}l_{kn}\left(\frac{x+t}{2}\right)\omega_{\phi\nu}\left(\frac{t-x}{2}\right),(x_{kn},\,t_{\phi\nu})\in G. \tag{42}$$

TO construct the function $f\left(\lambda,\,0\right)$, we put $x=0$ in (42) and get:

$$\tilde{K}\left(0,\,t\right)=\sum_{k=0}^{n}\sum_{\phi=0}^{\nu}c_{k\phi}l_{kn}\left(\frac{t}{2}\right)\omega_{\phi\nu}\left(\frac{t}{2}\right),\;\;(0,\,t_{\phi\nu})\in G. \tag{43}$$

Allowing for (43) in (6), we find the Jost function in the form of the polynomial

$$f\left(\lambda,0\right)=1+\int_0^2\sum_{k=0}^{n}\sum_{\phi=0}^{\nu}c_{k\phi}l_{kn}\left(\frac{t}{2}\right)\omega_{\phi\nu}\left(\frac{t}{2}\right)e^{i\lambda t}dt,\;\;(0,\,t_{\phi\nu})\in G. \tag{44}$$

The authors believe that it would be interesting to try to find the roots $\lambda_1,\lambda_2,...,\lambda_k$ of polynomial $f(\lambda,0)$.

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