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Some properties of (X_d, X_d^*) and (l^∞, X_d, X_d^*)-Bessel multipliers

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Abstract. We use the concepts of α and β-duals to define (X_d, X_d^*) and (l^∞, X_d, X_d^*)-Bessel multipliers in Banach spaces. We investigate the properties of these multipliers when the symbol $m \in l^\infty, X_d$. In particular, we study the possibility of compactness and invertibility of these multipliers depending on their symbols and corresponding sequences.

Key Words and Phrases: X_d-Bessel sequence, (X_d, X_d^*)-Bessel multiplier, (l^∞, X_d, X_d^*)-Bessel multiplier.
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1. Introduction

In [8], Schatten presented operators of the form $\sum m_k \phi_k \otimes \overline{\psi_k}$, where $\{\phi_k\}$ and $\{\psi_k\}$ are orthonormal families. Balazs replaced these orthonormal families with Bessel sequences to define Bessel multipliers [1]. Bessel multipliers for p-Bessel sequences in Banach spaces and for g-Bessel sequences in Hilbert spaces were introduced in [6] and [7], respectively. Multipliers play important roles in both pure and applied mathematics. Gabor multipliers which are also known as Gabor filters are used in the field of acoustics.

Throughout this paper, X is a Banach space, X_d is a complex sequence space; that is, a vector space whose elements are sequences of complex numbers. All sequence spaces will be assumed to include ϕ, the set of finitely nonzero sequences. A sequence space X_d is called a BK-space, if it is a Banach space and all of the coordinate functionals $\{a_k\} \to a_k$ are continuous. A BK-space is called solid if whenever $\{a_k\}$ and $\{b_k\}$ are sequences with $\{b_k\} \in X_d$ and $|a_k| \leq |b_k|$, for each $k \in \mathbb{N}$, then it follows that $\{a_k\} \in X_d$ and $\|\{a_k\}\|_{X_d} \leq \|\{b_k\}\|_{X_d}$. A sequence space X_d is called an AK-space if it is a topological vector space and $\{a_k\} = \lim_n p_n(\{a_k\})$ for each $\{a_k\} \in X_d$, where $p_n(\{a_k\}) = (a_1, a_2, ..., a_n, 0, ...)$.

In [4], Köthe has assigned for each sequence space X_d another sequence space X_d^α, α-dual (Köthe-dual) of X_d which is defined by:

$$X_d^\alpha = \left\{ \{a_k\} : \sum_{k=1}^{\infty} |a_k b_k| < \infty, \quad \forall \{b_k\} \in X_d \right\},$$

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and X_d^β for the β-dual of X_d defined by:

$$X_d^\beta = \left\{ \{a_k\} : \sum_{k=1}^{\infty} a_k b_k \ converges, \quad \forall \{b_k\} \in X_d \right\}.$$

It is evident that $X_d^\alpha \subseteq X_d^\beta$. We note that α and β-duals of a BK-space X_d are BK-spaces with respect to the norms

$$\|\{a_k\}\|_\alpha = \sup_{\|\{b_k\}\|_{X_d} \leq 1} \sum_{k=1}^{\infty} |a_k b_k|, \tag{1.1}$$

and

$$\|\{a_k\}\|_\beta = \sup_{\|\{b_k\}\|_{X_d} \leq 1} |\sum_{k=1}^{\infty} a_k b_k|, \tag{1.2}$$

respectively. Also if X_d is a solid BK-space, then $X_d^\alpha = X_d^\beta$ [5, 10].

Remark 1.1. We note that if X_d is a solid BK-space, the norms defined in (1.1) and (1.2) are equivalent by the open mapping theorem.

It is proved in [5, 10], that the spaces X_d^* and X_d^β are isometrically isomorphic with the norm defined in (1.2), when X_d is a BK-AK-space. So by Remark 1.1, we deduce that if X_d is a solid BK-AK-space, then the spaces X_d^* and X_d^α are isomorphic with the norm defined in (1.1) and there exist $K, K' > 0$ such that

$$K' \|\{a_k\}\|_{X_d^*} \leq \|\{a_k\}\|_\alpha \leq K \|\{a_k\}\|_{X_d^*}, \quad \{a_k\} \in X_d^* \simeq X_d^\alpha, \tag{1.3}$$

where K' can be set to 1.

Lemma 1.2. *[3] Let $\{e_k\}$ be a Schauder basis of a normed space X. The canonical projections $P_n : X \to X$, where $P_n(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{n} a_i e_i$, satisfy:*
(i) $\dim(P_n(X)) = n$;
(ii) $P_n P_m = P_m P_n = P_{\min(m,n)}$;
(iii) $P_n(x) \to x$ *in* X *for every* $x \in X$.

Definition 1.3. Let X be a Banach space and X_d be a BK-space. A countable sequence $\{g_k\}_{k=1}^{\infty}$ in the dual X^* is called an X_d-frame for X if
(i) $\{g_k(f)\} \in X_d, \quad f \in X$;
(ii) the norms $\|f\|_X$ and $\|\{g_k(f)\}\|_{X_d}$ are equivalent i.e., there exist constants $A, B > 0$ such that

$$A\|f\|_X \leq \|\{g_k(f)\}\|_{X_d} \leq B\|f\|_X, \quad f \in X. \tag{1.4}$$

The constants A and B are called lower and upper X_d-frame bounds, respectively. If (i) and the upper condition in (1.4), are satisfied, then $\{g_k\}$ is called an X_d-Bessel sequence for X with bound B. We call $\{g_k\}$ a tight X_d-frame if $A = B$ and a Parseval X_d-frame if $A = B = 1$.

Definition 1.4. Let $\{g_k\}$ be a sequence of elements in X^* and $\{m_k\} \subseteq \mathbb{C}$. We call $\{g_k\}$ a weighted X_d-frame for X, if the sequence $\{m_k g_k\}$ is an X_d-frame for X.

Proposition 1.5. *[2] Suppose that X_d is a BK-space for which the canonical unit vectors $\{e_k\}$ form a Schauder basis. Then $\{g_k\} \subseteq X^*$ is an X_d^*-Bessel sequence for X with bound B if and only if the operator*

$$T : \{d_k\} \to \sum_{k=1}^{\infty} d_k g_k,$$

is well defined (hence bounded) from X_d into X^* and $\|T\| \leq B$.

Definition 1.6. A sequence $\{f_k\} \subseteq X$ is called an X_d-Riesz basis for X, if it is complete in X and there exist constants $A, B > 0$ such that

$$A\|\{c_k\}\|_{X_d} \leq \|\sum_{k=1}^{\infty} c_k f_k\| \leq B\|\{c_k\}\|_{X_d}, \quad \{c_k\} \in X_d.$$

The constants A and B are called lower and upper X_d-Riesz basis bounds, respectively. If $\{f_k\}$ is an X_d-Riesz basis for $\overline{span}_k\{f_k\}$, then $\{f_k\}$ is called an X_d-Riesz sequence.

Proposition 1.7. *[9] Suppose that X_d is a reflexive BK-space for which the canonical unit vectors $\{e_k\}$ form a Schauder basis. Assume that $\{\psi_k\} \subseteq X^*$ is an X_d^*-Riesz basis for X^* with lower bound A and upper bound B. Then there exists a unique sequence $\{\tilde{\psi}_k\} \subseteq X$, which is an X_d-Riesz basis for X with lower bound $\frac{1}{B}$ and upper bound $\frac{1}{A}$, such that*

$$f = \sum_{k=1}^{\infty} \psi_k(f)\tilde{\psi}_k, \quad f \in X,$$

$$g = \sum_{k=1}^{\infty} g(\tilde{\psi}_k)\psi_k, \quad g \in X^*.$$

This sequence $\{\tilde{\psi}_k\}$ is the unique biorthogonal to $\{\psi_k\}$.

Throughout the following sections, X is a reflexive Banach space and X_d is a solid, reflexive, BK-space such that the canonical unit vectors $\{e_k\}$ form a Schauder basis for X_d.

2. Main Results

In the following theorem by the concepts of α and β-duals, we investigate boundedness of multipliers in two different cases:

Theorem 2.1. *Suppose that $\{\phi_k\} \subseteq X$ is an X_d^*-Bessel sequence for X^* with bound B'. Then the following statements hold:*

(i) Let $\{\psi_k\} \subseteq X^*$. Suppose that there exists $P > 0$ such that $\|\psi_k\| \leq P$ for each $k \in \mathbb{N}$, and $m = \{m_k\} \in X_d$. Then the operator $M = M_{m,(\phi_k),(\psi_k)} : X \to X$ defined by:

$$M_{m,(\phi_k),(\psi_k)}(f) = \sum_{k=1}^{\infty} m_k \psi_k(f)\phi_k, \quad f \in X,$$

is well defined and bounded.
(ii) Let $\{\psi_k\} \subseteq X^*$ be an X_d-Bessel sequence for X with bound B, and $m = \{m_k\} \in l^\infty$. Then the operator $M' = M'_{m,(\phi_k),(\psi_k)} : X \to X$ defined by:

$$M'_{m,(\phi_k),(\psi_k)}(f) = \sum_{k=1}^{\infty} m_k \psi_k(f)\phi_k, \quad f \in X,$$

is well defined and bounded.

Proof. (i) First, we prove that $\{\sum_{k=1}^{n} m_k \psi_k(f)\phi_k\}_{n=1}^{\infty}$ is Cauchy in X. Consider $m, n \in \mathbb{N}$, $m > n$. Then we have

$$\begin{aligned} \|\sum_{k=n+1}^{m} m_k \psi_k(f)\phi_k\| &= \sup_{g \in X^*, \|g\| \leq 1} |\sum_{k=n+1}^{m} m_k \psi_k(f)\phi_k(g)| \\ &\leq P\|f\| \sup_{g \in X^*, \|g\| \leq 1} \sum_{k=n+1}^{\infty} |m_k \phi_k(g)|, \end{aligned}$$

Now, by (1.1) and the proof of the first Proposition in [11], we have

$$\|\sum_{k=n+1}^{m} m_k \psi_k(f)\phi_k\| \leq P\|f\| \|\{m_k\} - p_n(\{m_k\})\|_{X_d} \sup_{g \in X^*, \|g\| \leq 1} \|\{\phi_k(g)\}\|_\alpha,$$

hence by (1.3), there exists $K > 0$ such that

$$\begin{aligned} \|\sum_{k=n+1}^{m} m_k \psi_k(f)\phi_k\| &\leq KP\|f\| \|\{m_k\} - p_n(\{m_k\})\|_{X_d} \sup_{g \in X^*, \|g\| \leq 1} \|\{\phi_k(g)\}\|_{X_d^*} \\ &\leq KPB'\|f\| \|\{m_k\} - p_n(\{m_k\})\|_{X_d}. \end{aligned}$$

Since the canonical unit vectors $\{e_k\}$ form a Schauder basis for X_d, by Lemma 1.2, $\lim_n \|\{m_k\} - p_n(\{m_k\})\|_{X_d} = 0$. Therefore $\{\sum_{k=1}^{n} m_k \psi_k(f)\phi_k\}_{n=1}^{\infty}$ is Cauchy in X and so M is well defined.
Now we show that M is bounded.

$$\begin{aligned} \|M(f)\| &= \|\sum_{k=1}^{\infty} m_k \psi_k(f)\phi_k\| = \sup_{g \in X^*, \|g\| \leq 1} |\sum_{k=1}^{\infty} m_k \psi_k(f)\phi_k(g)| \\ &\leq P\|f\| \sup_{g \in X^*, \|g\| \leq 1} \sum_{k=1}^{\infty} |m_k \phi_k(g)|, \end{aligned}$$

by (1.1) and (1.3), we have

$$\begin{aligned} \|M(f)\| &\leq P\|f\|\|\{m_k\}\|_{X_d} \sup_{g\in X^*,\|g\|\leq 1} \|\{\phi_k(g)\}\|_\alpha \\ &\leq KPB'\|f\|\|\{m_k\}\|_{X_d}, \quad f \in X. \end{aligned}$$

So, $\|M\| \leq KPB'\|\{m_k\}\|_{X_d}$.
(ii) Since $\{m_k\} \in l^\infty$, we have

$$|m_k\psi_k(f)| \leq |m_k||\psi_k(f)| \leq \|\{m_k\}\|_\infty |\psi_k(f)|, \quad k \in \mathbb{N}.$$

Now, since $\{\psi_k(f)\} \in X_d$ and X_d is a solid Bk-space, $\{m_k\psi_k(f)\} \in X_d$ and we have

$$\|\{m_k\psi_k(f)\}\|_{X_d} \leq \|\{m_k\}\|_\infty \|\{\psi_k(f)\}\|. \tag{2.1}$$

Now we prove that $\{\sum_{k=1}^n m_k\psi_k(f)\phi_k\}_{n=1}^\infty$ is Cauchy in X. Consider $m, n \in \mathbb{N}$, $m > n$. Then by (2.1) and (1.1), we have

$$\begin{aligned} \|\sum_{k=n+1}^m m_k\psi_k(f)\phi_k\| &= \sup_{g\in X^*,\|g\|\leq 1} |\sum_{k=n+1}^m m_k\psi_k(f)\phi_k(g)| \\ &\leq \|\{m_k\psi_k(f)\} - p_n(\{m_k\psi_k(f)\})\|_{X_d} \sup_{g\in X^*,\|g\|\leq 1} \|\{\phi_k(g)\}\|_\alpha. \end{aligned}$$

Similar to the proof of (i), $\{\sum_{k=1}^n m_k\psi_k(f)\phi_k\}_{n=1}^\infty$ is Cauchy in X_d. Therefore, M' is well defined.
By a similar argument we can show that $\|M'\| \leq KBB'\|m\|_\infty$. ◄

The operator M in Theorem 2.1, is called (X_d, X_d^*)-Bessel multiplier and M' is called (l^∞, X_d, X_d^*)-Bessel multiplier. The sequences $\{\phi_k\}$ and $\{\psi_k\}$ are called corresponding sequences of operators M and M' and the sequence $m = \{m_k\}$ is called the symbol of these operators.

Example 2.2. Let $X = X_d = l^p$, $1 < p < \infty$. Suppose that $\{E_k\}_{k=1}^\infty$ is the sequence of coefficient functionals associated to the canonical basis $\{e_k\}_{k=1}^\infty$ of X_d. Denote $\{\psi_k\}_{k=1}^\infty = \{\frac{1}{2}E_1, E_2, \frac{1}{2^2}E_1, E_3, \frac{1}{2^3}E_1, ...\}$, $\{\phi_k\}_{k=1}^\infty = \{e_1, e_2, e_3, e_4, e_5, ...\}$ and $\{m_k\}_{k=1}^\infty = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\}$. Then $\|\psi_k\| \leq 1$, for each $k \in \mathbb{N}$, $\{\phi_k\}_{k=1}^\infty \subseteq l^p$ is a Parseval l^q-frame for l^q and $\{m_k\}_{k=1}^\infty \in l^p$. Therefore, $M_{m,(\phi_k),(\psi_k)}$ is a (l^p, l^q) Bessel multiplier.

Example 2.3. Let $X = X_d = l^p$, $1 < p < \infty$. Suppose that $\{E_k\}_{k=1}^\infty$ is the sequence of coefficient functionals associated to the canonical basis $\{e_k\}_{k=1}^\infty$ of X. Denote $\{\psi_k\}_{k=1}^\infty = \{E_k\}_{k=1}^\infty$ and $\{\phi_k\}_{k=1}^\infty = \{e_k\}_{k=1}^\infty$. Then $M_{1,(\phi_k),(\psi_k)}$ is a (l^∞, l_p, l_q)-Bessel multiplier

Remark 2.4. We note that by the definition of (X_d, X_d^*)-Bessel multiplier, M can be expressed by:

$$M = T_{\phi_k} D_m U,$$

where T_{ϕ_k} is the synthesis operator of X_d^*-Bessel sequence $\{\phi_k\}$ and the mappings $D_m : l^\infty \to X_d$, $D_m(\{c_k\}) = \{m_k c_k\}$ and $U : X \to l^\infty$, $U(f) = \{\psi_k(f)\}$, are well defined operators. Also, by the definition of (l^∞, X_d, X_d^*) Bessel multiplier, M' can be shown by:

$$M' = T_{\phi_k} D_m U_{\psi_k},$$

where T_{ϕ_k} is the synthesis operator of X_d^*-Bessel sequence $\{\phi_k\}$. The mapping $D_m : X_d \to X_d$, $D_m(\{c_k\}) = \{m_k c_k\}$ is a well defined operator and U_{ψ_k} is the analysis operator of the X_d-Bessel sequence $\{\psi_k\}$. In this case, M' can also be written by:

$$M' = T_{\phi_k} U_{m_k \psi_k},$$

where T_{ϕ_k} is the synthesis operator of X_d^*-Bessel sequence $\{\phi_k\}$, and $U_{m_k\psi_k}$ is the analysis operator of the weighted X_d-Bessel sequence $\{\psi_k\}$, where $\{m_k\}$ is a sequence of weights.

3. COMPACTNESS AND INVERTIBILITY OF MULTIPLIERS

In this section, we investigate the compactness and invertibility of Bessel multipliers and determine the formula for $(M')^{-1}$ when M' is invertible.

Theorem 3.1. *The following assertions are true:*
(i) If M is an (X_d, X_d^*)-Bessel multiplier, then M is a compact operator.
(ii) If M' is a (l^∞, X_d, X_d^*)-Bessel multiplier and $m = \{m_k\} \in c_0$, then M' is a compact operator.

Proof. (i) We define the finite rank operator

$$M_K(f) = \sum_{k=1}^{K} m_k \psi_k(f) \phi_k.$$

Then we have

$$\begin{aligned} \|M - M_K\| &= \sup_{f\in X, \|f\|\leq 1} \sup_{g\in X^*, \|g\|\leq 1} \Big| \sum_{k=K+1}^{\infty} m_k \psi_k(f) \phi_k(g) \Big| \\ &\leq \sup_{f\in X, \|f\|\leq 1} \sup_{g\in X^*, \|g\|\leq 1} \sum_{k=K+1}^{\infty} |m_k \psi_k(f) \phi_k(g)|, \end{aligned}$$

now by (1.1), (1.3) and the proof of the first proposition in [11] we have

$$\begin{aligned} \|M - M_K\| &\leq P \|\{m_k\} - p_K(\{m_k\})\|_{X_d} \sup_{g\in X^*, \|g\|\leq 1} \|\{\phi_k(g)\}\|_\alpha \\ &\leq KPB' \|\{m_k\} - p_K(\{m_k\})\|_{X_d}. \end{aligned}$$

Since the canonical unit vectors $\{e_k\}$ form a Schauder basis for X_d, by Lemma 1.2, $\lim_l \|\{m_k\} - p_K(\{m_k\})\| = 0$ and so M is a compact operator.

(ii) For a given $m \in c_o$, let $m^{(l)} = (m_1, m_2, ..., m_l, 0, 0, ...)$. Then by part (ii) of Theorem 2.1, we have

$$\begin{aligned} \|M'_{m,(\phi_k),(\psi_k)} - M'_{m^{(l)},(\phi_k),(\psi_k)}\| &= \|M'_{m-m^{(l)},(\phi_k),(\psi_k)}\| \\ &\leq \|m - m^{(l)}\|_\infty KBB'. \end{aligned}$$

Since $m \in c_0$, $\lim_l \|m - m^{(l)}\|_\infty = 0$, and the proof is evident.◄

Here is an example which shows that a (l^∞, X_d, X_d^*)-Bessel multiplier may not be a compact operator, if $m = \{m_k\} \notin c_0$.

Example 3.2. Let $X = X_d = l^p$, $1 < p < \infty$. Suppose that $\{E_k\}_{k=1}^\infty$ is the sequence of coefficient functionals associated to the canonical basis $\{e_k\}_{k=1}^\infty$ of X. Denote $\{\psi_k\}_{k=1}^\infty = \{E_k\}_{k=1}^\infty$ and $\{\phi_k\}_{k=1}^\infty = \{e_k\}_{k=1}^\infty$. Then $M_{1,(\phi_k),(\psi_k)}$ is an (l^∞, l_p, l_q)-Bessel multiplier but it is not a compact operator.

Definition 3.3. The sequence $\{m_k\}$ is called semi-normalized, if

$$0 < \inf_k |m_k| \leq \sup_k |m_k| < \infty.$$

Theorem 3.4. *Suppose that $M'_{m,(\phi_k),(\psi_k)}$ is a (l^∞, X_d, X_d^*)-Bessel multiplier and $m = \{m_k\}$ is semi-normalized. Also assume that $\{\psi_k\} \subseteq X^*$ is an X_d^*-Riesz basis for X^* and $\{\phi_k\} \subseteq X$ is an X_d-Riesz basis for X. Then M' is an invertible operator.*
In this case $(M')^{-1} = M'_{(\frac{1}{m_k}),(\tilde{\psi_k}),(\tilde{\phi_k})}$, where $\{\tilde{\psi_k}\} \subseteq X$ and $\{\tilde{\phi_k}\} \subseteq X^*$ are X_d-Riesz basis for X and X_d^*-Riesz basis for X^*, respectively.

Proof. By Remark 2.4, $M' = T_{\phi_k} D_m U_{\psi_k}$. Suppose that $\{\psi_k\}$ and $\{\phi_k\}$ are X_d^* and X_d-Riesz basis for X^* and X, respectively. Then by Propositions 3.4, 4.5 and 4.7 in [9], T_{ϕ_k} and U_{ψ_k} are invertible and also D_m, since m is semi-normalized. Therefore M' is an invertible operator. By Proposition 1.7, there exist a unique X_d-Riesz basis $\{\tilde{\psi_k}\} \subseteq X$ and a unique X_d^*-Riesz basis $\{\tilde{\phi_k}\} \subseteq X^*$, which are biorthogonal to $\{\psi_k\}$ and $\{\phi_k\}$, respectively. Since m is semi-normalized, $\frac{1}{m} = \{\frac{1}{m_k}\} \in l^\infty$ and we have

$$\begin{aligned} M_{(\frac{1}{m}),(\tilde{\psi_k}),(\tilde{\phi_k})} \circ M_{m,(\phi_k),(\psi_k)}(f) &= M_{(\frac{1}{m}),(\tilde{\psi_k}),(\tilde{\phi_k})}(\sum_{k=1}^\infty m_k \psi_k(f) \phi_k) \\ &= \sum_{i=1}^\infty \frac{1}{m_i} \tilde{\phi_i}(\sum_{k=1}^\infty m_k \psi_k(f) \phi_k) \tilde{\psi_i} \\ &= \sum_{i=1}^\infty \frac{1}{m_i} \sum_{k=1}^\infty m_k \psi_k(f) \tilde{\phi_i}(\phi_k) \tilde{\psi_i} \\ &= \sum_{i=1}^\infty \psi_i(f) \tilde{\psi_i} \\ &= f, \quad f \in X. \end{aligned}$$

◄

Theorem 3.5. *Suppose that M is an (X_d, X_d^*)-Bessel multiplier on an infinite dimensional space X. Then M is not an invertible operator.*

Proof. Since by Theorem 3.1, every (X_d, X_d^*)-Bessel multiplier is a compact operator, M can not be an invertible operator on X.◀

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