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On the Number of Lattice Points in the Shifted Circles

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Abstract. In this work we study the question: for a given r, is there any point (x, y) on the plane such that the deviation of the number of lattice points, contained inside the circle of radius r centered at this point, from the area of the disc has order $O(r^{1/2+\varepsilon})$ for arbitrary positive number ε? We show that the answer for this question is positive.

Key Words and Phrases: circle problem, lattice points, shifted circle, deviation.

2010 Mathematics Subject Classifications: 11P21, 11L03, 11Y35

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1. Introduction

In number theory, the question on the number of lattice points enclosed in various domains on the plane is studied. Let $\Omega \subset R^2$ be some closed domain bounded by a smooth closed curve. Denote by $|\Omega|$ an area of the domain Ω. From geometric reasoning it is clear that the number of lattice points enclosed by the domain Ω is approximately equal to the area of the domain. In case the domain is a polygon with vertices at lattice points, the number T of lattice points can be represented in terms of the area of the polygon as follows:

$$T = |\Omega| - 1,$$

moreover, $T = \sum \delta$ is a sum taken along all lattice points on the polygon where we put $\delta = 1$ for interior points, and $\delta = 1/2$ for lattice points on boundary (see [14]). For other kinds of domains, we have a formula

$$T = |\Omega| + R,$$

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with some error term R. The problem consists in finding a best possible estimate of the error term. Note that here we consider all lattice points belonging to the closed domain with the boundary.

For the lattice point problem for the circle $x^2 + y^2 \leq r^2$ with center at origin, the following estimate has been proved using an elementary method:

$$R \leq 2\pi\sqrt{2}r, (K.Gauss(1834))$$

(see [2, p. 64]). V. Serpinskii (1906) ([3, 13]) improved this estimate using complicated analytic method to prove that

$$R << r^{2/3}.$$

In 1917 I. M. Vinogradov found an elementary proof of the estimate (see [13, p. 192.])

$$R << (r \log r)^{2/3}.$$

Despite that the method used by I. M. Vinogradov is elementary, Yarnik [8, 13] showed that Vinogradov's main result can't be improved in a similar way. The following results have been obtained later in this field (see [1-7]):

$$R << r^{\frac{326}{494}+\varepsilon}, \ by \ Walfisz \ (1927),$$

$$R << r^{\frac{15}{23}+\varepsilon}, \ by \ Titchmarsh \ (1935),$$

$$R << r^{\frac{13}{20}+\varepsilon}, \ by \ Hua \ (1942).$$

The best result for today is

$$R << r^{\frac{131}{208}+\varepsilon},$$

with arbitrary $\varepsilon > 0$, proven in 2003 by M. Huxley ([6]). There is a conjecture ([1, 13]) which states that

$$R << r^{\frac{1}{2}+\varepsilon}. \tag{1}$$

In [9] this conjecture has been verified for the large values (up to 250000) of the radius. Hardy and Landau ([1, 4, 11]) had independently shown that the estimate

$$R = o(r^2 \log r)^{\frac{1}{4}}$$

couldn't be proven.

In [1], the probabilistic aspects of the same problem for shifted circles has been considered. The authors in [1] investigate the fluctuations in $N_\alpha(R)$, the number of lattice points $n \in Z^2$ inside a circle of radius R centered at a fixed point $\alpha \in [0,1)^2$. Assuming that R is smoothly (e.g., uniformly) distributed on a segment $0 \le R \le T$, they proved that the random variable

$$\frac{N_\alpha(R) - \pi R^2}{\sqrt{R}}$$

has a limit distribution as $T \to \infty$, which is absolutely continuous (see [12, p. 364]) with respect to Lebesgue measure.

In this work we study the question: is there, for a given r, a point (x, y) on the plane such that the estimate (1) is valid for the circle of radius r with center at the point (x, y) ? We show that the answer for this question is positive. Proving this result in more exact form, we then find the points (x, y) for which the estimate (1) is most exact for various r.

2. Some auxiliary lemmas

The following lemma is known as Sonin's formula (see [3, p.200]).

Lemma 1. *Let a and b be real numbers, $a < b$. Then the equality*

$$\sum_{a<n\le b} f(n) = \int_a^b f(x)dx + \rho(b)f(b) - \rho(a)f(a) - \sigma(b)f'(b) +$$

$$+\sigma(a)f'(a) + \int_a^b f''(x)\sigma(x)dx,$$

holds, where $f(x)$ is a function defined in the interval $(a, b]$ with continuous derivative of second order

$$\rho(x) = \{x\} - 1/2,\ \sigma(x) = \int_0^x \rho(t)dt.$$

Lemma 2. *Let α be a non-integral real number. Then*

$$\{-\alpha\} = 1 - \{\alpha\}.$$

Proof. We have

$$\{-\alpha\} = \{1-\alpha\} = \{1 - ([\alpha] + \{\alpha\})\} = \{-[\alpha] + 1 - \{\alpha\}\} = 1 - \{\alpha\}.$$

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Lemma 3. *Let the integral $\int_a^b f(x)dx$ exist for $a < b$, and $\psi(x) \geq 0$ be monotonically non-increasing function in $[a, b]$. Suppose that*

$$\psi(a) \leq c_1,$$
$$\int_a^y f(x)dx \leq c_2$$

for $y \in (a, b)$. Then

$$\int_a^b f(x)\psi(x)dx \leq c_1 c_2.$$

Proof of Lemma 3 is given in [10].

Lemma 4. *Let the function $f'(x) \geq \delta > 0$ be monotonically non-increasing in $[a, b]$, and $f'(x) \geq \delta > 0$ in this segment. Then*

$$\left| \int_a^b e^{2\pi i f(x)} dx \right| \leq 4\delta^{-1}.$$

Proof. Using the equality

$$e^{i\varphi} = \cos\varphi + i \sin\varphi,$$

we can estimate two trigonometric integrals of the form

$$\int_a^b \cos(f(x))dx = \int_a^b \frac{1}{f'(x)} \cos(f(x))f'(x)dx.$$

Applying Theorem 403 of [10], we find

$$\left| \int_a^b \cos(f(x))dx \right| = \left| \frac{1}{f'(a)} \int_a^\xi \cos(f(x))f'(x)dx \right| \leq \frac{2}{\delta};\ \xi \in (a, b).$$

Lemma 3 is proven. ◄

The following lemma is Theorem 409 from [10 p. 374].

Lemma 5. *Let the condition*

$$f''(x) \geq A > 0$$

be satisfied for the function $f(x)$ on the interval $[a, b]$. Then

$$\left| \int_a^b e^{2\pi i f(x)} dx \right| \leq 12A^{-1/2}.$$

Lemma 6. *The function $\rho(x)$ has the following Fourier expansion:*

$$\rho(x) = \sum_{m=-\infty, m\neq 0}^{\infty} g_m e^{2\pi i m x}$$

with $g_m = -1/(2\pi i m)$. This lemma is evident.

3. Lattice points in shifted circles

The Gauss' circle problem consists in finding asymptotic relation for the number $N(r)$ of lattice points in the circle

$$x^2 + y^2 \leq r^2,$$

as $r \to \infty$, with the best possible error term.

Our goal is to get such an estimate for the case of very small shifting of the origin. We will show that in every subdomain of the unit quadrate $[-1, 0)^2$ there exists shifting of the origin to some point in this quadrate for which (1) is satisfied. Then we apply computer methods to search for the best shifting, providing numerical results. The program is written in Python.

Consider a shifted circle

$$(x + \theta)^2 + (y + \eta)^2 \leq r^2;\ 0 \leq \theta, \eta \leq 1 \tag{2}$$

with large radius r (see Fig. 1).

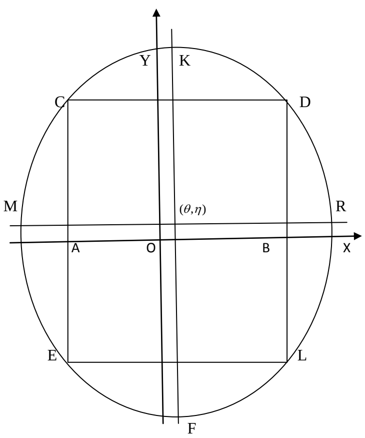

Figure 1.

Denote by $N(r; \theta, \eta)$ the number of lattice points in this circle.

Theorem 1. *Following inequality holds for sufficiently large $r > 0$:*

$$\int_0^1 \int_0^1 \left| N(r; \theta, \eta) - \pi r^2 \right|^2 d\theta d\eta \leq 256r.$$

Proof. If $1 > \theta > 0, 1 > \eta > 0$, then the lines $y = \theta, x = \eta$ do not contain lattices. The number $N(R; \theta, \eta)$ can be represented as a sum:

$$N(r; \theta, \eta) = N_1 + N_2 - N_3,$$

where N_1 is a number of lattice points contained in the union of quadrate $ECDL$ (see Fig. 1) and segments CKD and EFL, N_2 is a number of lattice points contained in the union of quadrate $ECDL$ and segments CME and DRL, and N_3 is a number of lattice points contained in the quadrate $ECDL$. The numbers of the lattice points N_1, N_2, and N_3 can be represented as follows:

$$N_1 = \sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta} \left[-\eta + \sqrt{r^2 - (n+\theta)^2}\right] + 1 + \left[r/\sqrt{2} - \theta\right] + \left[r/\sqrt{2} + \theta\right] +$$

$$+ \sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta} \left[\eta + \sqrt{r^2 - (n+\theta)^2}\right],$$

$$N_2 = \sum_{-\frac{r}{\sqrt{2}}-\eta<n<\frac{r}{\sqrt{2}}-\eta} \left[-\theta + \sqrt{r^2 - (n+\eta)^2}\right] + 1 + \left[r/\sqrt{2} - \eta\right] + \left[r/\sqrt{2} + \eta\right] +$$

$$+ \sum_{-\frac{r}{\sqrt{2}}-\eta<n<\frac{r}{\sqrt{2}}-\eta} \left[\theta + \sqrt{r^2 - (n+\eta)^2}\right],$$

$$N_3 = \left(1 + \left[r/\sqrt{2} - \theta\right] + \left[r/\sqrt{2} + \theta\right]\right) \left(1 + \left[r/\sqrt{2} - \eta\right] + \left[r/\sqrt{2} + \eta\right]\right).$$

To estimate the sum N_1, we write:

$$N_1 = \sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta} \left(-\eta + \sqrt{r^2 - (n+\theta)^2} - \left\{-\eta + \sqrt{r^2 - (n+\theta)^2}\right\}\right) +$$

$$+1 + \left[r/\sqrt{2} - \theta\right] + \left[r/\sqrt{2} + \theta\right] +$$

$$+ \sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta} \left(\eta + \sqrt{r^2 - (n+\theta)^2} - \left\{\eta + \sqrt{r^2 - (n+\theta)^2}\right\}\right). \qquad (3)$$

Below we integrate these sums with respect to θ and η, so we can assume that the limits of summation above are non-integral. Consider first the sum

$$V = \sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta} \left(-\eta + \sqrt{r^2 - (n+\theta)^2}\right) + \sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta} \left(\eta + \sqrt{r^2 - (n+\theta)^2}\right).$$

To estimate this sum, we will use Lemma 1. We have

$$V_1=\sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta}\left(-\eta+\sqrt{r^2-(n+\theta)^2}\right)$$

$$=\int_{-r/\sqrt{2}-\theta}^{r/\sqrt{2}-\theta}\left(-\eta+\sqrt{r^2-(x+\theta)^2}\right)dx+$$

$$+\rho\left(r/\sqrt{2}-\theta\right)\left(r/\sqrt{2}-\eta\right)-\rho\left(-r/\sqrt{2}-\theta\right)\left(r/\sqrt{2}-\eta\right)+$$

$$-\sigma\left(r/\sqrt{2}-\theta\right)+\sigma\left(-r/\sqrt{2}-\theta\right)+r^2\int_{-r/\sqrt{2}-\theta}^{r/\sqrt{2}-\theta}\frac{\sigma(x)dx}{(r^2-(x+\theta)^2)^{3/2}}=$$

$$=\int_{-r/\sqrt{2}-\theta}^{r/\sqrt{2}-\theta}\left(-\eta+\sqrt{r^2-(x+\theta)^2}\right)dx+\Delta+$$

$$+\rho\left(r/\sqrt{2}-\theta\right)\left(r/\sqrt{2}-\eta\right)-\rho\left(-r/\sqrt{2}-\theta\right)\left(r/\sqrt{2}-\eta\right). \tag{4}$$

By the evident relation $\sigma(x)\le 1/8$ (see [12, p.12]), we have

$$|\Delta|=\left|-\sigma\left(r/\sqrt{2}-\theta\right)+\sigma\left(-r/\sqrt{2}-\theta\right)+r^2\int_{-r/\sqrt{2}-\theta}^{r/\sqrt{2}-\theta}\frac{\sigma(x)dx}{(r^2-(x+\theta)^2)^{3/2}}\right|\le 1/2.$$

Similarly,

$$V_2=\sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta}\left(\eta+\sqrt{r^2-(n+\theta)^2}\right)=\int_{-r/\sqrt{2}-\theta}^{r/\sqrt{2}-\theta}\left(\eta+\sqrt{r^2-(x+\theta)^2}\right)dx+$$

$$+\rho\left(r/\sqrt{2}-\theta\right)\left(r/\sqrt{2}+\eta\right)-\rho\left(-r/\sqrt{2}-\theta\right)\left(r/\sqrt{2}+\eta\right)+$$

$$-\sigma\left(r/\sqrt{2}-\theta\right)+\sigma\left(-r/\sqrt{2}-\theta\right)+r^2\int_{-r/\sqrt{2}-\theta}^{r/\sqrt{2}-\theta}\frac{\sigma(x)dx}{(r^2-(x+\theta)^2)^{3/2}}=$$

$$=\int_{-r/\sqrt{2}-\theta}^{r/\sqrt{2}-\theta}\left(\eta+\sqrt{r^2-(x+\theta)^2}\right)dx+\Delta'+$$

$$+\rho\left(r/\sqrt{2}-\theta\right)\left(r/\sqrt{2}+\eta\right)-\rho\left(-r/\sqrt{2}-\theta\right)\left(r/\sqrt{2}+\eta\right), \tag{5}$$

and $|\Delta'| \leq 1/2$. The sum of the integrals on the right-hand sides of the equalities (4) and (5) expresses the area S_{EFLDKC} of the domain $EFLDKC$ (Fig. 1). From (3) we derive

$$N_1 = V_1 + V_2 + \sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta} \left(\frac{1}{2} - \left\{-\eta + \sqrt{r^2-(n+\theta)^2}\right\}\right) +$$

$$+ \sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta} \left(\frac{1}{2} - \left\{\eta + \sqrt{r^2-(n+\theta)^2}\right\}\right) =$$

$$= S_{ELDFKC} + \rho\left(r/\sqrt{2}-\theta\right)\left(r/\sqrt{2}-\eta\right) - \rho\left(-r/\sqrt{2}-\theta\right)\left(r/\sqrt{2}-\eta\right) +$$

$$+\rho\left(r/\sqrt{2}-\theta\right)\left(r/\sqrt{2}+\eta\right) - \rho\left(-r/\sqrt{2}+\theta\right)\left(r/\sqrt{2}-\eta\right) + \Delta + \Delta' +$$

$$+ \sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta} \left(\frac{1}{2} - \left\{-\eta + \sqrt{r^2-(n+\theta)^2}\right\}\right)$$

$$+ \sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta} \left(\frac{1}{2} - \left\{\eta + \sqrt{r^2-(n+\theta)^2}\right\}\right). \qquad (6)$$

Similarly, denoting

$$W_1 = \sum_{-\frac{r}{\sqrt{2}}-\eta<n<\frac{r}{\sqrt{2}}-\eta} \left(-\theta + \sqrt{r^2-(n+\eta)^2}\right),$$

$$W_2 = \sum_{-\frac{r}{\sqrt{2}}-\eta<n<\frac{r}{\sqrt{2}}-\eta} \left(\theta + \sqrt{r^2-(n+\eta)^2}\right),$$

we get:

$$N_2 = W_1 + W_2 + \sum_{-\frac{r}{\sqrt{2}}-\eta<n<\frac{r}{\sqrt{2}}-\eta} \left(\frac{1}{2} - \left\{-\theta + \sqrt{r^2-(n+\eta)^2}\right\}\right) +$$

$$+ \sum_{-\frac{r}{\sqrt{2}}-\eta<n<\frac{r}{\sqrt{2}}-\eta} \left(\frac{1}{2} - \left\{\theta + \sqrt{r^2-(n+\eta)^2}\right\}\right) =$$

$$= S_{CMEFRD} + \rho\left(r/\sqrt{2}-\eta\right)\left(r/\sqrt{2}-\theta\right) - \rho\left(-r/\sqrt{2}-\eta\right)\left(r/\sqrt{2}-\theta\right) +$$

$$+\rho\left(r/\sqrt{2}-\eta\right)\left(r/\sqrt{2}+\theta\right) - \rho\left(-r/\sqrt{2}-\eta\right)\left(r/\sqrt{2}+\theta\right) + \Delta'' + \Delta''' +$$

$$+ \sum_{-\frac{r}{\sqrt{2}}-\eta<n<\frac{r}{\sqrt{2}}-\eta} \left(\frac{1}{2} - \left\{-\theta + \sqrt{r^2-(n+\eta)^2}\right\}\right)$$

$$+ \sum_{-\frac{r}{\sqrt{2}}-\eta<n<\frac{r}{\sqrt{2}}-\eta} \left(\frac{1}{2} - \left\{\theta + \sqrt{r^2-(n+\eta)^2}\right\}\right), \tag{7}$$

where $|\Delta''|+|\Delta'''| \leq 1$. Then, summing up the obtained expressions and denoting $\Delta_0 = \Delta + \Delta' + \Delta'' + \Delta'''$, we find:

$$N(r;\theta,\eta) = N_1 + N_2 - N_3 =$$

$$= -\left(1 + \left[r/\sqrt{2}-\theta\right] + \left[r/\sqrt{2}+\theta\right]\right)\left(1 + \left[r/\sqrt{2}-\eta\right] + \left[r/\sqrt{2}+\eta\right]\right) +$$

$$+ S_{CMELRD} + S_{EFLDKC} + \left(-\left\{r/\sqrt{2}-\theta\right\} + \left\{-r/\sqrt{2}-\theta\right\}\right)\left(r/\sqrt{2}-\eta\right) +$$

$$+ \left(-\left\{r/\sqrt{2}-\theta\right\} + \left\{-r/\sqrt{2}-\theta\right\}\right)\left(r/\sqrt{2}+\eta\right) +$$

$$+ \sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta} \left(\frac{1}{2} - \left\{-\eta + \sqrt{r^2-(n+\theta)^2}\right\}\right) +$$

$$+ \sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta} \left(\frac{1}{2} - \left\{\eta + \sqrt{r^2-(n+\theta)^2}\right\}\right) +$$

$$+ \left(-\left\{r/\sqrt{2}-\eta\right\} + \left\{-r/\sqrt{2}-\eta\right\}\right)\left(r/\sqrt{2}-\theta\right) +$$

$$+ \left(-\left\{r/\sqrt{2}-\eta\right\} + \left\{-r/\sqrt{2}-\eta\right\}\right)\left(r/\sqrt{2}+\theta\right) + \Delta_0 +$$

$$+ \sum_{-\frac{r}{\sqrt{2}}-\eta<n<\frac{r}{\sqrt{2}}-\eta} \left(\frac{1}{2} - \left\{-\theta + \sqrt{r^2-(n+\eta)^2}\right\}\right) +$$

$$+ \sum_{-\frac{r}{\sqrt{2}}-\eta<n<\frac{r}{\sqrt{2}}-\eta} \left(\frac{1}{2} - \left\{\theta + \sqrt{r^2-(n+\eta)^2}\right\}\right).$$

It is easy to see that by Lemma 2

$$\left(1 + \left[r/\sqrt{2}-\theta\right] + \left[r/\sqrt{2}+\theta\right]\right)\left(1 + \left[r/\sqrt{2}-\eta\right] + \left[r/\sqrt{2}+\eta\right]\right) =$$

$$= \left(r\sqrt{2} - \left\{r/\sqrt{2}-\theta\right\} + \left\{-r/\sqrt{2}-\theta\right\}\right)$$

$$\left(r\sqrt{2} - \left\{r/\sqrt{2}-\eta\right\} + \left\{-r/\sqrt{2}-\eta\right\}\right) =$$

$$= 2r^2 + \left(-\left\{r/\sqrt{2}-\theta\right\} + \left\{-r/\sqrt{2}-\theta\right\}\right) r\sqrt{2}+$$

$$+ \left(-\left\{r/\sqrt{2}-\eta\right\} + \left\{-r/\sqrt{2}-\eta\right\}\right) r\sqrt{2} + \delta,$$

and

$$|\delta| = \left|\left(-\left\{r/\sqrt{2}-\theta\right\} + \left\{-r/\sqrt{2}-\theta\right\}\right)\left(-\left\{r/\sqrt{2}-\eta\right\} + \left\{-r/\sqrt{2}-\eta\right\}\right)\right| \le 1.$$

Denote

$$\delta_0 = \delta_0(r;\theta,\eta) =$$

$$\left(-\left\{r/\sqrt{2}-\theta\right\} + \left\{-r/\sqrt{2}-\theta\right\}\right)\left(-\left\{r/\sqrt{2}-\eta\right\} + \left\{-r/\sqrt{2}-\eta\right\}\right) + \Delta_0.$$

Now we have

$$N(r;\theta,\eta) = \pi r^2 + \sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta} \left(\frac{1}{2} - \left\{-\eta + \sqrt{r^2-(n+\theta)^2}\right\}\right) +$$

$$+ \sum_{-\frac{r}{\sqrt{2}}-\theta<n<\frac{r}{\sqrt{2}}-\theta} \left(\frac{1}{2} - \left\{\eta + \sqrt{r^2-(n+\theta)^2}\right\}\right)$$

$$+ \sum_{-\frac{r}{\sqrt{2}}-\eta<n<\frac{r}{\sqrt{2}}-\eta} \left(\frac{1}{2} - \left\{-\theta + \sqrt{r^2-(n+\eta)^2}\right\}\right) +$$

$$+ \sum_{-\frac{r}{\sqrt{2}}-\eta<n<\frac{r}{\sqrt{2}}-\eta} \left(\frac{1}{2} - \left\{\theta + \sqrt{r^2-(n+\eta)^2}\right\}\right) + \delta_0(r;\theta,\eta);\ |\delta_0| \le 3.$$

Therefore, by Cauchy inequality,

$$\int_0^1\int_0^1 \left|N(r;\theta,\eta) - \pi r^2 - \delta_0(r;\theta,\eta)\right|^2 d\theta d\eta \le$$

$$\le 4\int_0^1\int_0^1 \left|\sum_{\sqrt{r}<|n+\theta|<\frac{r}{\sqrt{2}}} \rho\left(-\eta + \sqrt{r^2-(n+\theta)^2}\right)\right|^2 d\theta d\eta +$$

$$+4\int_0^1\int_0^1 \left|\sum_{\sqrt{r}<|n+\theta|<\frac{r}{\sqrt{2}}} \rho\left(\eta + \sqrt{r^2-(n+\theta)^2}\right)\right|^2 d\theta d\eta +$$

$$+4\int_0^1\int_0^1\left|\sum_{\sqrt{r}<|n+\eta|<\frac{r}{\sqrt{2}}}\rho\left(-\theta+\sqrt{r^2-(n+\eta)^2}\right)\right|^2 d\theta d\eta+$$

$$+4\int_0^1\int_0^1\left|\sum_{\sqrt{r}<|n+\eta|<\frac{r}{\sqrt{2}}}\rho\left(\theta+\sqrt{r^2-(n+\eta)^2}\right)\right|^2 d\theta d\eta. \tag{8}$$

All of integrals on the right-hand side of the last inequality can be estimated in the same way. So, it suffices to estimate the first integral. We have

$$\int_0^1\int_0^1\left|\sum_{\sqrt{r}<|n+\theta|<\frac{r}{\sqrt{2}}}\rho\left(\eta+\sqrt{r^2-(n+\theta)^2}\right)\right|^2 d\theta d\eta \le$$

$$\le 2\int_0^1\int_0^1\left|\sum_{\sqrt{r}<n<\frac{r}{\sqrt{2}}}\rho\left(\eta+\sqrt{r^2-(n+\theta)^2}\right)\right|^2 d\theta d\eta+4. \tag{9}$$

By Parseval's equality we have

$$\int_0^1\int_0^1\left|\sum_{\sqrt{r}<n<\frac{r}{\sqrt{2}}}\rho\left(\eta+\sqrt{r^2-(n+\theta)^2}\right)\right|^2 d\theta d\eta =$$

$$=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\left|c_{pq}\right|^2,$$

where

$$c_{pq}=\int_0^1\int_0^1\left(\sum_{\sqrt{r}<n<\frac{r}{\sqrt{2}}}\rho\left(\eta+\sqrt{r^2-(n+\theta)^2}\right)\right)e^{-2\pi i(p\theta+q\eta)}d\theta d\eta =$$

$$=\int_0^1\sum_{m=-\infty,m\neq 0}^{\infty}g_m d\eta\int_0^1 d\theta\sum_{\sqrt{r}<n<r/\sqrt{2}}e^{2\pi i\left[m\left(\eta+\sqrt{r^2-(n+\theta)^2}\right)-q\eta-p\theta\right]}.$$

Since

$$\int_0^1 e^{2\pi i(m-q)\eta}d\eta=0$$

when $m \neq n$, and is equal to 1 when $m = n$, the previous integral is equal to

$$g_q \int_0^1 d\theta \sum_{\sqrt{r}<n<r/\sqrt{2}} e^{2\pi i\left[q\left(\sqrt{r^2-(n+\theta)^2}\right)-p(n+\theta)\right]} =$$

$$= g_q \int_{[\sqrt{r}]+1}^{[r/\sqrt{2}]+1} e^{2\pi i\left[q\left(\sqrt{r^2-(n+\theta)^2}\right)-p\theta\right]} d\theta.$$

So, we have

$$\int_0^1 \int_0^1 \left| \sum_{\sqrt{r}<|n+\theta|<\frac{r}{\sqrt{2}}} \rho\left(\eta + \sqrt{r^2-(n+\theta)^2}\right) \right|^2 d\theta d\eta \leq \sum_{q=-\infty, q\neq 0}^{\infty} \sum_{p=-\infty}^{\infty} |c_{qp}|^2 \leq$$

$$\leq \sum_{q=-\infty, q\neq 0}^{\infty} \sum_{p=-\infty}^{\infty} |g_q|^2 \left| \int_{[\sqrt{r}]+1}^{[r/\sqrt{2}]+1} e^{2\pi i\left[q\left(\sqrt{r^2-(n+\theta)^2}\right)-p\theta\right]} d\theta \right|^2. \qquad (10)$$

To estimate the inner integral, apply Lemmas 4 and 5 with $f(\theta) = q\sqrt{r^2-(n+\theta)^2} - p\theta$. Since

$$\left|f''(\theta)\right| \geq qr^{-1},$$

by Lemma 5 we have

$$\left| \int_{[\sqrt{r}]+1}^{[r/\sqrt{2}]+1} e^{2\pi i\left[q\left(\sqrt{r^2-(n+\theta)^2}\right)-p\theta\right]} d\theta \right|^2 \leq \frac{12r}{q},$$

when $|p| \leq 2q$. When $|p| > 2q$ we have

$$\left|f'(\theta)\right| = \left| \frac{-q(n+\theta)}{\sqrt{r^2-(n+\theta)^2}} - p \right| \geq p/2,$$

and by Lemma 4

$$\left| \int_{[\sqrt{r}]+1}^{[r/\sqrt{2}]+1} e^{2\pi i\left[q\left(\sqrt{r^2-(n+\theta)^2}\right)-p\theta\right]} d\theta \right|^2 \leq \frac{64}{p^2}.$$

Therefore, for the last sum in (10) we have the bound

$$\sum_{q=-\infty, q\neq 0}^{\infty} \sum_{p=-\infty}^{\infty} |g_q|^2 \left| \int_{[\sqrt{r}]+1}^{[r/\sqrt{2}]+1} e^{2\pi i\left[q\left(\sqrt{r^2-(n+\theta)^2}\right)-p\theta\right]} d\theta \right|^2 \leq$$

$$\leq 48r \sum_{q=-\infty,q\neq 0}^{\infty} |g_q|^2 + 32 \sum_{q=-\infty,q\neq 0}^{\infty} |g_q|^2 \sum_{|p|>2q}^{\infty} p^{-2} \leq$$

$$\leq \frac{96r+64}{4\pi^2} \sum_{q=1}^{\infty} q^{-2} = 4r+3.$$

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Now from the relations (8)-(10) the statement of the theorem follows easily.

Corollary 1. *There exists a point* $(\theta,\eta) \in [0,1)$ *such that*

$$\left|N(r;\theta,\eta) - \pi r^2\right| \leq 16\sqrt{r},$$

for all $r \geq 2$.

Proof. From Theorem 1 we deduce for some (θ,η):

$$\left|N(r;\theta,\eta) - \pi r^2\right| \leq 8\sqrt{r+3/4} + 3 \leq$$

$$\leq \sqrt{2(64(r+3/4)+9)} \leq \sqrt{128r+128} \leq 16\sqrt{r}$$

when $r \geq 2$.

The proved theorem and its corollary give no information about the location of the point (θ,η) inside the quadrate $[-1,0)$. If we use the theorem to localize the point (θ,η) considering a small quadrate centered at this point, then the coefficient on the right-hand side of the inequality stated in the corollary will grow in inverse proportion to the side of the quadrate.

To get some empirical information on the location of the points (θ,η), we need computer calculations. We divide the unit quadrate into several small quadrates and consider circles of radius r centered at the vertex points (θ,η) of these quadrates. The program written in Python allows us to compute the number of lattice points in the considered circles and to estimate the relative deviation of the number of lattice points from the area of the disc. The software is given below.

```
import math
R = int(input("insert a radius of a circle"))
H = int(input("insert the number of shifting steps"))
def Disc (k,l):
  i=0
  for x in range(-R,R+1):
    for y in range(-R,R+1):
      while (H*x-k)**2+(H*y-l)**2<=(H*R)**2:
```

```
            i+=1
            break
        y+=1
    x+=1
print ("the number of lattice points in the circle is N(R)=", i)
print ("the relative deviation is",(math.pi*R**2-i)/(math.sqrt(R)) )
for k in range(H):
    for l in range(H):
        print (k,l)
        Disc (k,l)
        l+=1
    k+=1
```

Here H is the number of parts the interval $(0,1)$ is divided into. The r denotes the radius of the circle. The radii and the number of division points are entered from keyboard. We enter $H=5$ for the number of steps. For r we used the values: $500, 600, 800, 1000$, $2000, 5000$ and 6400. In the last case we assume $H=6$. For every pair (u,v) $(0 \le u < H,\ 0 \le v < H)$ we compute the number of lattice points $N = N(u,v;\,r)$ in the circles with center at the point $(u/H,\ v/H)$ and the value of relative deviation $d = \frac{\pi r^2 - N(u,v;\,r)}{\sqrt{r}}$. For the results of calculations we preserve 8 digits after the decimal point. Since the value of math.pi is taken with an error of at most 10^{-11}, the nominator of the fraction d is known with an error of at most 10^{-3}. Since $\sqrt{r} < 100$, at least one of these 8 decimal digits after the decimal point is a right digit. This is sufficient for establishing approximate shifting for which the relative deviation d is small or large.

Computer calculation analysis was made for cases of radii $r = 500, 600, 800$, $1000, 2000, 5000$ and 6400. In each case, a relative (d_{min}) deviation with minimal absolute value, a relative (d_{max}) deviation with maximal absolute value, and the centers at which these values are reached have been found.

In the case of $r = 500$: $d_{min} = 0.052$ is reached at the centers $(1/5, 2/5)$, $(1/5, 4/5), (2/5, 1/5), (2/5, 4/5), (3/5, 1/5), (3/5, 4/5), (4/5, 2/5), (4/5, 3/5)$; $d_{max} = 2.1987$ is reached at the origin.

In the case of $r = 600$: $d_{min} = -0.14188$ is reached at the centers $(1/5, 1/5)$, $(1/5, 3/5), (4/5, 1/5), (4/5, 4/5)$; $d_{max} = -24.6147$ is reached at $(2/5, 3/5)$.

In the case of $r = 800$: $d_{min} = 0.2934$ is reached at the centers $(2/5, 1/5)$, $(2/5, 4/5), (3/5, 1/5), (3/5, 4/5), (4/5, 2/5)$; $d_{max} = 8.2983$ is reached at $(1/5, 2/5)$, $(1/5, 3/5)$.

In the case of $r = 1000$: $d_{min} = -0.1058$ is reached at the centers $(1/5, 1/5)$, $(1/5, 4/5), (4/5, 1/5), (4/5, 4/5)$; $d_{max} = 1.6651$ is reached at $(0, 1/5), (0, 4/5)$, $(1/5, 0), (4/5, 0)$.

In the case of $r = 2000$: $d_{min} = -0.2099$ is reached at the centers $(1/5, 2/5)$, $(1/5, 3/5), (2/5, 0), (3/5, 1/5), (4/5, 2/5), (4/5, 3/5)$; $d_{max} = 1.6237$ is reached at $(0, 2/5), (0, 3/5), (2/5, 1/5)$, $(2/5, 4/5), (3/5, 0), (3/5, 4/5)$.

In the case of $r = 5000$: $d_{min} = 0.1604$ is reached at the centers $(2/5, 2/5)$, $(2/5, 3/5), (3/5, 2/5), (3/5, 3/5)$; $d_{max} = 1.9706$ is reached at the origin.

In the case of $r = 6400$: $d_{min} = 0.0761$ is reached at the centers $(1/3, 1/3)$, $(1/3, 2/3), (1/3, 5/6), (2/3, 1/3), (2/3, 2/3)$; $d_{max} = 5.1761$ is reached at the origin. ◄

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