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# Solving Two Initial-Boundary Value Problems Including Fractional Partial Differential Equations By Spectral and Contour Integral Methods

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**Abstract.** Initial-boundary value problems including fractional partial differential equations are the mathematical models of physical problems and natural phenomena. In this paper, at first we consider a fractional partial differential equation which has no mixed term derivative with respect to spatial and time variable. We first consider the spectral problem, then its eigenvalues and eigenfunctions are calculated. After that the eigenvalues and eigenfunctions of the adjoint problem are calculated. By using these eigenfunctions and Mittag-Leffler functions the approximate solution is constructed. In second section, we consider differential equation which has a mixed term derivative. In this case, by using Laplace transformation, the analytic solution and approximate solution are calculated as integral expression over suitable closed contours by contour integral method. At the end, some examples are presented for several cases of different distributions of eigenvalues in complex plane.

**Key Words and Phrases**: initial-boundary value problem, fractional partial differential equation, spectral problem, contour integral, closed contour.

**2010 Mathematics Subject Classifications**: 35R11, 26A33

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## 1. Introduction

Initial-boundary value problems have been considered in more classical cases: Fourier (separation of variables) and Fourier-Birkhoff methods are applied to obtain their spectral problems in classical cases [1, 2, 3]. When the given PDEs (partial differential equations) have mixed term derivatives with respect to time variable and spatial variables, the above-mentioned methods are not applicable. Because of mixed term derivative, we can not separate the given partial differential equation and given boundary conditions with respect to time variable and

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spatial variables [4]. In addition, there are many important differential equations with mixed term derivative and they appear in engineering and physical problems as well. In nonlinear case of these equations, we can mention to the Camassa-Holm equation of the form

$$u_t - u_{xxt} + 2ku_x + 3uu_x = 2u_x u_{xx} + uu_{xxx},$$

and the Degasperis-Procesi equation given by

$$u_t - u_{xxt} + 2ku_x + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}.$$

Both of these equations are third order PDEs with mixed term derivatives [5]. These and many other equations have been considered more often and have been solved by numerical methods such as methods of Adomian decomposition in [6, 7] and $\frac{G'}{G}$ in [8, 9, 10]. Also in the linear case, there are important linear PDEs having mixed terms like this:

$$\frac{\partial^3 u(x,t)}{\partial x^3} + a\frac{\partial^2 u(x,t)}{\partial x^2} + b\frac{\partial^2 u(x,t)}{\partial x \partial t} + c\frac{\partial u}{\partial t} = 0.$$

Because of mixed term derivative for linear case, we can not solve them via above-mentioned classical methods either. Furthermore, in linear case, it is not possible to use the Laplace transformation method when the eigenvalues of the spectral problem are distributed in the whole of complex plane or in the case of the spectral problem which has no eigenvalue. In this cases, we should use contour integral method.

This paper involves three parts. In first part, the given PDE has no mixed term derivative and in second part, the given PDE has a mixed term derivative. Third part presents some suitable examples.

## 2. Fractional equation without mixed term derivative

### 2.1. Main problem and its spectral problem

We consider the following fractional equation with initial and boundary conditions:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu, \quad x \in (0,1), t > 0, 0 < \alpha < 1, \tag{1}$$

$$u(0,t) = u(1,t) = 0, \qquad t > 0, \tag{2}$$

$$u(x,0) = \varphi(x), \qquad x \in [0,1], \tag{3}$$

where $u(x,t)$ is an unknown function and $\varphi(x)$ is a known continuous function. We consider

$$u(x,t) = y(x)T(t),$$

and substituting this into the equation (1), we have:

$$yT^{\alpha}(t) = aTy'' + bTy' + cTy.$$

Then the following spectral problem is obtained:

$$ay'' + by' + (c - \lambda^2)y = 0, \quad x \in (0,1), \tag{4}$$

$$y(0) = 0, \qquad y(1) = 0. \tag{5}$$

Furthermore, the fractional equation with respect to $t$ is:

$$T^{\alpha} - \lambda^2 T = 0, \qquad t > 0. \tag{6}$$

We will compute the solutions of fractional equation with respect to $t$ by using Mittag-Leffler functions.

**Remark 1.** *Without lost of generality, the spectral problem of (4)-(5) can be considered in the following form:*

$$y''(x) + 2ay'(x) - \lambda^2 y(x) = 0, \; x \in (0,1), \qquad y(0) = 0, \; y(1) = 0. \tag{7}$$

*Then the characteristic equation of the equation (7) is*

$$\theta^2 + 2a\theta - \lambda^2 = 0 \implies \Delta = 4a^2 + 4\lambda^2,$$
$$\theta_1 = -a - \sqrt{a^2+\lambda^2}, \qquad \theta_2 = -a + \sqrt{a^2+\lambda^2}.$$

*and the general solution of the equation (7) is as follows:*

$$y(x) = c_1 e^{(-a-\sqrt{a^2+\lambda^2})x} + c_2 e^{(-a+\sqrt{a^2+\lambda^2})x}, \tag{8}$$

*where $c_1, c_2$ are arbitrary constants.*

Imposing boundary conditions (5) on the solution (8) yields the following algebraic system:

$$\begin{cases} c_1 + c_2 = 0, \\ c_1 e^{-a-\sqrt{a^2+\lambda^2}} + c_2 e^{-a+\sqrt{a^2+\lambda^2}} = 0. \end{cases}$$

To obtain non-trivial solutions, coefficent determinant should be zero:

$$\begin{vmatrix} 1 & 1 \\ e^{-a-\sqrt{a^2+\lambda^2}} & e^{-a+\sqrt{a^2+\lambda^2}} \end{vmatrix} = e^{-a+\sqrt{a^2+\lambda^2}} - e^{-a-\sqrt{a^2+\lambda^2}} = 0,$$

therefore, the eigenvalues of spectral problem will be in the following form:

$$e^{2\sqrt{a^2+\lambda^2}} = e^{2k\pi i}\ ,\ a^2+\lambda^2 = -k^2\pi^2 \Rightarrow \lambda_k^2 = -(a^2+k^2\pi^2) = (a^2+k^2\pi^2)i^2,$$

$$\lambda_k = \pm\sqrt{(a^2+k^2\pi^2)}i \quad , \quad k \in \mathbb{Z}, \tag{9}$$

and the related eigenfunctions are as follows:

$$y_k(x) = A_k[e^{-(a+\sqrt{a^2+\lambda_k^2})x} - e^{-(a-\sqrt{a^2+\lambda_k^2})x}]$$

$$= A_k e^{-ax}[e^{-k\pi i x} - e^{k\pi i x}] = -2iA_k e^{-ax}\sin k\pi x,$$

$$y_k(x) = e^{-ax}\sin k\pi x\ , \quad k \in \mathbb{N}. \tag{10}$$

Eigenfunctions (10) are not orthogonal. In this case, the eigenfunctions can not form a complete basis system and we should use the eigenfunctions of related adjoint problem.

### 2.2. Adjoint problem and its eigenvalues and eigenfunctions

For adjoint problem, we have:

$$Z'' - 2aZ' - \rho^2 Z = 0, \quad x \in (0,1), \tag{11}$$

$$Z(0) = 0\ ,\ Z(1) = 0, \tag{12}$$

and the characteristic equation of adjoint equation (11) is as follows:

$$\mu^2 - 2a\mu - \rho^2 = 0 \quad \Rightarrow \quad \mu = a \pm \sqrt{a^2+\rho^2}.$$

The general solution of the equation (11) is as follows:

$$Z(x,\rho) = c_3 e^{ax-\sqrt{a^2+\rho^2}x} + c_4 e^{ax+\sqrt{a^2+\rho^2}x}. \tag{13}$$

Imposing boundary conditions (12) on the solution (13) yields the following algebraic system:

$$\begin{cases} c_3 + c_4 = 0, \\ c_3 e^{a-\sqrt{a^2+\rho^2}} + c_4 e^{a+\sqrt{a^2+\rho^2}} = 0, \end{cases}$$

$$\begin{vmatrix} 1 & 1 \\ e^{a-\sqrt{a^2+\rho^2}} & e^{a+\sqrt{a^2+\rho^2}} \end{vmatrix} = e^{a+\sqrt{a^2+\rho^2}} - e^{a-\sqrt{a^2+\rho^2}} = 0,$$

where $c_3, c_4$ are arbitrary constants.

The eigenvalues of adjoint problem are computed here:

$$e^{2\sqrt{a^2+\rho^2}} = e^{2k\pi i} \ , \ a^2 + \rho_k^2 = -k^2\pi^2 \Rightarrow \rho_k^2 = -(a^2 + k^2\pi^2) = (a^2 + k^2\pi^2)i^2,$$

$$\rho_k = \pm\sqrt{(a^2 + k^2\pi^2)}i \quad , \quad k \in \mathbb{Z}, \tag{14}$$

and the eigenfunctions of adjoint problem are as follows:

$$\begin{aligned} Z_k(x,\rho) &= B_k[e^{ax-\sqrt{a^2+\rho_k^\alpha}x} - e^{ax+\sqrt{a^2+\rho_k^\alpha}x}] \\ &= B_k e^{ax}[e^{-k\pi ix} - e^{k\pi ix}] = -2iB_k e^{ax}\sin k\pi x, \quad k \in \mathbb{N}. \end{aligned} \tag{15}$$

Note that the eigenfunctions of main problem are not equal to the eigenfunctions of adjoint problem. We will show that the eigenfunctions of (7) and the eigenfunctions of (11) are biorthogonal, that is:

$$\langle y_m(x,\lambda_m), Z_n(x,\rho_n)\rangle = \int_0^1 e^{-ax}\sin mx.e^{ax}\sin nx dx = \begin{cases} 0 & m \neq n, \\ 1 & m = n. \end{cases}$$

Then the approximate solution of problem (1)-(3) is constructed as follows:

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) y_k(x,\lambda_k) = \sum_{k=1}^{\infty} T_k h_\alpha(t,\lambda_k) y_k(x,\lambda_k), \tag{16}$$

where $h_\alpha(t,\lambda_k)$ are Mittag-Leffler function and we compute $T_k$ by imposing initial condition

$$\varphi(x) = u(x,0) = \sum_{k=1}^{\infty} T_k h_\alpha(0,\lambda_k) y_k(x,\lambda_k),$$

$$\langle \varphi, Z_m(x,\rho_m)\rangle = \int_0^1 \varphi(x) Z_m(x,\rho_m) dx$$

$$= \sum_{k=1}^{\infty} T_k h_\alpha(0,\lambda_k) \int_0^1 y_k(x,\lambda_k) Z_m(x,\rho_m)dx. \qquad (17)$$

Because of the following relation:

$$\int_0^1 y_k(x,\lambda_k) Z_m(x,\rho_m)dx = \begin{cases} 0 & m \neq k, \\ 1 & m = k, \end{cases}$$

and (15), the coefficients $T_m$ are computed as follows:

$$T_m = \frac{\langle \varphi, Z_m(x,\rho_m)\rangle}{h_\alpha(0,\lambda_m)}.$$

**Remark 2.** *The modified form of Mittag-Leffler function is as follows:*

$$h_p(t) = \sum_{k=1}^{\infty} \frac{t^{kp-1}}{(kp-1)!} = \frac{t^{p-1}}{(p-1)!} + \frac{t^{2p-1}}{(2p-1)!} + \frac{t^{3p-1}}{(3p-1)!} + \ldots \ . \qquad (18)$$

*We assume that this function is invariant with respect to the fractional derivative of order* $p$ *, that is:* $D^{(np)}h_p(t) = h_p(t)$.

*For solving the fractional equation with respect to* $t$*, we use the general form of Mittag-Leffler function:*

$$T(t) = h_p(t,r) = \sum_{k=1}^{\infty} \frac{r^k t^{kp-1}}{(kp-1)!}, \qquad (19)$$

$$T^{(n)}(t) = D^{(n)}h_p(t,r) = r^n h_p(t,r). \qquad (20)$$

**Remark 3.** *By using Mittag-Leffler function, we solve the fractional equations*

$$a_m T^{\frac{m}{n}}(t) + a_{m-1}T^{\frac{m-1}{n}}(t) + \ldots + a_1 T^{\frac{1}{n}}(t) + a_0 T = 0. \qquad (21)$$

*The related characteristics equation is:*

$$a_m r^m + a_{m-1} r^{m-1} + \ldots + a_1 r + a_0 = 0. \qquad (22)$$

*Then the general solution of the fractional equation is as follows:*

$$T(t) = c_1 h_p(t,r_1) + c_2 h_p(t,r_2) + \ldots + c_m h_p(t,r_m), \qquad (23)$$

*where* $p = \frac{1}{n}$ *is the order of fractional derivative [12].*

**Example 1.** Consider the following initial-boundary value fractional problem:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a\frac{\partial^3 u}{\partial x^3} + b\frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x} + du, \quad x \in (0,1), t > 0, 0 < \alpha < 1. \tag{24}$$

Suppose $a = b = 0, c = d = 1$. Then we have:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial u}{\partial x} + u. \tag{25}$$

Let the approximate solution of (25) be as follows:

$$\begin{aligned} u(x,t) = \sum_{k=1}^{4} A_k(x)\frac{t^{k\alpha-1}}{(k\alpha-1)!} &= A_1(x)\frac{t^{\alpha-1}}{(\alpha-1)!} + A_2(x)\frac{t^{2\alpha-1}}{(2\alpha-1)!} \\ &+ A_3(x)\frac{t^{3\alpha-1}}{(3\alpha-1)!} + A_4(x)\frac{t^{4\alpha-1}}{(4\alpha-1)!}. \end{aligned} \tag{26}$$

Now, we compute $D^\alpha u(x,t)$, $\frac{\partial u(x,t)}{\partial x}$ as follows:

$$D^\alpha u(x,t) = A_2(x)\frac{t^{\alpha-1}}{(\alpha-1)!} + A_3(x)\frac{t^{2\alpha-1}}{(2\alpha-1)!} + A_4(x)\frac{t^{3\alpha-1}}{(3\alpha-1)!}, \tag{27}$$

$$\begin{aligned} \frac{\partial u(x,t)}{\partial x} = A_1'(x)\frac{t^{\alpha-1}}{(\alpha-1)!} + A_2'(x)\frac{t^{2\alpha-1}}{(2\alpha-1)!} + A_3'(x)\frac{t^{3\alpha-1}}{(3\alpha-1)!} \\ + A_4'(x)\frac{t^{4\alpha-1}}{(4\alpha-1)!}. \end{aligned} \tag{28}$$

By substituting the relations (26),(27),(28) in (25), we have:

$$\begin{aligned} A_2(x)\frac{t^{\alpha-1}}{(\alpha-1)!} + A_3(x)\frac{t^{2\alpha-1}}{(\alpha-1)!} + A_4(x)\frac{t^{3\alpha-1}}{(3\alpha-1)!} \\ = A_1'(x)\frac{t^{\alpha-1}}{(\alpha-1)!} + A_2'(x)\frac{t^{2\alpha-1}}{(2\alpha-1)!} + A_3'(x)\frac{t^{3\alpha-1}}{(3\alpha-1)!} + A_4'(x)\frac{t^{4\alpha-1}}{(4\alpha-1)!} \\ + A_1(x)\frac{t^{\alpha-1}}{(\alpha-1)!} + A_2(x)\frac{t^{2\alpha-1}}{(2\alpha-1)!} + A_3(x)\frac{t^{3\alpha-1}}{(3\alpha-1)!} + A_4(x)\frac{t^{4\alpha-1}}{(4\alpha-1)!}, \end{aligned} \tag{29}$$

From (29), we obtain:

$$\begin{gathered} A_2(x) = A_1'(x) + A_1(x), \quad A_3(x) = A_2'(x) + A_2(x), \\ A_4(x) = A_3'(x) + A_3(x), \quad A_4'(x) + A_4(x) = 0. \end{gathered}$$

Consequently, we get

$$A_1(x) = \frac{x^3}{3!}e^{-x}\ ,\ A_2(x) = \frac{x^2}{2!}e^{-x}\ ,\ A_3(x) = \frac{x}{1!}e^{-x}\ ,\ A_4(x) = e^{-x}. \tag{30}$$

## 3. Fractional equation with mixed term derivative

### 3.1. Main problem and its spectral problem

In this section, we consider the fractional PDE with mixed term derivative

$$D_x^2 u(x,t) + aD_x D_t^\alpha u(x,t) + bD_x u(x,t) + cD_t^\alpha u(x,t) + du(x,t) = 0,$$
$$x \in (0,1)\ ,\ t > 0\ ,\ 0 < \alpha < 1, \tag{31}$$

with boundary and initial conditions

$$u(0,t) = 0\ ,\ u(1,t) = 0, \quad t > 0, \tag{32}$$

$$u(x,0) = \varphi(x), \quad x \in [0,1]. \tag{33}$$

Since we can not apply the separation of variables method, the spectral problem is obtained by Laplace transformation method.

$$\widehat{u}''(x,\lambda) + aD_x[\lambda^\alpha \widehat{u}(x,\lambda) - u(x,0)] + b\widehat{u}'(x,\lambda) +$$
$$+ c[\lambda^\alpha \widehat{u}(x,\lambda) - u(x,0)] + d\widehat{u}(x,\lambda) = 0,$$

$$\widehat{u}''(x,\lambda) + (a\lambda^\alpha + b)\widehat{u}'(x,\lambda) + (c\lambda^\alpha + d)\widehat{u}(x,\lambda) = a\varphi'(x) + c\varphi(x),\ x \in (0,1), \tag{34}$$

$$\widehat{u}(0,\lambda) = \widehat{u}(1,\lambda) = 0. \tag{35}$$

We suppose

$$y(x,\lambda) = \widehat{u}(x,\lambda) \quad , \quad f(x) = a\varphi'(x) + c\varphi(x).$$

Then the spectral problem is as follows:

$$y''(x,\lambda) + (a\lambda^\alpha + b)y'(x,\lambda) + (c\lambda^\alpha + d)y(x) = f(x), \quad x \in (0,1), \tag{36}$$

$$y(0) = y(1) = 0. \tag{37}$$

The homogeneous type of equation (36) is

$$y'' + (a\lambda^\alpha + b)y' + (c\lambda^\alpha + d)y = 0, \tag{38}$$

and its characteristic equation is

$$\theta^2 + (a\lambda^\alpha + b)\theta + (c\lambda^\alpha + d) = 0\ ,\ \theta(\lambda^\alpha) = \frac{-(a\lambda^\alpha + b) \pm \sqrt{(a\lambda^\alpha + b)^2 - 4(c\lambda^\alpha + d)}}{2}.$$

Let us suppose $\lambda^\alpha = \rho$. Then we have:

$$\theta(\lambda^\alpha) = \theta(\rho) = \frac{-(a\rho+b) \pm \sqrt{(a\rho+b)^2 - 4(c\rho+d)}}{2},$$

$$\theta_1(\rho) = \frac{-(a\rho+b) - \sqrt{(a\rho+b)^2 - 4(c\rho+d)}}{2},$$

$$\theta_2(\rho) = \frac{-(a\rho+b) + \sqrt{(a\rho+b)^2 - 4(c\rho+d)}}{2},$$

$$y(x,\lambda) = c_1 e^{\theta_1(\rho)x} + c_2 e^{\theta_2(\rho)x}. \tag{39}$$

### 3.2. The asymptotic expansion of roots of characteristic equation and eigenvalues

Now, we compute the asymptotic expansion of above mentioned roots [13]:

$$\theta_1(\rho) = -\frac{a\rho+b}{2} - \sqrt{(\frac{a\rho+b}{2})^2 - (c\rho+d)}$$

$$= -\frac{a\rho+b}{2} - \sqrt{\frac{1}{4}a^2\rho^2 + (\frac{1}{2}ab - c)\rho + (\frac{1}{4}b^2 - d)}.$$

The square part has the following asymptotic expansion:

$$\sqrt{(\frac{a}{2})^2\rho^2 + (\frac{ab}{2} - c)\rho + (\frac{b^2}{4} - d)} = \frac{a}{2}\rho + A_0 + \frac{A_{-1}}{\rho} + \frac{A_{-2}}{\rho^2} + ...,$$

$$(\frac{a}{2})^2\rho^2 + (\frac{ab}{2} - c)\rho + (\frac{b^2}{4} - d) \equiv (\frac{a}{2})^2\rho^2 + aA_0\rho + (A_0^2 + aA_{-1}) + (2A_0A_{-1} + aA_{-2})\frac{1}{\rho} + ....$$

By equalizing the coefficients of powers of $\rho$ , we have:

$$A_0 = \frac{b}{2} - \frac{c}{a}, \quad A_{-1} = \frac{bc}{a^2} - \frac{c^2}{a^3} - \frac{d}{a}.$$

By assuming $2A_0A_{-1} + aA_{-2} = 0$, we have:

$$A_{-2} = -\frac{b}{a}(\frac{bc}{a^2} - \frac{c^2}{a^3} - \frac{d}{a}) + \frac{2c}{a^2}(\frac{bc}{a^2} - \frac{c^2}{a^3} - \frac{d}{a}).$$

Therefore, the asymptotic expansion of $\theta_1(\rho)$ will be in the following form:

$$\theta_1(\rho) = -\frac{a\rho+b}{2} - [\frac{a\rho}{2} + (\frac{b}{2} - \frac{c}{a}) + \frac{\frac{bc}{a^2} - \frac{c^2}{a^3} - \frac{d}{a}}{\rho} + \frac{A_{-2}}{\rho^2} + ...]$$

$$= -a\rho - (b - \frac{c}{a}) - \frac{\frac{bc}{a^2} - \frac{c^2}{a^3} - \frac{d}{a}}{\rho} - \frac{A_{-2}}{\rho^2}.$$

Similarly, the asymptotic expansion of $\theta_2(\rho)$ is:

$$\theta_2(\rho) = -\frac{a\rho + b}{2} + [\frac{a\rho}{2} + (\frac{b}{2} - \frac{c}{a}) + \frac{\frac{bc}{a^2} - \frac{c^2}{a^3} - \frac{d}{a}}{\rho} + \frac{A_{-2}}{\rho^2} + ...]$$

$$= -\frac{c}{a} - \frac{\frac{bc}{a^2} - \frac{c^2}{a^3} - \frac{d}{a}}{\rho} - \frac{A_{-2}}{\rho^2}.$$

We see that the asymptotic expansion of $\theta_1(\rho)$ includes the terms with $\rho$ and its negative powers, while the asymptotic expansion of $\theta_2(\rho)$ includes the term of constant value and negative powers of $\rho$ . Therefore, from the point of view of convergence, we focus only on $\theta_1(\rho)$ .

$$\theta_1(\lambda^\alpha) = -a\lambda^\alpha - \frac{ab - c}{a} - \frac{abc - c^2 - da^2}{a^3\lambda^\alpha} - \frac{A_{-2}}{\lambda^{2\alpha}} + ... \ ,$$

$$\theta_2(\lambda^\alpha) = -\frac{c}{a} + \frac{abc - c^2 - da^2}{a^3\lambda^\alpha} + \frac{A_{-2}}{\lambda^{2\alpha}} + ... \ , \tag{40}$$

consequently, the solutions of spectral problem are

$$y_1(x, \lambda^\alpha) = e^{-a\lambda^\alpha x}[e^{-\frac{ab-c}{a}x}], \qquad y_2(x, \lambda^\alpha) = [e^{-\frac{c}{a}x}]. \tag{41}$$

Now, we compute the solution of non-homogeneous part of spectral problem by Lagrange method, that is:

$$y(x) = c_1 e^{\theta_1(\rho)x} + c_2 e^{\theta_2(\rho)x}. \tag{42}$$

We suppose the particular solution has the following form:

$$y(x) = c_1(x) e^{\theta_1(\rho)x} + c_2(x) e^{\theta_2(\rho)x}. \tag{43}$$

By substituting values $y, y', y''$ in (36), we have the following algebraic system for unknowns $c_1(x)$ *and* $c_2(x)$:

$$\begin{cases} c_1'(x)e^{\theta_1 x} + c_2'(x)e^{\theta_2 x} = 0, \\ c_1'(x)\theta_1 e^{\theta_1 x} + c_2'\theta_2 e^{\theta_2 x} = a\varphi'(x) + c\varphi(x), \end{cases}$$

$$W(x) = \begin{vmatrix} e^{\theta_1 x} & e^{\theta_2 x} \\ \theta_1 e^{\theta_1 x} & \theta_2 e^{\theta_2 x} \end{vmatrix} = (\theta_2 - \theta_1)e^{(\theta_1+\theta_2)x},$$

$$c_1'(x) = -\frac{1}{W(x)}(a\varphi'(x) + c\varphi(x))e^{\theta_2 x} = -\frac{(a\varphi'(x) + c\varphi(x))e^{-\theta_1 x}}{\theta_2 - \theta_1},$$

$$c_2'(x) = \frac{1}{W(x)}(a\varphi'(x) + c\varphi(x))e^{\theta_1 x} = \frac{(a\varphi'(x) + c\varphi(x))e^{-\theta_2 x}}{\theta_2 - \theta_1},$$

$$\begin{aligned} c_1(x) &= c_1 - \int_{x_1}^{x} \frac{(a\varphi'(\xi) + c\varphi(\xi))e^{-\theta_1 \xi}}{\theta_2 - \theta_1} d\xi, \\ c_2(x) &= c_2 + \int_{x_2}^{x} \frac{(a\varphi'(\xi) + c\varphi(\xi))e^{-\theta_2 \xi}}{\theta_2 - \theta_1} d\xi. \end{aligned}$$

If we suppose $x_1 = x_2 = 0$ , then we have:

$$\begin{aligned} y(x) = c_1 e^{\theta_1 x} + c_2 e^{\theta_2 x} &- \int_0^x \frac{(a\varphi'(\xi) + c\varphi(\xi))e^{\theta_1 (x-\xi)}}{\theta_2 - \theta_1} d\xi \\ &+ \int_0^x \frac{(a\varphi'(\xi) + c\varphi(\xi))e^{\theta_2 (x-\xi)}}{\theta_2 - \theta_1} d\xi, \end{aligned}$$

$$y(x) = c_1 e^{\theta_1 x} + c_2 e^{\theta_2 x} + \int_0^x \frac{a\varphi'(\xi) + c\varphi(\xi)}{\theta_2 - \theta_1} [e^{\theta_2 (x-\xi)} - e^{\theta_1 (x-\xi)}] d\xi. \tag{44}$$

Considering boundary conditions (37) of spectral problem in (44) yeilds

$$\begin{cases} c_1 + c_2 = 0, \\ c_1 e^{\theta_1} + c_2 e^{\theta_2} + \int_0^1 \frac{a\varphi'(\xi) + c\varphi(\xi)}{\theta_2 - \theta_1} [e^{\theta_2 (1-\xi)} - e^{\theta_1 (1-\xi)}] d\xi = 0, \end{cases}$$

$$c_1 = \frac{1}{e^{\theta_2} - e^{\theta_1}} \int_0^1 \frac{a\varphi'(\xi) + c\varphi(\xi)}{\theta_2 - \theta_1} [e^{\theta_2 (1-\xi)} - e^{\theta_1 (1-\xi)}] d\xi \ , \quad c_2 = -c_1,$$

$$\begin{aligned} y(x) = &\frac{e^{\theta_1 x}}{e^{\theta_2} - e^{\theta_1}} \int_0^1 \frac{a\varphi'(\xi) + c\varphi(\xi)}{\theta_2 - \theta_1} [e^{\theta_2 (1-\xi)} - e^{\theta_1 (1-\xi)}] d\xi \\ &- \frac{e^{\theta_2 x}}{e^{\theta_2} - e^{\theta_1}} \int_0^1 \frac{a\varphi'(\xi) + c\varphi(\xi)}{\theta_2 - \theta_1} [e^{\theta_2 (1-\xi)} - e^{\theta_1 (1-\xi)}] d\xi \\ &+ \int_0^x \frac{a\varphi'(\xi) + c\varphi(\xi)}{\theta_2 - \theta_1} [e^{\theta_2 (x-\xi)} - e^{\theta_1 (x-\xi)}] d\xi. \end{aligned} \tag{45}$$

From the relation (45), we have:

$$g(x,\xi)=\begin{cases}\frac{e^{\theta_2(x-\xi)}-e^{\theta_1(x-\xi)}}{\theta_2-\theta_1}, & x>\xi,\\ 0, x<\xi, & \end{cases} \Rightarrow g(x,\xi)=\frac{e^{\theta_2(x-\xi)}-e^{\theta_1(x-\xi)}}{\theta_2-\theta_1}\Theta(x-\xi),$$

$$\Theta(x-\xi)=\begin{cases}1\ , & x>\xi,\\ 0\ , & x<\xi,\end{cases}$$

where $\Theta(x-\xi)$ is the Heaviside function and $g(x,\xi,\lambda^\alpha)$ is a fundamental solution of the equation (38),[2].
Consequently, we have:

$$y(x,\lambda)=c_1e^{\theta_1x}+c_2e^{\theta_2x}+\int_0^1 g(x,\xi,\lambda^\alpha)f(\xi)d\xi. \tag{46}$$

For computing the eigenvalues, by considering boundary conditions (37) in (46), we obtain:

$$\begin{cases}c_1+c_2=-\int_0^1 g(0,\xi,\lambda^\alpha)f(\xi)d\xi,\\ c_1e^{\theta_1}+c_2e^{\theta_2}=-\int_0^1 g(1,\xi,\lambda^\alpha)f(\xi)d\xi,\end{cases}$$

$$\Delta(\lambda^\alpha)=\begin{vmatrix}1 & 1\\ e^{\theta_1} & e^{\theta_2}\end{vmatrix}=e^{\theta_2}-e^{\theta_1}=[e^{-\frac{c}{a}}]-e^{-a\lambda^\alpha}[e^{-\frac{ab-c}{a}}]=0.$$

Therefore, we have:

$$e^{-a\lambda^\alpha}=e^{b-\frac{2c}{a}},\quad -a\lambda_k^\alpha=b-\frac{2c}{a}+2k\pi i\ \Rightarrow\ \lambda_k^\alpha=-\frac{b}{a}+\frac{2c}{a^2}-\frac{2k\pi i}{a},\ k\in\mathbb{Z}. \tag{47}$$

Now, we compute the related eigenfunctions. For this, we consider:

$$c_2=-c_1-\int_0^1 g(0,\xi,\lambda^\alpha)f(\xi)d\xi.$$

Therefore

$$y_k(x,\lambda_k^\alpha)=c_1[e^{\theta_1(\lambda_k^\alpha)x}-e^{\theta_2(\lambda_k^\alpha)x}]-e^{\theta_2(\lambda_k^\alpha)x}\int_0^1 g(0,\xi,\lambda_k^\alpha)f(\xi)d\xi$$

$$+\int_0^1 g(x,\xi,\lambda_k^\alpha)f(\xi)d\xi\quad,\quad k\in\mathbb{Z}.$$

## 4. The analytic and approximate solutions of main problem

According to the distribution of eigenvalues (47) in complex plane

$$\lambda_k^\alpha = -\frac{b}{a} + \frac{2c}{a^2} \pm \frac{2k\pi i}{a}.$$

In the following theorem, we consider different cases for the approximate solution of the initial-boundary value problem (31)-(33) and its spectral problem (36)-(37).

**Theorem 1.** *Let the following conditions hold for initial-boundary value problem (31)-(33):*

$$\varphi \in \mathbb{C}^2(0,1), \qquad \varphi(0) = \varphi'(0) = 0, a \geq 0, b, c, d \in \mathbb{R}.$$

*Then the initial-boundary value problem (31)-(33) has a unique analytic solution in the form of (48) and its approximate solution is in the form of (49).*

*Proof.* As the locations of eigenvalues in the complex plane may be different, we give the proof of this theorem for different cases:

**A)** If the eigenvalues lie in left-hand side of Laplace line $L = (c - i\infty, c + i\infty)$ in complex plane, where $c > 0$, then we have two cases:

**A.1:** If the eigenvalues are distributed according to Figure 1, then by choosing a suitable closed contour which includes finite eigenvalues in left-hand side of Laplace line, we compute the solution by using residual theory in complex analysis [11,14,15].

If the closed contours $\Gamma_\nu = [a_\nu, b_\nu] \cup c_\nu$ according to Figure 1, then the analytic solution will be as follows:

$$u(x,t) = \int_L e^{\lambda t} y(x,\lambda) d\lambda = \lim_{\nu\to\infty} \int_{\Gamma_\nu} e^{\lambda t} y(x,\lambda) d\lambda = \sum_{k=0}^{\infty} e^{\lambda_k t} y(x,\lambda_k) d\lambda, \qquad (48)$$

where $[a_\nu, b_\nu] \subset (c - i\infty, c + i\infty)$ , $\nu \in \mathbb{N}$.

And the approximate solution is computed using contour integral method by choosing series (48).

$$u(x,t) \cong \sum_{k=0}^{N} e^{\lambda_k t} y(x,\lambda_k) d\lambda. \qquad (49)$$

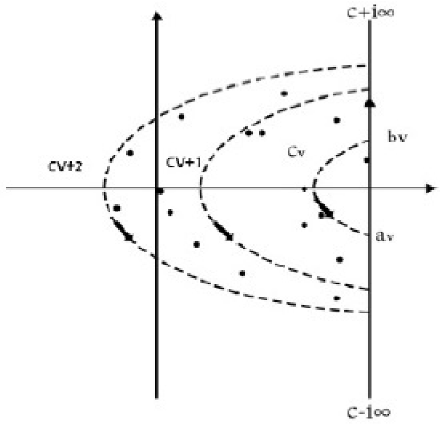


**Figure 1.** The eigenvalues lie in left-hand side of Laplace line

**Remark 4.** *If the real part of $\lambda^{\alpha}$ is zero, that is:*

$$\frac{2c}{a^2} - \frac{b}{a} = 0 \Longrightarrow \lambda_k^{\alpha} = \pm \frac{2k\pi i}{a},$$

*then the eigenvalues are on imaginary axis (according to Figure 2), and we can write the analytic solution and related approximate solution by choosing a closed contour integral as (48) and (49), respectively.*

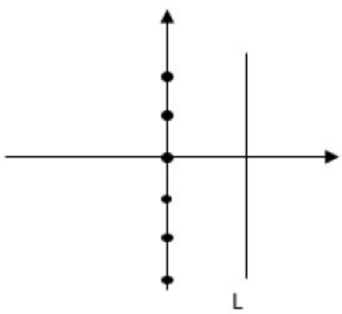


**Figure 2.** The eigenvalues lie on imaginary axis

**A.2:** If the eigenvalues are distributed in left-hand side of Laplace line and the contour $M_{\nu}$ (Figure 3), the contour $M_{\nu}$ is chosen such that there is no eigenvalue between this contour and Laplace line. In this case, we can compute the solution over contour $M_\nu$ and the approximate solution can be calculated by Laplace transformation method [14,16,17].

As $B_\nu$ is a closed contour, by the Cauchy integral theorem we have:

$$\int_{B_\nu} e^{\lambda t} y(x,\lambda) d\lambda = 0. \tag{50}$$

Consider the following closed contours $B_{\nu_i}$ :

$$\begin{aligned} B_{\nu_1} &= L_{\nu_1} \cup C_{\nu_1} \cup D_{\nu_1} \cup M_{\nu_1}, \\ B_{\nu_2} &= L_{\nu_2} \cup C_{\nu_2} \cup D_{\nu_2} \cup M_{\nu_2}, \\ &\vdots \\ B_{\nu_i} &= L_{\nu_i} \cup C_{\nu_i} \cup D_{\nu_i} \cup M_{\nu_i}. \end{aligned}$$

According to the relation (50), we have:

$$0 = \int_{B_{\nu_i}} e^{\lambda t} y(x,\lambda) d\lambda = \int_{L_{\nu_i}} e^{\lambda t} y(x,\lambda) d\lambda + \int_{C_{\nu_i}} e^{\lambda t} y(x,\lambda) d\lambda$$

$$+ \int_{D_{\nu_i}} e^{\lambda t} y(x,\lambda) d\lambda + \int_{M_{\nu_i}} e^{\lambda t} y(x,\lambda) d\lambda.$$

Note that the curves $C_{\nu_i} and\ D_{\nu_i}$ have opposite directions:

$$\int_{C_{\nu_i}} e^{\lambda t} y(x,\lambda) d\lambda = - \int_{D_{\nu_i}} e^{\lambda t} y(x,\lambda) d\lambda,$$

therefore,

$$\int_{L_{\nu_i}} e^{\lambda t} y(x,\lambda) d\lambda = - \int_{-M_{\nu_i}} e^{\lambda t} y(x,\lambda) d\lambda = \int_{M_{\nu_i}} e^{\lambda t} y(x,\lambda) d\lambda$$

$$\Longrightarrow \int_{L_\nu} e^{\lambda t} y(x,\lambda) d\lambda = \int_{M_\nu} e^{\lambda t} y(x,\lambda) d\lambda, \tag{51}$$

$$u(x,t) = \int_{c-i\infty}^{c+i\infty} e^{\lambda t} y(x,\lambda) d\lambda = \lim_{\nu_i \to \infty} \int_{L_{\nu_i}} e^{\lambda t} y(x,\lambda) d\lambda$$

$$\cong \int_{L_\nu} e^{\lambda t} y(x,\lambda) d\lambda = \int_{M_\nu} e^{\lambda t} y(x,\lambda) d\lambda. \tag{52}$$

This completes the proof. ◄

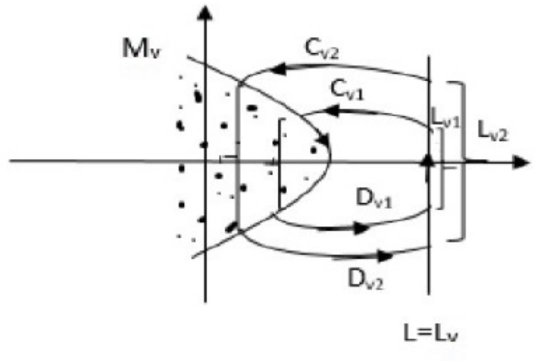


**Figure 3.** The eigenvalues lie in the left-hand side of curve $M_\nu$

**B)** If the eigenvalues lie in right-hand side of Laplace line, then the initial-boundary value problem has no solution. Because the series solution (48) is not convergent for positive and infinite values of real parts of eigenvalues.

**Example 2.** As we saw in *(47)*:

$$\lambda_k^\alpha = -\frac{b}{a} + \frac{2c}{a^2} \pm \frac{2k\pi i}{a},$$

$$\lambda_k^\alpha \sim \frac{2k\pi i}{a} \Longrightarrow \lambda_k \sim \sqrt[\alpha]{\frac{2k\pi i}{a}} = \sqrt[\alpha]{\frac{2\pi}{a}}(ki)^{\frac{1}{\alpha}}. \tag{53}$$

Due to (53), we can consider two cases.

If $\alpha = \frac{1}{2(2n+1)}$, then the eigenvalues lie on the negative part of real axes and the approximate solution is computed in the form of (A.2).

If $\alpha = \frac{1}{4(2n)}$, then the eigenvalues lie on the positive part of real axes. If they lie in left-hand side of Laplace line, the approximate solution is computed in the form of (A.1) and if they lie in right-hand side of Laplace line, the problem has no solution (case B).

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