Abstract. Our goal in this paper is to find a characterization of \( n \)-dimensional bilinear Hardy inequalities
\[
\left\| \int_{B(0, \cdot)} f \cdot \int_{B(0, \cdot)} g \right\|_{q,u,(0,\infty)} \leq C \|f\|_{p_1,v_1,\mathbb{R}^n} \|g\|_{p_2,v_2,\mathbb{R}^n}, \quad f, g \in \mathcal{M}^+(\mathbb{R}^n),
\]
when \( 0 < q \leq \infty \), \( 1 \leq p_i \leq \infty \) and \( u \) and \( v_i \) are weight functions on \((0,\infty)\) and \( \mathbb{R}^n \), respectively. Obtained results are new when \( p_i = 1 \) or \( p_i = \infty \), \( i = 1, 2 \), or \( 0 < q \leq 1 \) even in 1-dimensional case.

Since the solution of the first inequality can be obtained from the characterization of the second one by usual change of variables we concentrate our attention on characterization of the latter. The characterization of this inequality is easily obtained for \( p_1 \leq q \) using the characterizations of multidimensional weighted Hardy-type inequalities while in the case \( q < p_1 \) the problem is reduced to the solution of multidimensional weighted iterated Hardy-type inequality.

To achieve our goal, we characterize the validity of multidimensional weighted iterated Hardy-type inequality
\[
\left\| \int_{B(0,s)} h(z)dz \right\|_{p,u,(0,t)} \leq c\|h\|_{\theta,v,(0,\infty)}, \quad h \in \mathcal{M}^+(\mathbb{R}^n)
\]
where \( 0 < p, q < \infty \), \( 1 \leq \theta \leq \infty \), \( u \in \mathcal{W}(0,\infty) \), \( v \in \mathcal{W}(\mathbb{R}^n) \) and \( \mu \) is a non-negative Borel measure on \((0,\infty)\). We are able to obtain the characterization under the additional condition that the measure \( \mu \) is non-degenerate with respect to \( U^{q/p} \).

Key Words and Phrases: multidimensional bilinear operators, multidimensional iterated Hardy inequalities, weights

2010 Mathematics Subject Classifications: 26D10, 26D15
1. Introduction

The aim of this paper is to study the boundedness of $n$-dimensional bilinear Hardy operators $H^2_n : L^{p_1}(w_1) \times L^{p_2}(w_2) \rightarrow L^q(u)$ and $(H^2_n)^* : L^{p_1}(w_1) \times L^{p_2}(w_2) \rightarrow L^q(u)$, defined for all $f_1, f_2 \in \mathcal{M}^+(\mathbb{R}^n)$ by

$$H^2_n(f_1, f_2)(t) := \int_{B(0,t)} f_1(x) \, dx \cdot \int_{B(0,t)} f_2(x) \, dx, \quad t > 0,$$

and

$$(H^2_n)^*(f_1, f_2)(t) := \int_{B(0,t)} f_1(x) \, dx \cdot \int_{B(0,t)} f_2(x) \, dx, \quad t > 0,$$

that is, to investigate the validity of $n$-dimensional bilinear Hardy inequalities

$$\left\| \int_{B(0,t)} f \cdot \int_{B(0,t)} g \right\|_{q,u,(0,\infty)} \leq C \|f\|_{p_1,v_1,\mathbb{R}^n} \|g\|_{p_2,v_2,\mathbb{R}^n}, \quad f, g \in \mathcal{M}^+(\mathbb{R}^n), \quad (1)$$

and

$$\left\| \int_{B(0,t)} f \cdot \int_{B(0,t)} g \right\|_{q,u,(0,\infty)} \leq C \|f\|_{p_1,v_1,\mathbb{R}^n} \|g\|_{p_2,v_2,\mathbb{R}^n}, \quad f, g \in \mathcal{M}^+(\mathbb{R}^n). \quad (2)$$

The motivation for the investigation of $n$-dimensional $m$-linear Hardy inequalities can be explained, for instance, by the paper [19], where a weight theory has been developed for a new multi(sub)linear maximal function

$$\mathcal{M}(f_1, \cdots, f_m)(x) := \sup_{Q \ni x} \prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i(y_i)| \, dy_i, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ containing $x$ with sides parallel to the coordinate axes, introduced in order to control the multilinear Calderón-Zygmund operators. Recall that, this operator is strictly smaller that the $m$-fold product of $M$, that is, the operator $\prod_{i=1}^m Mf_i$, where $M$ is the Hardy-Littlewood maximal operator. Drawing parallels between linear and $m$-linear theories, in our opinion, it will be useful to have a characterization of weight functions for which $n$-dimensional $m$-linear Hardy operator

$$H^m_n(f_1, \cdots, f_m)(t) := \int_{B(0,t)} f_1(x) \, dx \cdots \int_{B(0,t)} f_m(x) \, dx, \quad t > 0,$$
is bounded from $L_{p_1}(w_1) \times \cdots \times L_{p_m}(w_m)$ into $L_p(u)$, that is, the inequality
$$
\|H_m^n(f_1, \cdots, f_m)\|_{L_p(u)} \leq C\|f_1\|_{L_{p_1}(w_1)} \cdots \|f_m\|_{L_{p_m}(w_m)}
$$
holds.

In one-dimensional case, the bilinear Hardy operator $H_2 \equiv H^2_2$, acting on $\mathcal{M}^+(0, \infty) \times \mathcal{M}^+(0, \infty)$, is defined by
$$
H_2(f, g)(x) = \int_0^x f(t) \, dt \cdot \int_0^x g(t) \, dt.
$$
As far as we know, the boundedness of $H_2 : \mathcal{M}^+(0, \infty) \times \mathcal{M}^+(0, \infty) \to L_q(u)$, that is, the bilinear Hardy inequality
$$
\left( \int_0^\infty \left( \int_0^x f \cdot \int_0^x g \right)^q u(x) \, dx \right)^{1/q} \leq C \left( \int_0^\infty f_{p_1 v_1} \left( \int_0^\infty g_{p_2 v_2} \right)^{1/p_2} \, dx \right)^{1/p_1}, \quad f, g \in \mathcal{M}^+(0, \infty)
$$
has not been considered previously in the literature, apart from [4] and [16], where general bilinear operators were considered and their boundedness was characterized in the case $1/q \geq 1/p_1 + 1/p_2$, by means of a Schur-type criterion. The boundedness of $H_2 : L_{p_1}(v_1) \times L_{p_2}(v_2) \to L_q(u)$ was characterized recently in [1] via the discretization method, and in [17, 2, 24] using the iteration method. The range of exponents in all papers was $1 < p_1, p_2, q < \infty$.

As in 1-dimensional case (cf. [17]), the characterization of $n$-dimensional bilinear Hardy inequalities can be easily obtained using the characterizations of multidimensional weighted Hardy-type inequalities, when $p_1 \leq q$ (see Theorems 3, 5 and 6). In the most difficult case where $q < p_1$, interchanging the suprema and applying the multidimensional weighted Hardy-type inequalities, by integrating by parts, we get that inequality (2) is equivalent to the inequality
$$
\left( \int_0^\infty \left( \int_0^x \left( \int_{B(0,t)} g \right)^q u(t) \, dt \right)^{r_1/q} \, d \left[ - \|v_{-1/p_1 \|}^{r_1}_{p_1 v_1} \right] \right)^{1/r_1} \leq C \|g\|_{p_2 v_2, \mathbb{R}^n}, \quad g \in \mathcal{M}^+(\mathbb{R}^n)
$$
with $1/r_1 = 1/q - 1/p_1$ (see Theorem 4).

In this paper we characterize the validity of the multidimensional weighted iterated Hardy-type inequality
$$
\left\| \int_{B(0,s)} h(z) \, dz \right\|_{p_1 u, (0, \ell)} \leq c \|h\|_{\theta, \mu, \mathbb{R}^n}, \quad h \in \mathcal{M}^+(\mathbb{R}^n),
$$
(4)
where $0 < p, q < \infty$, $1 \leq \theta \leq \infty$, $u \in W(0, \infty)$, $v \in W(\mathbb{R}^n)$ and $\mu$ is a non-negative Borel measure on $(0, \infty)$ (see Theorem 2). We are able to obtain the characterization under the additional condition that the measure $\mu$ is non-degenerate with respect to $U^{q/p}$, that is, conditions (14) are satisfied.

In 1-dimensional case there exist different solutions of iterated Hardy-type inequalities

$$\left\| \left\| \int_{t}^{\infty} h(\tau) d\tau \right\|_{p,u,(0,\cdot)} \right\|_{q,w,(0,\infty)} \leq C \| h \|_{\theta,v,(0,\infty)}, \ h \in \mathcal{M}^+(0, \infty), \ (5)$$

where $0 < p, q \leq \infty$, $1 \leq \theta \leq \infty$ and $u, w, v \in W(0, \infty)$.

Note that the inequality (5) have been considered in the case $p = 1$ in [6] (see also [7]), where the result was presented without proof, in the case $p = \infty$ in [13] and in the case $\theta = 1$ in [8] and [25], where the special type of weight function $v$ was considered. Recall that the inequality has been completely characterized in [9] and [10] in the case $0 < p < \infty$, $0 < q \leq \infty$, $1 \leq \theta \leq \infty$ by using discretization and anti-discretization methods. Another approach to get the characterization of inequalities (5) was presented in [23]. But these characterizations involve auxiliary functions, which make conditions more complicated. The characterization of the inequality can be reduced to the characterization of the weighted Hardy inequality on the cones of non-increasing functions (see, [11, 12]). Different approach to solve iterated Hardy-type inequalities has been given in [20]. In order to characterize inequality (4) we will use the technique of [9] and [10].

It should be noted that none of the above would ever have existed if it wasn’t for the (now classical) well-known characterizations of weights for which the Hardy inequality holds. This subject, which is, incidentally, exactly one hundred years old, is absolutely indispensable in this area of mathematics (cf. [22, 18]). In our proof below multidimensional analogues of such results from [3, 5, 21] will be heavily used.

The paper is organized as follows. We start with some notations and preliminaries in Section 2. The discretization and anti-discretization methods for solution of inequalities (4) are given in Sections 3 and 4, respectively. Finally, the solutions of multidimensional bilinear Hardy inequalities are presented in Section 5.

## 2. Notations and Preliminaries

Throughout the paper, we denote by $c$ or $C$ a positive constant, which is independent of main parameters, but may vary from line to line. However a constant with subscript such as $c_1$ does not change in different occurrences. By
a \lesssim b (b \gtrsim a) we mean that \( a \leq \lambda b \), where \( \lambda > 0 \) depends only on inessential parameters. If \( a \lesssim b \) and \( b \lesssim a \), we write \( a \approx b \) and say that \( a \) and \( b \) are equivalent. Throughout the paper we use the abbreviation \( \text{LHS}(\ast) \) (\( \text{RHS}(\ast) \)) for the left (right) hand side of the relation \( \ast \). By \( \chi_Q \) we denote the characteristic function of a set \( Q \). Unless a special remark is made, the differential element \( dx \) is omitted when the integrals under consideration are the Lebesgue integrals.

For \( x \in \mathbb{R}^n \) and \( r > 0 \), let \( B(x, r) := \{ y \in \mathbb{R}^n : |x - y| < r \} \) be the open ball centered at \( x \) of radius \( r \) and \( ^\circ B(x, r) := \mathbb{R}^n \backslash B(x, r) \). We define \( S[a, b) := \{ x \in \mathbb{R}^n : a \leq |x| < b \} = ^\circ B(0, a) \backslash B(0, b) \), where \( 0 \leq a < b < \infty \).

Let \( \mu \) be a non-negative measure on \( A \subset \mathbb{R}^n \), \( n \geq 1 \). By \( \mathcal{M}(A, \mu) \) we denote the set of all \( \mu \)-measurable functions on \( A \). The symbol \( \mathcal{M}^+(A, \mu) \) stands for the collection of all \( f \in \mathcal{M}(A, \mu) \) which are non-negative on \( A \). The family of all weight functions (also called just weights) on \( A \), that is, non-negative functions locally integrable with respect to measure \( \mu \) on \( A \), is given by \( \mathcal{W}(A, \mu) \). If the measure \( \mu \) is the Lebesgue measure on \( A \), then we omit the symbol \( \mu \) in the notation.

For \( p \in (0, \infty] \) and \( w \in \mathcal{M}^+(A, \mu) \), we define the functional \( \| \cdot \|_{p,w,A,\mu} \) on \( \mathcal{M}(A, \mu) \) by

\[
\| f \|_{p,w,A,\mu} := \begin{cases} 
\left( \int_A |f(x)|^p w(x) \, d\mu(x) \right)^{1/p} & \text{if } p < \infty, \\
\mu - \text{ess sup}_A |f(x)| w(x) & \text{if } p = \infty.
\end{cases}
\]

If, in addition, \( w \in \mathcal{W}(A, \mu) \), then the weighted Lebesgue space \( L^p(w, A, \mu) \) is given by

\[
L^p(w, A, \mu) = \{ f \in \mathcal{M}(A, \mu) : \| f \|_{p,w,A,\mu} < \infty \}
\]

and it is equipped with the quasi-norm \( \| \cdot \|_{p,w,A,\mu} \).

When \( w \equiv 1 \) on \( A \), we write simply \( L^p(A) \) and \( \| \cdot \|_{p,A} \) instead of \( L^p(w, A, \mu) \) and \( \| \cdot \|_{p,w,A,\mu} \) respectively.

For \( u \in \mathcal{W}(0, \infty), v \in \mathcal{W}(\mathbb{R}^n) \) and \( 1 \leq \theta \leq \infty \), we denote

\[
U(t) := \int_0^t u(s) \, ds, \quad V_\theta(t) := \left\{ \begin{array}{ll}
\| u^{1/\theta} \|_{\theta', B(0,t)}, & \text{when } \theta < \infty, \\
\| u^{-1} \|_{1', B(0,t)}, & \text{when } \theta = \infty,
\end{array} \right. \quad t \in (0, \infty),
\]

and assume that \( U(t) > 0, t \in (0, \infty) \).

We adopt the following usual conventions.

**Convention 1.** (i) Throughout the paper we put \( 0 \cdot \infty = 0, \infty/\infty = 0 \) and \( 0/0 = 0 \).

(ii) If \( \theta \in [1, +\infty] \), we define \( \theta' \) by \( 1/\theta + 1/\theta' = 1 \).

(iii) If \( I = (a, b) \subset \mathbb{R} \) and \( g \) is a monotone function on \( I \), then by \( g(a) \) and \( g(b) \) we mean the limits \( \lim_{x \to a^+} g(x) \) and \( \lim_{x \to b^-} g(x) \), respectively.
Let us now recall some definitions and basic facts concerning discretization and anti-discretization which can be found in [14], [15] and [8].

**Definition 1.** Let \( \{ a_k \} \) be a sequence of positive real numbers. We say that \( \{ a_k \} \) is geometrically increasing or geometrically decreasing and write \( a_k \uparrow \uparrow \) or \( a_k \downarrow \downarrow \) when
\[
\inf_{k \in \mathbb{Z}} \frac{a_{k+1}}{a_k} > 1 \quad \text{or} \quad \sup_{k \in \mathbb{Z}} \frac{a_{k+1}}{a_k} < 1,
\]
respectively.

**Definition 2.** Let \( u \) be a continuous strictly increasing function on \([0, \infty)\) such that \( u(0) = 0 \) and \( \lim_{t \to \infty} u(t) = \infty \). Then we say that \( u \) is admissible.

**Definition 3.** Let \( u \) be an admissible function. A function \( h \) is called \( u \)-quasiconcave if \( h \) is equivalent to a non-decreasing function on \((0, \infty)\) and \( h/u \) is equivalent to a non-increasing function on \((0, \infty)\).

**Definition 4.** A \( u \)-quasiconcave function \( h \) is called non-degenerate if
\[
\lim_{t \to 0^+} h(t) = \lim_{t \to \infty} \frac{1}{h(t)} = \lim_{t \to \infty} \frac{h(t)}{u(t)} = \lim_{t \to 0^+} \frac{u(t)}{h(t)} = 0.
\]
The family of non-degenerate \( u \)-quasiconcave functions will be denoted by \( \Omega_u \).

In view of [14, Lemma 2.7], we give the definition of discretizing sequence as follows:

Let \( u \) be an admissible function and \( h \in \Omega_u \). Let \( \alpha > 4 \). A sequence \( \{ x_k \} \) defined by \( x_0 = 1 \) and
\[
x_{k+1} := \inf \left\{ t; \min \left\{ \frac{h(t)}{h(x_k)}, \frac{h(x_k)u(t)}{u(x_k)h(t)} \right\} = \alpha \right\}, \quad \text{when} \quad k \geq 0;
x_{k-1} := \inf \left\{ t; \min \left\{ \frac{h(x_k)}{h(t)}, \frac{h(t)u(x_k)}{u(t)h(x_k)} \right\} = \alpha \right\}, \quad \text{when} \quad k \leq 0,
\]
is called a discretizing sequence for \( h \) with respect to \( u \).

Let \( Z_1, Z_2 \) be defined by
\[
Z_1 := \{ k \in \mathbb{Z}; \alpha h(x_k) = h(x_{k+1}) \}, \quad Z_2 := \mathbb{Z} \setminus Z_1.
\]

Then
(i) \( u(x_k) \uparrow \uparrow \);
(ii) \( h(x_k) \uparrow \uparrow \) and \( \frac{h(x_k)}{u(x_k)} \downarrow \downarrow \);
(iii) \( Z = Z_1 \cup Z_2, Z_1 \cap Z_2 = \emptyset \) and for every \( t \in [x_k, x_{k+1}] \)
\[
h(x_k) \approx h(t) \quad \text{if} \quad k \in Z_1,
h(x_k) \approx \frac{h(t)}{u(t)} \quad \text{if} \quad k \in Z_2.
\]
Theorem 1. [14, Theorem 2.11] Let \( u \) be \( [0, \infty) \). We say that the function \( h \) defined as
\[
h(x) = u(x) \int_{(0, \infty)} \frac{d\mu(t)}{u(t) + u(x)}, \quad x \in (0, \infty),
\]
is the fundamental function of the measure \( \mu \) with respect to \( u \). We will also say that \( \mu \) is a representation measure of \( h \) with respect to \( u \).

We say that \( \mu \) is non-degenerate with respect to \( u \) if the following conditions are satisfied:
\[
\int_{[0, \infty)} \frac{d\mu(t)}{u(t) + u(x)} < \infty, \quad x \in (0, \infty), \quad \int_{[0, 1]} \frac{d\mu(t)}{u(t)} = \int_{[1, \infty)} d\mu(t) = \infty.
\]

Let \( u \) be an admissible function and let \( \mu \) be a non-negative Borel measure on \([0, \infty)\). Assume that \( \mu \) is non-degenerate with respect to \( u \). Let \( h \) be the fundamental function of \( \mu \) with respect to \( u \). Then \( h \in \Omega_u \) (see [14, Remark 2.10]).

We recall some known results from [14]. Our formulations of the following statements, which are more convenient for our future applications, are not exactly the same as in the mentioned paper. But by following the proofs of these theorems in [14], it is not difficult to see that such formulations are also true.

Theorem 1. [14, Theorem 2.11] Let \( p, q, r \in (0, \infty) \). Let \( u \) be an admissible function. Let \( \mu \) be a non-negative Borel measure on \([0, \infty)\) non-degenerate with respect to \( u^q \), and let \( h \) be the fundamental function of \( \mu \) with respect to \( u^q \). Let \( \sigma \) be \( u^p \)-quasiconcave. Let \( \{x_k\}_{k \in \mathbb{Z}} \) be a discretizing sequence for \( h \) with respect to \( u^q \). Then
\[
\int_{[0, \infty)} \frac{h(t)^{r/q - 1}}{\sigma(t)^{r/p}} \frac{d\mu(t)}{u(t)} \approx \sum_{k \in \mathbb{Z}} \frac{h(x_k)^{r/q}}{\sigma(x_k)^{r/p}}.
\]

Corollary 1. [14, Corollary 2.13] Let \( q \in (0, \infty) \), let \( u \) be an admissible function, let \( f \) be \( u \)-quasiconcave, let \( \mu \) be a non-negative Borel measure on \([0, \infty)\) non-degenerate with respect to \( u^q \), and let \( h \) be the fundamental function of \( \mu \) with respect to \( u^q \). Let \( \{x_k\}_{k \in \mathbb{Z}} \) be a discretizing sequence for \( h \) with respect to \( u^q \). Then
\[
\left( \int_{[0, \infty)} \left( \frac{f(t)}{u(t)} \right)^q \frac{d\mu(t)}{u(t)} \right)^{1/q} \approx \left( \sum_{k \in \mathbb{Z}} \left( \frac{f(x_k)}{u(x_k)} \right)^q h(x_k) \right)^{1/q}.
\]

Lemma 1. [14, Lemma 3.5] Let \( p, q, r \in (0, \infty) \). Let \( u \) be an admissible function. Let \( h \in \Omega_{u^q} \) and \( g \) be \( u^p \)-quasiconcave function on \((0, \infty)\). Let \( \{x_k\}_{k \in \mathbb{Z}} \) be a discretizing sequence of \( h \) with respect to \( u^q \). Then
\[
\sup_{t \in (0, \infty)} \frac{h(t)^{1/q}}{g(t)^{1/p}} \approx \sup_{k \in \mathbb{Z}} \frac{h(x_k)^{1/q}}{g(x_k)^{1/p}}.
\]
If \( q \in (0, +\infty) \) and \( \{w_k\} = \{w_k\}_{k \in \mathbb{Z}} \) is a sequence of positive numbers, we denote by \( \ell^q(\{w_k\}, \mathbb{Z}) \) the following discrete analogue of a weighted Lebesgue space: if \( 0 < q < +\infty \), then

\[
\ell^q(\{w_k\}, \mathbb{Z}) = \left\{ \{a_k\}_{k \in \mathbb{Z}} : \|a_k\|_{\ell^q(\{w_k\}, \mathbb{Z})} := \left( \sum_{k \in \mathbb{Z}} |a_k w_k|^q \right)^{1/q} < +\infty \right\}
\]

and

\[
\ell^\infty(\{w_k\}, \mathbb{Z}) = \left\{ \{a_k\}_{k \in \mathbb{Z}} : \|a_k\|_{\ell^\infty(\{w_k\}, \mathbb{Z})} := \sup_{k \in \mathbb{Z}} |a_k w_k| < +\infty \right\}.
\]

If \( w_k = 1 \) for all \( k \in \mathbb{Z} \), we write simply \( \ell^q(\mathbb{Z}) \) instead of \( \ell^q(\{w_k\}, \mathbb{Z}) \).

We quote some known results (see, for instance, [14, Lemma 3.1 and 3.2]).

**Lemma 2.** Let \( q \in (0, +\infty] \). If \( \{\tau_k\}_{k \in \mathbb{Z}} \) is a geometrically decreasing sequence, then

\[
\left\| \tau_k \sum_{m \leq k} a_m \right\|_{\ell^q(\mathbb{Z})} \approx \|\tau_k a_k\|_{\ell^q(\mathbb{Z})}
\]

and

\[
\left\| \tau_k \sup_{m \leq k} a_m \right\|_{\ell^q(\mathbb{Z})} \approx \|\tau_k a_k\|_{\ell^q(\mathbb{Z})}
\]

for all non-negative sequences \( \{a_k\}_{k \in \mathbb{Z}} \).

Let \( \{\sigma_k\}_{k \in \mathbb{Z}} \) be a geometrically increasing sequence. Then

\[
\left\| \sigma_k \sum_{m \geq k} a_m \right\|_{\ell^q(\mathbb{Z})} \approx \|\sigma_k a_k\|_{\ell^q(\mathbb{Z})}
\]

and

\[
\left\| \sigma_k \sup_{m \geq k} a_m \right\|_{\ell^q(\mathbb{Z})} \approx \|\sigma_k a_k\|_{\ell^q(\mathbb{Z})}
\]

for all non-negative sequences \( \{a_k\}_{k \in \mathbb{Z}} \).

Given two (quasi-)Banach spaces \( X \) and \( Y \), we write \( X \hookrightarrow Y \) if \( X \subset Y \) and if the natural embedding of \( X \) in \( Y \) is continuous.

The following statement is a discrete version of the classical Landau resonance theorem. Proof can be found, for example, in [14].
Proposition 1. ([14, Proposition 4.1]) Let \(0 < \theta, q \leq +\infty\) and let \(\{v_k\}_{k \in \mathbb{Z}}\) and \(\{w_k\}_{k \in \mathbb{Z}}\) be two sequences of positive numbers. Assume that
\[
\ell^0(\{v_k\}, \mathbb{Z}) \hookrightarrow \ell^0(\{w_k\}, \mathbb{Z}).
\] (7)

Then
\[
\|\{w_k u_k^{-1}\}\|_{\ell^p(\mathbb{Z})} \leq C,
\]
where \(1/p := (1/q - 1/\theta)_+\) \(^\dagger\) and \(C\) stands for the norm of embedding (7).

We shall use the following inequality, which is a simple consequence of the discrete H"older inequality:
\[
\|\{a_k b_k\}\|_{\ell^q(\mathbb{Z})} \leq \|\{a_k\}\|_{\ell^p(\mathbb{Z})} \|\{b_k\}\|_{\ell^q(\mathbb{Z})}.
\] (8)

Finally, we recall the following "an integration by parts" formula.

Proposition 2. Let \(g\) be a non-negative function on \((0, \infty)\) such that \(0 < \int_0^x g(t) \, dt < \infty, x > 0\). Assume that \(f\) is a non-negative non-increasing right-continuous function on \((0, \infty)\) such that \(\lim_{x \to \infty} f(x) = 0\). Let \(\alpha > 0\). Then
\[
\int_0^\infty \left( \int_0^x g \right)^\alpha g(x) f(x) \, dx < \infty \iff \int_0^\infty \left( \int_0^x g \right)^{\alpha + 1} d[-f(x)] < \infty.
\]

Moreover,
\[
\int_0^\infty \left( \int_0^x g \right)^\alpha g(x) f(x) \, dx \approx \int_0^{(0, \infty)} \left( \int_0^x g \right)^{\alpha + 1} d[-f(x)].
\]

3. Discretization of Inequality (4)

In this section we discretize the inequality
\[
\left( \int_{(0, \infty)} \left( \frac{1}{U(t)} \int_0^t \left( \int_{\mathbb{R}(0,y)} h(z) \, dz \right)^p u(y) \, dy \right)^{q/p} d\mu(t) \right)^{1/q} \leq c \|h\|_{\theta,v,R^\alpha}. \tag{9}
\]

At first we do the following remarks.

\(^\dagger\)For any \(a \in \mathbb{R}\) we denote \(a_+ = a\) when \(a > 0\) and \(a_+ = 0\) when \(a \leq 0\).
Remark 1. Recall that, if $F$ is a non-negative non-increasing function on $(0, \infty)$, then
\[
\text{ess sup}_{t \in (0, \infty)} F(t) G(t) = \text{ess sup}_{t \in (0, \infty)} F(t) \text{ess sup}_{\tau \in (0, t)} G(\tau); \tag{10}
\]
likewise, when $F$ is a non-negative non-decreasing function on $(0, \infty)$, then
\[
\text{ess sup}_{t \in (0, \infty)} F(t) G(t) = \text{ess sup}_{t \in (0, \infty)} F(t) \text{ess sup}_{\tau \in (t, \infty)} G(\tau) \tag{11}
\]
(see, for instance, [15, p. 85]).

Given a non-negative non-decreasing function $b$ on $(0, \infty)$, denote
\[
B(x, t) := \frac{b(x)}{b(x) + b(t)} \quad (x > 0, t > 0).
\]
Observe that
\[
B(x, t) \approx \min \left\{ 1, \frac{b(x)}{b(t)} \right\}.
\]
It is easy to see that $B(x, t)$ is a $b$-quasiconcave function of $x$ for any fixed $t > 0$. It has been shown in [15, p. 85] that the relation
\[
\text{ess sup}_{t \in (0, \infty)} B(x, t) g(t) \approx \text{ess sup}_{t \in (0, \infty)} g(t) \min \left\{ 1, \frac{b(x)}{b(t)} \right\} = b(x) \text{ess sup}_{t \in (x, \infty)} \frac{1}{b(t)} \text{ess sup}_{\tau \in (0, t)} g(\tau) \tag{12}
\]
holds for any $g \in M^+(0, \infty)$. Consequently, $\text{ess sup}_{t \in (0, \infty)} B(x, t) g(t)$ is a $b$-quasiconcave function.

Lemma 3. Let $0 < p, q < \infty$ and let $u \in W(0, \infty), v \in W(\mathbb{R}^n)$. Assume that $u$ is such that $U$ is admissible. Suppose that non-negative Borel measure $\mu$ on $(0, \infty)$ is non-degenerate with respect to $U^{q/p}$. Let $\{x_k\}$ be any discretizing sequence for the fundamental function $\varphi$ of $\mu$ with respect to $U^{q/p}$. Then
\[
\text{LHS (9)} \approx \left\| \left\{ \left\| \int_{S[y,x_k]} h(z) dz \right\|_{p,u,I_k} \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\|_{\ell^q(\mathbb{Z})} \right\|_{\ell^q(\mathbb{Z})} + \right.
\]
\[
+ \left\| \left\{ \varphi(x_k)^{1/q} \int_{S[x_k,x_{k+1}]} h(z) dz \right\|_{\ell^q(\mathbb{Z})} \right\|_{\ell^q(\mathbb{Z})}.
\]
Proof. Let $0 < p, q < \infty$. Suppose that $U$ is admissible on $(0, \infty)$. Assume that $\mu$ is a non-negative Borel measure on $(0, \infty)$ and $\varphi$ is the fundamental function of $\mu$ with respect to $U^{q/p}$, that is,

$$\varphi(x) := \int_{(0, \infty)} U(x, t)^{q/p} d\mu(t) \quad \text{for all} \quad x \in (0, \infty),$$

where

$$U(x, t) := \frac{U(x)}{U(t) + U(x)}.$$

Assume that the measure $\mu$ is non-degenerate with respect to $U^{q/p}$:

$$\int_{(0, \infty)} U(t)^{q/p} d\mu(t) + \int_{(0, \infty)} U(x)^{q/p} d\mu(x) < \infty, \quad x \in (0, \infty),$$

and there exists a discretizing sequence for $\varphi$ with respect to $U^{q/p}$. Let $\{x_k\}$ be one such sequence. Then $\varphi(x_k) \uparrow$ and $\varphi(x_k) U^{-q/p}(x_k) \downarrow$.

Furthermore, there is a decomposition $Z = Z_1 \cup Z_2$, $Z_1 \cap Z_2 = \emptyset$ such that for every $k \in Z_1$ and $t \in [x_k, x_{k+1})$, $\varphi(x_k) \approx \varphi(t)$ and for every $k \in Z_2$ and $t \in [x_k, x_{k+1})$, $\varphi(x_k) U(x_k)^{-q/p} \approx \varphi(t) U(t)^{-q/p}$.

Applying Corollary 1 to the $U$-quasiconcave function

$$f(t) = \int_0^t \left( \int_{B(0, y)} h(z) dz \right)^p u(y) dy,$$

we get

$$\text{LHS (9)} \approx \left\| \left\{ \int_{S[0, y, \infty)} h(z) dz \right\}_{y \geq 0}^{x_k} \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\|_{\ell^q(Z)}.$$

Using Lemma 2

$$\text{LHS (9)} \approx \left\| \left\{ \int_{S[y, \infty)} h(z) dz \right\}_{y \geq 0}^{x_k} \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\|_{\ell^q(Z)} \approx \left\| \left\{ \int_{S[y, x_k]} h(z) dz + \int_{S[x_k, \infty)} h(z) dz \right\}_{y \geq 0}^{x_k} \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\|_{\ell^q(Z)}.$$
where $I_k := [x_{k-1}, x_k)$, $k \in \mathbb{Z}$. Since $\|1\|_{p,u,I_k} \approx U(x_k)$, we obtain

$$\text{LHS} \ (9) \approx \left\| \left\{ \int_{S[y,x_k]} h(z)dz \right\}_{p,u,I_k} \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\|_{\ell^q(Z)} + \left\| \left\{ \varphi(x_k)^{1/q} \int_{S[x_k,\infty)} h(z)dz \right\}_{p,u,I_k} \right\|_{\ell^q(Z)}.$$ 

By using Lemma 2 on the second term, we arrive at

$$\text{LHS} \ (9) \approx \left\| \left\{ \int_{S[y,x_k]} h(z)dz \right\}_{p,u,I_k} \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\|_{\ell^q(Z)} + \left\| \left\{ \varphi(x_k)^{1/q} \int_{S[x_k,x_{k+1}]} h(z)dz \right\}_{p,u,I_k} \right\|_{\ell^q(Z)}.$$ 

\[\uparrow\]

**Lemma 4.** Let $0 < p, q < \infty$, $1 \leq \theta \leq \infty$, $1/p = (1/q - 1/\theta)_+$, and let $u \in \mathcal{W}(0, \infty)$ and $v \in \mathcal{W}(\mathbb{R}^n)$ be such that $U$ is admissible and $V_{\theta}(t) < \infty$, $t \in (0, \infty)$ with $\lim_{t \to \infty} V_{\theta}(t) = 0$. Suppose that non-negative Borel measure $\mu$ on $(0, \infty)$ is non-degenerate with respect to $U^{q/p}$. Let $\{x_k\}$ be any discretizing sequence for the fundamental function $\varphi$ of the measure $\mu$ with respect to $U^{q/p}$. Then inequality (9) holds for every $h \in \mathcal{M}^+(\mathbb{R}^n)$ if and only if

$$A := \left\| \left\{ \varphi(x_k)^{1/q} \frac{B(x_{k-1}, x_k)}{U(x_k)^{1/p}} \right\}_{p,u,I_k} \right\|_{\ell^p(Z)} + \left\| \left\{ \varphi(x_k)^{1/q} C(x_k, x_{k+1}) \right\}_{p,u,I_k} \right\|_{\ell^p(Z)} < \infty,$$ (15)

where

$$B(x_{k-1}, x_k) := \sup_{h \in \mathcal{M}^+(S[x_{k-1}, x_k])} \frac{\| \int_{S[t,x_k]} h(z)dz \|_{p,u,[x_{k-1},x_k]}}{\| h \|_{\theta,v,S[x_{k-1},x_k]}},$$ (16)
and

\[ C(x_k, x_{k+1}) := \sup_{h \in \mathcal{M}^+(S[x_k, x_{k+1}])} \frac{\|h\|_{1, S[x_k, x_{k+1}]}}{\|\theta, v, S[x_k, x_{k+1}]\|}. \]  

(17)

Moreover, the best constant in inequality (9) satisfies \( c \approx A \).

**Proof. Sufficiency.** In view of (16) and inequality (8), we have

\[
\left\{ \left\{ \left\{ \int_{S[y, x_k]} h(z)dz \right\}_{p, u, I_h} \varphi(x_k)^{1/q} \frac{U(x_k)^{1/p}}{} \right\}_{\ell^q(Z)} \right. \\
\leq \left\{ \left\{ B(x_{k-1}, x_k) \varphi(x_k)^{1/q} \frac{U(x_k)^{1/p}}{} \|h\|_{\theta, v, S[x_{k-1}, x_k]} \right\}_{\ell^p(Z)} \right. \\
\leq \left\{ \left\{ B(x_{k-1}, x_k) \varphi(x_k)^{1/q} \frac{U(x_k)^{1/p}}{} \right\}_{\ell^p(Z)} \left\| \|h\|_{\theta, v, S[x_{k-1}, x_k]} \right\|_{\ell^q(Z)} \right. \\
= \left\{ \left\{ B(x_{k-1}, x_k) \varphi(x_k)^{1/q} \frac{U(x_k)^{1/p}}{} \right\}_{\ell^p(Z)} \|h\|_{\theta, v, \mathbb{R}^n}. \right. 
\]  

(18)

By (17) and (8), we get

\[
\left\{ \left\{ \varphi(x_k)^{1/q} \int_{S[x_k, x_{k+1}]} h(z)dz \right\}_{\ell^q(Z)} \right. \\
\leq \left\{ \left\{ \varphi(x_k)^{1/q} C(x_k, x_{k+1}) \|h\|_{\theta, v, S[x_k, x_{k+1}]} \right\}_{\ell^q(Z)} \right. \\
\leq \left\{ \left\{ \varphi(x_k)^{1/q} C(x_k, x_{k+1}) \right\}_{\ell^p(Z)} \|h\|_{\theta, v, \mathbb{R}^n}. \right. 
\]  

(19)

By Lemma 3, using (18) and (19), we obtain

\[
\text{LHS (9)} \lesssim \left( \left\{ \left\{ \varphi(x_k)^{1/q} B(x_{k-1}, x_k) \right\}_{\ell^p(Z)} \right. \right. \\
+ \left. \left. \left\{ \varphi(x_k)^{1/q} C(x_k, x_{k+1}) \right\}_{\ell^p(Z)} \right\|\|h\|_{\theta, v, \mathbb{R}^n} = A \right\|\|h\|_{\theta, v, \mathbb{R}^n}. \right. 
\]

Consequently, (9) holds provided that \( A < \infty \) and \( c \leq A \).
Necessity. Assume that the inequality (9) holds with \( c < \infty \). By (16), there are \( h_k \in \mathfrak{M}^+(\mathbb{R}^n) \), \( k \in \mathbb{Z} \), such that \( \text{supp} \, h_k \subset S[x_{k-1}, x_k) \),

\[
\|h_k\|_{\theta,v,S[x_{k-1},x_k)} = 1, \quad \left\| \int_{S[y,x_k)} h_k(z) \, dz \right\|_{p,u, I_k} \geq \frac{1}{2} B(x_{k-1}, x_k) \text{ for all } k \in \mathbb{Z}. \quad (20)
\]

Define

\[
h = \sum_{m \in \mathbb{Z}} a_m h_m,
\]

where \( \{a_k\}_{k \in \mathbb{Z}} \) is any sequence of positive numbers. Then, by Lemma 3, we have

\[
\text{LHS} \, (9) \gtrsim \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\} \right\|_{\ell^q(\mathbb{Z})}
\]

\[
\gtrsim \left\| \left\{ a_k B(x_{k-1}, x_k) \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\} \right\|_{\ell^q(\mathbb{Z})}. \quad (22)
\]

Moreover,

\[
\text{RHS} \, (9) = c \left\| \sum_{m \in \mathbb{Z}} a_m h_m \right\|_{\theta,v,\mathbb{R}^n} = c \|\{a_k\}\|_{\ell^q(\mathbb{Z})}. \quad (23)
\]

By (9), (22) and (23), we obtain

\[
\left\| \left\{ a_k B(x_{k-1}, x_k) \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\} \right\|_{\ell^q(\mathbb{Z})} \lesssim c \|\{a_k\}\|_{\ell^q(\mathbb{Z})}. \quad (24)
\]

Then, by Proposition 1 we arrive at

\[
\left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} B(x_{k-1}, x_k) \right\} \right\|_{\ell^q(\mathbb{Z})} \lesssim c. \quad (25)
\]

On the other hand, by (17), there are \( \psi_k \in \mathfrak{M}^+(\mathbb{R}^n) \), \( k \in \mathbb{Z} \), such that \( \text{supp} \, \psi_k \subset S[x_k, x_{k+1}) \),

\[
\|\psi_k\|_{\theta,v,S[x_k,x_{k+1})} = 1 \quad \text{and} \quad \|\psi_k\|_{1,S[x_k,x_{k+1})} \geq \frac{1}{2} C(x_k, x_{k+1}) \text{ for all } k \in \mathbb{Z}. \quad (26)
\]
Define
\[ h = \sum_{m \in \mathbb{Z}} b_m \psi_m, \]  
where \( \{b_k\}_{k \in \mathbb{Z}} \) is any sequence of positive numbers. Then, by Lemma 3, we have
\[
\text{LHS (9)} \gtrsim \left\| \left\{ \varphi(x_k)^{1/q} \int_{S[x_k, x_{k+1}]} \sum_{m \in \mathbb{Z}} b_m \psi_m \right\} \right\|_{\ell^q(\mathbb{Z})}
\]
\[
\gtrsim \left\| \left\{ b_k \varphi(x_k)^{1/q} C(x_k, x_{k+1}) \right\} \right\|_{\ell^q(\mathbb{Z})}.
\]
We also have
\[
\text{RHS (9)} = c \left\| \sum_{m \in \mathbb{Z}} b_m \psi_m \right\|_{\theta,v,\mathbb{R}^n} = c \left\| \{b_k\} \right\|_{\ell^\theta(\mathbb{Z})}.
\]
Consequently
\[
\left\| \left\{ b_k \varphi(x_k)^{1/q} C(x_k, x_{k+1}) \right\} \right\|_{\ell^q(\mathbb{Z})} \lesssim c \left\| \{b_k\} \right\|_{\ell^\theta(\mathbb{Z})}.
\]
Then, applying Proposition 1, we get
\[
\left\| \left\{ \varphi(x_k)^{1/q} C(x_k, x_{k+1}) \right\} \right\|_{\ell^p(\mathbb{Z})} \lesssim c.
\]  
Combining (25) and (28), we arrive at \( A \lesssim c. \)

**Remark 2.** Let \( 1 \leq \theta \leq \infty. \) Note that
\[
C(x_k, x_{k+1}) = \begin{cases} 
\left\| v^{-1/\theta} \right\|_{\theta', S[x_k, x_{k+1}]}^\dagger & \text{when } \theta < \infty, \\
\left\| v^{-1} \right\|_{1, S[x_k, x_{k+1}]}^\dagger & \text{when } \theta = \infty,
\end{cases}
\]

If \( \theta < \infty, \) then, in view of Lemma 2, it is evident that
\[
\left\| \left\{ \varphi(x_k)^{1/q} C(x_k, x_{k+1}) \right\} \right\|_{\ell^p(\mathbb{Z})} \approx \left\| \left\{ \varphi(x_k)^{1/q} \right\} \right\|_{\ell^p(\mathbb{Z})} \left\| v^{-1/\theta} \right\|_{\theta', S[x_k, x_{k+1}]}.
\]
Monotonicity of \( \left\| v^{-1/\theta} \right\|_{\theta', S[t, \infty]} \) implies that
\[
\left\| \left\{ \varphi(x_k)^{1/q} \right\} \right\|_{\ell^p(\mathbb{Z})} \leq \lim_{t \to \infty} \left\| v^{-1/\theta} \right\|_{\theta', S[t, \infty]}.
Since \( \{ \varphi(x_k)^{1/q} \} \) is geometrically increasing, we obtain
\[
\left\{ \varphi(x_k)^{1/q} \right\}_q \| v^{-1/\theta} \|_{\theta, S[x_k, \infty)} \geq \varphi(\infty)^{1/q} \lim_{t \to \infty} \| v^{-1/\theta} \|_{\theta, S[t, \infty)}.
\]

This inequality shows that \( \lim_{t \to \infty} \| v^{-1/\theta} \|_{\theta, S[t, \infty)} \) must be equal to 0, because \( \varphi(\infty) \) is always equal to \( \infty \) by our assumptions on the function \( \varphi \).

Similarly, \( \lim_{t \to \infty} \| v^{-1/\theta} \|_{1, S[t, \infty)} \) must be equal to 0, when \( \theta = \infty \).

Therefore, throughout the paper we consider weight functions \( v \) such that 
\[
\lim_{t \to \infty} V_{\theta}(t) = 0.
\]

Note also that the condition \( V_{\theta}(t) < \infty, t \in (0, \infty) \) implies \( \lim_{t \to \infty} V_{\theta}(t) = 0 \), when \( 1 < \theta \leq \infty \).

4. Anti-discretization of conditions

In this section we anti-discretize the conditions obtained in Lemma 4.

Lemma 5. Let \( 0 < p, q < \infty, 1 \leq \theta \leq \infty, 1/p = (1/q - 1/\theta)_+ \) and let \( u \in W(0, \infty) \) and \( v \in W(\mathbb{R}^n) \) be such that \( U \) is admissible and \( V_{\theta}(t) < \infty, t \in (0, \infty) \) with \( \lim_{t \to \infty} V_{\theta}(t) = 0 \). Suppose that non-negative Borel measure \( \mu \) on \( (0, \infty) \) is non-degenerate with respect to \( U^{q/p} \). Let \( \{ x_k \} \) be any discretizing sequence for the fundamental function \( \varphi \) of the measure \( \mu \) with respect to \( U^{q/p} \).

(a) If \( \theta \leq p \), then \( A \approx A^* \), where
\[
A^* := \left\| \left\{ \varphi(x_k)^{1/q} \left( \sup_{t \in (0, \infty)} U(t, x_k)^{1/p} V_{\theta}(t) \right) \right\} \right\|_{L^p(\mathbb{Z})}.
\]

(b) If \( p < \theta \) and \( 1/r = 1/p - 1/\theta \), then
\[
A \approx B^*,
\]

where
\[
B^* := \left\| \left\{ \varphi(x_k)^{1/q} \left( \int_{(0, \infty)} U(t, x_k)^{r/p} d(-V_{\theta}(t-))^r \right)^{1/r} \right\} \right\|_{L^p(\mathbb{Z})}.
\]

Here
\[
V_{\theta}(t-) := \lim_{\tau \to t-} V_{\theta}(\tau).
\]
Proof. (a) By [21, Theorem 2.2, (a) and (f)], from Lemma 4 we have

\[
A \approx \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \sup_{t \in I_k} \left( \int_{x_{k-1}}^{t} u(s) ds \right)^{1/p} \left\| v^{-1/\theta} \right\|_{\theta',S(t,x_k)} \right\} \left\| v^{-1/\theta} \right\|_{\theta',S(x_k,x_k+1)} \right\|_{\ell^p(\mathbb{Z})}.
\]

By Lemma 2, we get

\[
A \lesssim \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \sup_{t \in I_k} U(t)^{1/p} \left\| v^{-1/\theta} \right\|_{\theta',S(t,\infty)} \right\} \left\| v^{-1/\theta} \right\|_{\theta',S(x_k,x_k+1)} \right\|_{\ell^p(\mathbb{Z})}.
\]

\[
= \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \sup_{t \in (0,x_k]} U(t)^{1/p} \left\| v^{-1/\theta} \right\|_{\theta',S(t,\infty)} \right\} \left\| v^{-1/\theta} \right\|_{\theta',S(x_k,x_k+1)} \right\|_{\ell^p(\mathbb{Z})}.
\]

We now prove the reverse estimate. We have

\[
A^* \approx \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \sup_{t \in I_k} \left( \int_{x_{k-1}}^{t} u(s) ds \right)^{1/p} \left\| v^{-1/\theta} \right\|_{\theta',S(t,x_k)} \right\} \left\| v^{-1/\theta} \right\|_{\theta',S(x_k,x_k+1)} \right\|_{\ell^p(\mathbb{Z})}.
\]
Lemma 4, we have

\[ \left\| \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left( \sup_{t \in I_k} \left( \int_{x_{k-1}}^{x_k} u(s) \, ds \right)^{1/p} \right) \left( \int_{x_{k-1}}^{x_k} u(s) \, ds \right)^{1/p} \right\|_{\ell^p(\mathbb{Z})} \]

+ \left\{ \left( \int_{x_{k-1}}^{x_k} u(s) \, ds \right)^{1/p} \right\}_{\ell^p(\mathbb{Z})} \approx A.

(b) Assume that \( \theta < \infty \). By [21, Theorem 2.2, (b) and (g)], and (29), from Lemma 4, we have

\[ A \approx \left\| \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left( \sup_{t \in I_k} \left( \int_{x_{k-1}}^{x_k} u(s) \, ds \right)^{r/\theta} u(t) \left( \int_{x_{k-1}}^{x_k} u(s) \, ds \right)^{1/\theta} \left( \int_{x_{k-1}}^{x_k} u(s) \, ds \right)^{1/\theta} \right) \right\|_{\ell^p(\mathbb{Z})} \]

Since

\[ \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^{x_k} u(s) \, ds \right)^{r/\theta} u(t) \right)^{1/r} \approx U(x_k)^{1/p}, \]

it is easy to see that

\[ A \approx \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left( \sup_{t \in I_k} \left( \int_{x_{k-1}}^{x_k} u(s) \, ds \right)^{r/\theta} u(t) \left( \int_{x_{k-1}}^{x_k} u(s) \, ds \right)^{1/\theta} \left( \int_{x_{k-1}}^{x_k} u(s) \, ds \right)^{1/\theta} \right) \right\}_{\ell^p(\mathbb{Z})} \]
By Lemma 2, in view of Remark 2, we have

\[
\begin{aligned}
&+ \left\| \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left( \int_{x_{k-1}}^{x_k} \left( \int_{x_{k-1}}^t u(s) ds \right)^{r/\theta} u(t) \right) u(t) \right\|_{l^p(Z)} \left\| v^{-1/\theta} \right\|_{\theta', S[x_k, x_{k+1}]}^{1/r} \\
\end{aligned}
\]

Integrating by parts, we arrive at

\[
A \leq \left\| \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left( \int_{x_{k-1}}^{x_k} U(t)^{r/p} d \left( - \left\| v^{-1/\theta} \right\|_{\theta', S[t, \infty]}^{r} \right) \right) \right\|_{l^p(Z)} + \left\| \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left\| v^{-1/\theta} \right\|_{\theta', S|x_k, -\infty]} \right\|_{l^p(Z)}
\]

By Lemma 2, in view of Remark 2, we have

\[
\begin{aligned}
&= \left\| \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left( \left\| v^{-1/\theta} \right\|_{\theta', S|x_k, -\infty]} - \lim_{m \to \infty} \left\| v^{-1/\theta} \right\|_{\theta', S|x_m, -\infty]}^{r} \right) \right\|_{l^p(Z)} \\
&\simeq \left\| \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left( \left\| v^{-1/\theta} \right\|_{\theta', S|x_k, -\infty]} - \left\| v^{-1/\theta} \right\|_{\theta', S|x_{k+1}, -\infty]}^{r} \right) \right\|_{l^p(Z)} \\
&\simeq \left\| \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left( \int_{x_{k}, x_{k+1}} d \left( - \left\| v^{-1/\theta} \right\|_{\theta', S[t, \infty]}^{r} \right) \right) \right\|_{l^p(Z)} \\
\end{aligned}
\]
Using the latter in (30) and applying Lemma 2, we arrive at

\[ A \lesssim \left\| \frac{\varphi(x_k)\|1/q}{U(x_k)^{1/p}} \left( \int_{(0,x_k)} U(t)^{r/p} d\left( -\|v^{-1/\theta}\|^{r}_{\theta',S(t,\infty)} \right) \right) \right\|_{\ell^r(\mathbb{Z})} \]

\[ + \left\| \left\{ \varphi(x_k)^{1/q} \left( \int_{[x_k,\infty)} d\left( -\|v^{-1/\theta}\|^{r}_{\theta',S(t,\infty)} \right) \right) \right\|_{\ell^r(\mathbb{Z})} \]

\[ \approx \left\| \left\{ \varphi(x_k)^{1/q} \left( \int_{(0,\infty)} U(t,x_k)^{r/p} d\left( -\|v^{-1/\theta}\|^{r}_{\theta',S(t,\infty)} \right) \right) \right\|_{\ell^r(\mathbb{Z})} = B^*. \]

Consequently, \( A \lesssim B^* \).

Conversely, by Lemma 2, in view of Remark 2, we have

\[ B^* \approx \left\| \left\{ \varphi(x_k)^{1/q} \left( \int_{[x_k-1,x_k)} U(t)^{r/p} d\left( -\|v^{-1/\theta}\|^{r}_{\theta',S(t,\infty)} \right) \right) \right\|_{\ell^r(\mathbb{Z})} \]

\[ + \left\| \left\{ \varphi(x_k)^{1/q} \left( \int_{[x_k-1,x_k)} U(t)^{r/p} d\left( -\|v^{-1/\theta}\|^{r}_{\theta',S(t,\infty)} \right) \right) \right\|_{\ell^r(\mathbb{Z})} \]

\[ \lesssim \left\| \left\{ \varphi(x_k)^{1/q} \left( \int_{[x_k-1,x_k)} U(t)^{r/p} d\left( -\|v^{-1/\theta}\|^{r}_{\theta',S(t,\infty)} \right) \right) \right\|_{\ell^r(\mathbb{Z})} \]
Integrating by parts yields

\[ B^* \lesssim \left\| \left\{ \varphi'(x_k)^{1/q} \| v^{-1/\theta'} \|_{\theta', S[x_k, \infty]} \right\} \right\|_{L^p(Z)} \]

Since

\[ = \left\| \left\{ \varphi'(x_k)^{1/q} \| v^{-1/\theta'} \|_{\theta', S[x_k, \infty]} \right\} \right\|_{L^p(Z)} \]

we arrive at

\[ B^* \lesssim \left\| \left\{ \varphi(x_k)^{1/q} \left( \int_{x_k}^{t} u(s) \, ds \right)^{r/\theta} u(t) \right\|_{\theta', S[t, \infty]} \right\|_{L^p(Z)} \]

\[ \approx \left\| \left\{ \varphi(x_k)^{1/q} \left( \int_{x_k}^{t} u(s) \, ds \right)^{r/\theta} u(t) \right\|_{\theta', S[t, t_k]} \right\|_{L^p(Z)} \]

\[ + \left\| \left\{ \varphi(x_k)^{1/q} \| v^{-1/\theta'} \|_{\theta', S[x_k, \infty]} \right\} \right\|_{L^p(Z)} \approx A. \]
Now assume that $\theta = \infty$. In this case the proof can be done in the same line and we leave it to the reader. The only difference is that one should apply \cite[Theorem 2.2, (e)]{21} and take into account that $C(x_k, x_{k+1}) = \|v^{-1}\|_{1, S[x_k, x_{k+1}]}$, $k \in \mathbb{Z}$. ▶

We are now in a position to characterize inequality (9).

**Theorem 2.** Let $0 < p, q < \infty$, $1 \leq \theta \leq \infty$, $1/\rho = (1/q - 1/\theta)$, and let $u \in \mathcal{W}(0, \infty)$ and $v \in \mathcal{W}(\mathbb{R}^n)$ be such that $U$ is admissible and $V_\theta(t) < \infty$, $t \in (0, \infty)$ with $\lim_{t \to \infty} V_\theta(t) = 0$. Suppose that non-negative Borel measure $\mu$ on $(0, \infty)$ is non-degenerate with respect to $U^{q/p}$. Then the inequality (9) holds for every measurable function on $\mathbb{R}^n$ if and only if

(i) $\theta \leq \min\{p, q\}$ and

$$N_1 := \sup_{x \in (0, \infty)} \left( \int_{(0, \infty)} U(x, t)^{q/p} d\mu(t) \right)^{1/q} \sup_{t \in (0, \infty)} U(t, x)^{1/p} V_\theta(t) < \infty.$$ 

Moreover, the best constant in (9) satisfies $c \approx N_1$.

(ii) $q < \theta < p$ and

$$N_2 := \left( \int_{(0, \infty)} \left( \int_{(0, \infty)} U(x, t)^{q/p} d\mu(t) \right)^{\rho/\theta} \right) \times$$

$$\times \left( \sup_{t \in (0, \infty)} U(t, x)^{1/p} V_\theta(t) \right)^{\rho} \sup_{x \in (0, \infty)} U(x, t)^{q/p} d\mu(t) \right)^{1/p} < \infty.$$ 

Moreover, the best constant in (9) satisfies $c \approx N_2$.

(iii) $p < \theta \leq q$, $r = \theta p/(\theta - p)$ and

$$N_3 := \sup_{x \in (0, \infty)} \left( \int_{(0, \infty)} U(x, t)^{q/p} d\mu(t) \right)^{1/q} \times$$

$$\times \left( \int_{(0, \infty)} U(t, x)^{r/p} d\left( - V_\theta(t) \right)^{1/r} \right) < \infty.$$ 

Moreover, the best constant in (9) satisfies $c \approx N_3$.

(iv) $\max\{p, q\} < \theta$, $r = \theta p/(\theta - p)$ and

$$N_4 := \left( \int_{(0, \infty)} \left( \int_{(0, \infty)} U(x, t)^{q/p} d\mu(t) \right)^{\rho/\theta} \right) \times$$

$$\times \left( \int_{(0, \infty)} U(t, x)^{r/p} d\left( - V_\theta(t) \right)^{1/r} \right) < \infty.$$ 

Moreover, the best constant in (9) satisfies $c \approx N_4$. 

"
\[
\times \left( \int_{(0,\infty)} U(t,x)^{r/p} d \left( -V_\theta(t)^r \right) \right)^{\rho/r} d\mu(x) < \infty.
\]

Moreover, the best constant in (9) satisfies \( c \approx N_4 \).

(v) \( \theta = \infty \) and

\[
N_5 := \left( \int_{(0,\infty)} \left( \int_{(0,\infty)} U(t,x) d \left( -V_\infty(t)^p \right) \right)^{q/p} d\mu(x) \right)^{1/q} < \infty.
\]

Moreover, the best constant in (9) satisfies \( c \approx N_5 \).

Proof.

(i) The proof follows by Lemma 4, Lemma 5, (a), and Lemma 1.

(ii) The proof follows by Lemma 4, Lemma 5, (a) and Theorem 1.

(iii) The proof follows by Lemma 4, Lemma 5, (b), and Lemma 1.

(iv) The proof follows by Lemma 4, Lemma 5, (b), and Theorem 1.

(v) The proof follows by Lemma 4, Lemma 5, (b), and Theorem 1. \( \blacktriangleleft \)

5. Characterization of \( n \)-dimensional bilinear Hardy inequalities

In this section we give characterization of \( n \)-dimensional bilinear Hardy inequalities (1) and (2).

The following note allows us to concentrate our attention only on characterization of (2).

Remark 3. Note that the inequality

\[
\left( \int_0^\infty \left( \int_{B(0,t)} f \cdot \int_{B(0,t)} g \right)^q u(t)dt \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} f^{p_1 v_1} \right)^{1/p_1} \left( \int_{\mathbb{R}^n} g^{p_2 v_2} \right)^{1/p_2}
\]

is equivalent to the inequality

\[
\left( \int_0^\infty \left( \int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} g \right)^q \tilde{u}(t)dt \right)^{1/q}
\]
\[ \leq C \left( \int_{\mathbb{R}^n} f^{p_1} v_1 \right)^{1/p_1} \left( \int_{\mathbb{R}^n} g^{p_2} \tilde{v}_2 \right)^{1/p_2}, \quad (32) \]

where
\[ \tilde{u}(t) = u(t^{-1}) \tau^{-2}, \tilde{v}_1(x) = v_1(|x|^{-2} x |x|^{-2n(1-p_1)}), \tilde{v}_2(x) = v_2(|x|^{-2} x |x|^{-2n(1-p_2)}). \]

**Indeed:** Since any \( f \in \mathcal{M}^+(\mathbb{R}^n) \) can be uniquely represented as \( f(x) = g(|x|^{-2} x |x|^{-2n}), g \in \mathcal{M}^+(\mathbb{R}^n) \), then inequality (31) is equivalent to the following inequality:

\[
\left( \int_0^\infty \left( \int_{B(0,t)} f(|y|^{-2} y |y|^{-2n} dy \int_{B(0,t)} g(|y|^{-2} y |y|^{-2n} dy \right)^q u(t)dt \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} f(|y|^{-2} y |y|^{-2n} dy \right)^{1/p_1} \times \left( \int_{\mathbb{R}^n} g(|y|^{-2} y |y|^{-2n} dy \right)^{1/p_2}. \quad (33) \]

Using the substitution \( x = |y|^{-2} y \) in multidimensional integrals, we see that (33) is equivalent to the inequality

\[
\left( \int_0^\infty \left( \int_{B(0,1/t)} f(x)dx \int_{B(0,1/t)} g(x)dx \right)^q u(t)dt \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} f(x)^{p_1} v_1(|x|^{-2} x |x|^{-2n(1-p_1)} dx \right)^{1/p_1} \times \left( \int_{\mathbb{R}^n} g(x)^{p_2} v_2(|x|^{-2} x |x|^{-2n(1-p_2)} dx \right)^{1/p_2},
\]

and finally applying \( \tau = 1/t \), we conclude that the latter is equivalent to

\[
\left( \int_0^\infty \left( \int_{B(0,\tau)} f(x)dx \int_{B(0,\tau)} g(x)dx \right)^q u(\tau^{-1}) \tau^{-2} d\tau \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} f(x)^{p_1} v_1(|x|^{-2} x |x|^{-2n(1-p_1)} dx \right)^{1/p_1} \times \left( \int_{\mathbb{R}^n} g(x)^{p_2} v_2(|x|^{-2} x |x|^{-2n(1-p_2)} dx \right)^{1/p_2}.\]
Moreover, the best constant $C$ in (2) satisfies $C \approx B_1$.

(b) $1 \leq p_2 < \infty$, $0 < q < p_2$, $1/r_2 = 1/q - 1/p_2$, and

$$B_2 := \sup_{t \in (0, \infty)} \left\{ \left| v_1^{1/p_1} \left( \int_0^t U(y)^{r_2/p_2} u(y) \left| v_2^{1/p_2} \right|^{r_2} \right)^{1/r_2} \right\} < \infty.$$  
Moreover, the best constant $C$ in (2) satisfies $C \approx B_2$.

(c) $p_2 = \infty$, and

$$B_3 := \sup_{t \in (0, \infty)} \left\{ \left| v_1^{1/p_1} \left( \int_0^t u(y) \left| v_2^{1/p_2} \right|^{q} \right)^{1/q} \right\} < \infty.$$  
Moreover, the best constant $C$ in (2) satisfies $C \approx B_3$.

Proof. Interchanging the suprema, we obtain

$$\sup_{f,g \in \mathcal{W}^+ (\mathbb{R}^n)} \left\{ \left| \int_{B(0, \cdot)} f \cdot \int_{B(0, \cdot)} g \right|_{q,u,(0, \infty)} \right\}_{q,u,(0, \infty)}$$

$$= \sup_{g \in \mathcal{W}^+ (\mathbb{R}^n)} \left. \frac{1}{\| g \|_{2,p_2,\mathbb{R}^n}} \sup_{f \in \mathcal{W}^+ (\mathbb{R}^n)} \left\{ \left| \int_{B(0, \cdot)} f \cdot \int_{B(0, \cdot)} g \right|_{q,u,(0, \infty)} \right\}_{q,u,(0, \infty)} \right. \right. . \ \ \ (34)$$

By [21, Theorem 2.2, (a) and (f)], we get

$$\sup_{f,g \in \mathcal{W}^+ (\mathbb{R}^n)} \left\{ \left| \int_{B(0, \cdot)} f \cdot \int_{B(0, \cdot)} g \right|_{q,u,(0, \infty)} \right\}_{q,u,(0, \infty)}$$

$$\approx \sup_{g \in \mathcal{W}^+ (\mathbb{R}^n)} \left. \frac{1}{\| g \|_{2,p_2,\mathbb{R}^n}} \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_{B(0, \tau)} g \right)^q u(\tau) d\tau \right)^{1/q} \left| v_1^{1/p_1} \right|_{p_1, \mathcal{E}B(0,t)} \right. \right. . \ \ \ (35)$$
(a) Let $1 \leq p_2 \leq q < \infty$. Again, interchanging the suprema, by \[21, \text{Theorem 2.2, (a) and (f)}, \text{on using (10)}, \text{we get}

\[
\sup_{f, g \in \mathcal{M}^+(\mathbb{R}^n)} \frac{\left\| \int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} g \right\|_{q,u,(0,\infty)}}{\left\| f \right\|_{p_1,v_1,\mathbb{R}^n} \left\| g \right\|_{p_2,v_2,\mathbb{R}^n}} \\
= \sup_{t \in (0,\infty)} \left\| v_1^{-1/p_1} \right\|_{p_1',\mathbb{R}^n} \sup_{g \in \mathcal{M}^+(\mathbb{R}^n)} \left( \int_0^t \left( \int_0^y \chi(0,t)(\tau)u(\tau) d\tau \right)^q \right)^1/q \left\| g \right\|_{p_2,v_2,\mathbb{R}^n} \\
\approx \sup_{t \in (0,\infty)} \left\| v_1^{-1/p_1} \right\|_{p_1',\mathbb{R}^n} \sup_{y \in (0,\infty)} \left( \int_0^y u \right)^1/q \left\| v_2^{-1/p_2} \right\|_{p_2',\mathbb{R}^n} \\
= \sup_{t \in (0,\infty)} U(t)^{1/q} \left\| v_1^{-1/p_1} \right\|_{p_1',\mathbb{R}^n} \left\| v_2^{-1/p_2} \right\|_{p_2',\mathbb{R}^n}.
\]

(b) Let $1 \leq p_2 < \infty$, $0 < q < p_2$ and $1/r_2 = 1/q - 1/p_2$. Interchanging the suprema, by \[21, \text{Theorem 2.2, (b) and (g)}, \text{we obtain}

\[
\sup_{f, g \in \mathcal{M}^+(\mathbb{R}^n)} \frac{\left\| \int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} g \right\|_{q,u,(0,\infty)}}{\left\| f \right\|_{p_1,v_1,\mathbb{R}^n} \left\| g \right\|_{p_2,v_2,\mathbb{R}^n}} \\
\approx \sup_{t \in (0,\infty)} \left\| v_1^{-1/p_1} \right\|_{p_1',\mathbb{R}^n} \sup_{g \in \mathcal{M}^+(\mathbb{R}^n)} \left( \int_0^t \left( \int_0^y \chi(0,t)(\tau)u(\tau) d\tau \right)^q \right)^1/q \left\| g \right\|_{p_2,v_2,\mathbb{R}^n} \\
\approx \sup_{t \in (0,\infty)} \left\| v_1^{-1/p_1} \right\|_{p_1',\mathbb{R}^n} \times \\
\times \left( \int_0^y \chi(0,t)(\tau)u(\tau) d\tau \right)^{r_2/p_2} \chi(0,t)(y)u(y) \left\| v_2^{-1/p_2} \right\|_{p_2',\mathbb{R}^n} \left. dy \right|_{y}^{r_2} \\
= \sup_{t \in (0,\infty)} \left\| v_1^{-1/p_1} \right\|_{p_1',\mathbb{R}^n} \left( \int_0^t U(y)^{r_2/p_2} u(y) \left\| v_2^{-1/p_2} \right\|_{p_2',\mathbb{R}^n} \left. dy \right|_{y}^{r_2} \right).
Moreover, the best constant
\[ C = \lim_{\|f\|_{p_2,\mathbb{R}^n} \to 0} \left( \int_0^{\infty} \left( \int_{\mathbb{R}^n} f \cdot f_{\mathbb{R}^n}(\xi) \right)^{\frac{q}{r_1}} d\xi \right)^{1/q} \]
\[ = \sup_{f,g \in \mathcal{B}(\mathbb{R}^n)} \frac{\left( \int_0^{\infty} \left( \int_{\mathbb{R}^n} f \cdot f_{\mathbb{R}^n}(\xi) \right)^{\frac{q}{r_1}} d\xi \right)^{1/q}}{\|f\|_{p_1,\mathbb{R}^n} \|g\|_{p_2,\mathbb{R}^n}} \]
\[ \approx \sup_{t \in (0,\infty)} \|v_{1,t}^{-1/p_1}\|_{p_1,\mathbb{R}^n} \sup_{g \in \mathcal{B}(\mathbb{R}^n)} \left( \int_0^{\infty} \left( \int_0^{\infty} \frac{\chi(t)}{U^{r_1/q}} dt \right)^{1/q} \right) \cdot \left( \int_0^{\infty} \left( \int_0^{\infty} \frac{\chi(t)}{U^{r_1/q}} dt \right)^{1/q} \right)^{1/q} \]
Moreover, the best constant $C$ in (2) satisfies $C \approx A_2$.

(iii) $r_1 < p_2 < \infty$, $1/r_2 = 1/q - 1/p_2$, $1/l = 1/r_1 - 1/p_2$, and

\[
A_3 := \left( \int_{(0,\infty)} \left( \int_{(0,\infty)} U(x, t)^{r_1/q} U(t)^{r_1/q} d \left( - \| v_1^{-1/p_1} \|^2_{p_1', cB(0,t)} \right) \right)^{(1-r_1)/r_1} \right)
\times \left( \int_{(0,\infty)} U(t, x)^{r_2/q} d \left( - \| v_2^{-1/p_2} \|^2_{p_2', cB(t,\infty)} \right) \right)^{1/r_2}
\times U(x)^{r_1/q} d \left[ - \| v_1^{-1/p_1} \|^2_{p_1', cB(0,x)} \right]^{1/l} < \infty.
\]

Moreover, the best constant $C$ in (2) satisfies $C \approx A_3$.

(iv) $p_2 = \infty$, and

\[
A_4 := \left( \int_{(0,\infty)} \left( \int_{(0,\infty)} U(t, x) d \left( - \| v_2^{-1/q} \|^q_{1, S[t,\infty]} \right) \right)^{r_1/q} \right)
\times U(x)^{r_1/q} d \left[ - \| v_1^{-1/p_1} \|^2_{p_1', cB(0,x)} \right]^{1/r_1} < \infty.
\]

Moreover, the best constant $C$ in (2) satisfies $C \approx A_4$.

Proof. Assume that $1 \leq p_1, p_2 < \infty$, $0 < q < p_1$ and $1/r_1 = 1/q - 1/p_1$. By [21, Theorem 2.2, (b) and (g)], (34) yields

\[
\sup_{f, g \in \mathcal{M}^+(\mathbb{R}^n)} \left( \int_{0}^{\infty} \left( \int_{S(0,t)} f \cdot f_{\mathcal{C}(0,t)} g \right)^q u(t) dt \right)^{1/q}
\frac{1}{\left( \int_{\mathbb{R}^n} f^{p_1 v_1} \right)^{1/p_1} \left( \int_{\mathbb{R}^n} g^{p_2 v_2} \right)^{1/p_2}}
= \sup_{g \in \mathcal{M}^+(\mathbb{R}^n)} \left( \int_{0}^{\infty} \left( \int_{0}^{t} \Psi(t) dt \right)^{r_1/p_1} \Psi(x) \left\| v_1^{-1/p_1} \|^2_{p_1', cB(0,x)} \right) \right)^{1/r_1}
\frac{1}{\left( \int_{\mathbb{R}^n} g^{p_2 v_2} \right)^{1/p_2}},
\]

where

\[
\Psi(t) := \left( \int_{S(0,t)} g \right)^q u(t).
\]
By Proposition 2, in view of \( \lim_{t \to \infty} \| v_1^{-1/p_1} \|_{p_1, B(0,t)} = 0 \), we have

\[
\left( \int_0^\infty \left( \int_0^x \Psi(t) \, dt \right)^{r_1/p_1} \Psi(x) \| v_1^{-1/p_1} \|_{p_1, B(0,x)} \, dx \right)^{1/r_1} \\
\approx \left( \int_{(0,\infty)} \left( \int_0^x \Psi(t) \, dt \right)^{r_1/q} d \left[ - \| v_1^{-1/p_1} \|_{p_1, B(0,x)} \right]^{1/r_1}.
\]

Thus

\[
\sup_{f,g \in \mathcal{M}^+(\mathbb{R}^n)} \left( \int_0^\infty \left( \int_{B(0,t)} f \cdot \int_{B(0,t)} g \right)^q u(t) \, dt \right)^{1/q} \\
= \sup_{g \in \mathcal{M}^+(0,\infty)} \left( \int_{\mathbb{R}^n} g^{p_2} v_2 \right)^{1/p_2} \\
= \sup_{g \in \mathcal{M}^+(0,\infty)} \left( \int_{\mathbb{R}^n} g^{p_2} v_2 \right)^{1/p_2}.
\]

(i) The statement follows by Theorem 2, (i).

(ii) The statement follows by Theorem 2, (iii).

(iii) The statement follows by Theorem 2, (iv).

(iv) The statement follows by Theorem 2, (v).

In the limiting case when \( p_1 = \infty \) we obtain the following statement.

**Theorem 5.** Let \( 1 \leq p_2 \leq \infty \), \( 0 < q < \infty \), and let \( u \in \mathcal{W}(0, \infty) \), \( v_1, v_2 \in \mathcal{W}(\mathbb{R}^n) \). Then inequality

\[
\left( \int_0^\infty \left( \int_{B(0,t)} f \cdot \int_{B(0,t)} g \right)^q u(t) \, dt \right)^{1/q} \leq C \| f \|_{\infty, v_1, \mathbb{R}^n} \| g \|_{p_2, v_2, \mathbb{R}^n}
\]

holds for all \( f, g \in \mathcal{M}^+(\mathbb{R}^n) \) if and only if:
(i) \( p_2 \leq q \), and
\[
D_1 := \sup_{t \in (0, \infty)} \left( \int_0^t u(y) \left\| v_1^{-1} \right\|_{1, \mathcal{V}^1_B(0,y)} \, dy \right)^{1/q} \left\| v_2^{-1/p_2} \right\|_{p_2', \mathcal{V}^1_B(0,t)} < \infty.
\]
Moreover, the best constant \( C \) in (35) satisfies \( C \approx D_1 \).

(ii) \( q < p_2 < \infty \), \( 1/r_2 = 1/q - 1/p_2 \), and
\[
D_2 := \left( \int_0^\infty \left( \int_0^t u(\tau) \left\| v_1^{-1} \right\|_{1, \mathcal{V}^1_B(0,\tau)} \, d\tau \right)^{r_2} \frac{1}{r_2} \times \right) \left( \int_0^t u(t) \left\| v_1^{-1} \right\|_{1, \mathcal{V}^1_B(0,t)} \left\| v_2^{-1/p_2} \right\|_{p_2', \mathcal{V}^1_B(0,t)} \, dt \right)^{1/q} < \infty.
\]
Moreover, the best constant \( C \) in (35) satisfies \( C \approx D_2 \).

(iii) \( p_2 = \infty \), and
\[
D_3 := \left( \int_0^\infty u(t) \left\| v_1^{-1} \right\|_{1, \mathcal{V}^1_B(0,t)} \left\| v_2^{-1/p_2} \right\|_{p_2', \mathcal{V}^1_B(0,t)} \, dt \right)^{1/q} < \infty.
\]
Moreover, the best constant \( C \) in (35) satisfies \( C \approx D_3 \).

**Proof.** By [21, Theorem 2.2, (e)], (34) yields
\[
\sup_{f, g \in \mathcal{M}^+(\mathbb{R}^n)} \left( \int_0^\infty \left( \int_0^t f \cdot g \right)^q u(t) \, dt \right)^{1/q} = \sup_{f, g \in \mathcal{M}^+(\mathbb{R}^n)} \left( \int_0^\infty \left( \int_0^t g \right)^q u(t) \left\| v_1^{-1} \right\|_{1, \mathcal{V}^1_B(0,t)} \, dt \right)^{1/q}.
\]
The proof follows by application of [21, Theorem 2.2].

We have the following statement when \( q = \infty \).

**Theorem 6.** Let \( 1 \leq p_1, p_2 \leq \infty \), and let \( u \in \mathcal{W}(0, \infty) \), \( v_1, v_2 \in \mathcal{W}(\mathbb{R}^n) \). Then inequality
\[
\text{ess sup}_{t \in (0, \infty)} \left( \int_{\mathcal{V}^1_B(0,t)} f \cdot \int_{\mathcal{V}^1_B(0,t)} g \right) u(t) \leq C \left\| f \right\|_{p_1, \mathbb{R}^n} \left\| g \right\|_{p_2, \mathbb{R}^n} \tag{36}
\]
holds for all \( f, g \in \mathcal{M}^+(\mathbb{R}^n) \) if and only if:
Moreover, the best constant $C$ in (36) satisfies $C \approx E_1$.

(b) $p_1 < \infty$, $p_2 = \infty$, and

$$ E_2 := \sup_{t \in (0, \infty)} u(t) \| v_1^{-1/p_1} \| p_1', \ast B(0, t) \| v_2^{-1/p_2} \| p_2', \ast B(0, t) < \infty. $$

Moreover, the best constant $C$ in (36) satisfies $C \approx E_2$.

(c) $p_1 = p_2 = \infty$, and

$$ E_3 := \sup_{t \in (0, \infty)} u(t) \| v_1^{-1} \| p_1', \ast B(0, t) \| v_2^{-1} \| p_2', \ast B(0, t) < \infty. $$

Moreover, the best constant $C$ in (36) satisfies $C \approx E_3$.

Proof. (a) and (b): Let $p_1 < \infty$. By [21, Theorem 2.2, (c) and (h)], (34) yields

$$ \sup_{f, g \in \mathbb{R}^n} \frac{\text{ess sup}_{t \in (0, \infty)} \left( \int f B(0, t) f \cdot f B(0, t) g \right) u(t)}{\| f \|_{p_1, \mathbb{R}_n} \| g \|_{p_2, \mathbb{R}_n}^n} $$

$$ \approx \sup_{g \in \mathbb{R}^n} \frac{1}{\| g \|_{p_2', \mathbb{R}_n}} \sup_{t \in (0, \infty)} \left( \text{ess sup}_{\tau \in (0, t)} \left( \int f B(0, \tau) g \right) u(\tau) \right) \| v_1^{-1/p_1} \| p_1', \ast B(0, t)'^{\ast}. $$

Interchanging the suprema, by duality, on using (10), we arrive at

$$ \sup_{f, g \in \mathbb{R}^n} \frac{\text{ess sup}_{t \in (0, \infty)} \left( \int f B(0, t) f \cdot f B(0, t) g \right) u(t)}{\| f \|_{p_1, \mathbb{R}_n} \| g \|_{p_2, \mathbb{R}_n}^n} $$

$$ \approx \sup_{t \in (0, \infty)} \| v_1^{-1/p_1} \| p_1', \ast B(0, t) \left( \text{ess sup}_{\tau \in (0, t)} \left( \sup_{g \in \mathbb{R}^n} \frac{\int f B(0, \tau) g}{\| g \|_{p_2, \mathbb{R}_n}^n} \right) \right) $$

$$ = \sup_{t \in (0, \infty)} \| v_1^{-1/p_1} \| p_1', \ast B(0, t) \left( \text{ess sup}_{\tau \in (0, t)} \| v_2^{-1/p_2} \| p_2', \ast B(0, \tau) \right) $$

$$ = \text{ess sup}_{t \in (0, \infty)} u(t) \| v_1^{-1/p_1} \| p_1', \ast B(0, t) \| v_2^{-1/p_2} \| p_2', \ast B(0, t). $$
when \( p_2 < \infty \), and
\[
\sup_{f,g \in \mathcal{M}^+ (\mathbb{R}^n)} \text{ess sup}_{t \in (0, \infty)} \left( \frac{\int_{B(0,t)} f \cdot \int_{B(0,t)} g}{\|f\|_{p_1, v_1, \mathbb{R}^n} \|g\|_{\infty, v_2, \mathbb{R}^n}} \right) u(t)
\]
\[
\approx \text{ess sup}_{t \in (0, \infty)} u(t) \|v_1^{-1/p_1}\|_{1, B(0,t)} \|v_2^{-1}\|_{1, B(0,t)}.
\]

when \( p_2 = \infty \).

(c) Let \( p_1 = p_2 = \infty \). By [21, Theorem 2.2, (d)], (34) yields
\[
\sup_{f,g \in \mathcal{M}^+ (\mathbb{R}^n)} \text{ess sup}_{t \in (0, \infty)} \left( \frac{\int_{B(0,t)} f \cdot \int_{B(0,t)} g}{\|f\|_{\infty, v_1, \mathbb{R}^n} \|g\|_{\infty, v_2, \mathbb{R}^n}} \right) u(t)
\]
\[
\approx \sup_{g \in \mathcal{M}^+ (\mathbb{R}^n)} \frac{1}{\|g\|_{\infty, v_2, \mathbb{R}^n}} \sup_{t \in (0, \infty)} \left( \text{ess sup}_{\tau \in (0, t)} \left( \int_{B(0,\tau)} g \right) u(\tau) \right) \|v_1^{-1}\|_{1, B(0,t)}.
\]

Interchanging the suprema, by duality, on using (10), we arrive at
\[
\sup_{f,g \in \mathcal{M}^+ (\mathbb{R}^n)} \text{ess sup}_{t \in (0, \infty)} \left( \frac{\int_{B(0,t)} f \cdot \int_{B(0,t)} g}{\|f\|_{\infty, v_1, \mathbb{R}^n} \|g\|_{\infty, v_2, \mathbb{R}^n}} \right) u(t)
\]
\[
\approx \sup_{t \in (0, \infty)} \|v_1^{-1}\|_{1, B(0,t)} \left( \text{ess sup}_{\tau \in (0, t)} \left( \sup_{g \in \mathcal{M}^+ (\mathbb{R}^n)} \frac{\int_{B(0,\tau)} g}{\|g\|_{\infty, v_2, \mathbb{R}^n}} \right) u(\tau) \right) \|v_1^{-1}\|_{1, B(0,\tau)} \|v_2^{-1}\|_{1, B(0,\tau)}
\]
\[
= \sup_{t \in (0, \infty)} \|v_1^{-1}\|_{1, B(0,t)} \left( \text{ess sup}_{\tau \in (0, t)} u(\tau) \|v_2^{-1}\|_{1, B(0,\tau)} \right)
\]
\[
= \text{ess sup}_{t \in (0, \infty)} u(t) \|v_1^{-1}\|_{1, B(0,t)} \|v_2^{-1}\|_{1, B(0,t)}. \blacktriangleleft
\]

Acknowledgement

The research of Rza Mustafayev is supported by the grant of Karamanoğlu Mehmetbey University Scientific Research Project (FEF.09-M-18).
References


Nevin Bilgiçi
Republic of Turkey Ministry of National Education,
Kırıkkale High School, 71100, Kırıkkale, Turkey
E-mail: nevinbilgicli@gmail.com
Multidimensional Bilinear Hardy Inequalities

Rza Mustafayev
Department of Mathematics, Faculty of Science,
Karamanoğlu Mehmetbey University, Karaman, 70100, Turkey
E-mail: rzamustafayev@gmail.com

Tuğçe Ünver
Department of Mathematics, Faculty of Science and
Arts, Kirikkale University, 71450 Yahşihan, Kirikkale, Turkey
E-mail: tugceunver@gmail.com

Received 09 September 2019
Accepted 27 October 2019