On Conditions of the Completeness of Some Systems of Bessel Functions in the Space $L^2((0; 1); x^{2p}dx)$

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Abstract. We establish the necessary and sufficient conditions for the completeness of the system $(x^{-p-1}\sqrt{\rho_k}J_{\nu}(x\rho_k) : k \in \mathbb{N})$ in the space $L^2((0; 1); x^{2p}dx)$, where $J_{\nu}$ is the Bessel function of the first kind of index $\nu \geq 1/2$, $p \in \mathbb{R}$ and $(\rho_k : k \in \mathbb{N})$ is an arbitrary sequence of distinct nonzero complex numbers.

Key Words and Phrases: Bessel function, entire function, completeness, minimality, basis, Jensen formula.

2010 Mathematics Subject Classifications: 42A65, 42C30, 33C10, 30B60, 30D20, 44A15

1. Introduction

Let $L^2((0; 1); t^\alpha dt)$, $\alpha \in \mathbb{R}$, denote the weighted Lebesgue space of all measurable functions $f : (0; 1) \to \mathbb{C}$, for which

$$\int_0^1 t^\alpha |f(t)|^2 dt < +\infty.$$ 

Let

$$J_{\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k! \Gamma(\nu + k + 1)}$$

be the Bessel function of the first kind of index $\nu$. For $\nu > -1$ the function $J_{\nu}$ has an infinite set $\{\rho_k : k \in \mathbb{N}\}$ of real roots, among which $\rho_k$, $k \in \mathbb{N}$, are the positive roots and $\rho_{-k} := -\rho_k$, $k \in \mathbb{N}$, are the negative roots (see [5, p. 94], [8, p. 350]). All roots are simple except, perhaps, the root $\rho_0 = 0$. A system $(e_k : k \in \mathbb{N})$ of elements of the Banach space $H$ is called complete if the closure of the linear span of this system coincides with the whole $H$ (see [6, p. 131], [7, p. 4258]).
system of elements \((e_k : k \in \mathbb{N}) \in H\) is said to be minimal if for each \(k \in \mathbb{N}\) the element \(e_k\) does not belong to the closure of the linear span of all other elements (see [6, p. 131], [7, p. 4258]).

It is known that the approximation properties of the system \((\sqrt{\pi}J_\nu(x\rho_k) : k \in \mathbb{N})\) with \(\nu > -1\) depends on the properties of the sequence \((\rho_k : k \in \mathbb{N})\). The classical results relate mainly to the case where \((\rho_k : k \in \mathbb{N})\) is a sequence of positive zeros of the function \(J_\nu\) (see, for instance, [1], [2], [4], [8]). In particular, it is well known that the system \((\sqrt{\pi}J_\nu(x\rho_k) : k \in \mathbb{N})\) is an orthogonal basis for the space \(L^2((0; 1))\) if \((\rho_k : k \in \mathbb{N})\) is a sequence of positive zeros of \(J_\nu\) (see [1], [4], [8, pp. 355–357]). From this it follows that if \(\nu > -1\) and \((\rho_k : k \in \mathbb{N})\) is a sequence of positive zeros of \(J_\nu\), then the system \((x^{-\nu}J_\nu(x\rho_k) : k \in \mathbb{N})\) is complete and minimal in \(L^2(((0; 1)); x^{2\nu+1}dx)\). The system \((\sqrt{\pi}J_\nu(x\rho_k) : k \in \mathbb{N})\) is also complete in \(L^2((0; 1))\) if \((\rho_k : k \in \mathbb{N})\) is a sequence of zeros of the function \(J_\nu\) (see [8, pp. 347, 356]). From [2] it follows that if \(\nu > -1/2\) and \((\rho_k : k \in \mathbb{N})\) is a sequence of distinct positive numbers such that \(\rho_k \leq \pi(k + \nu/2)\) for all sufficiently large \(k \in \mathbb{N}\), then the system \((\sqrt{\pi}J_\nu(x\rho_k) : k \in \mathbb{N})\) is complete in \(L^2((0; 1))\).

It is important to study the approximation properties of the above systems of Bessel functions if \((\rho_k : k \in \mathbb{N})\) is an arbitrary sequence of complex numbers. The necessary and sufficient conditions have been obtained in [9] for the completeness and minimality of system \((\sqrt{\pi}p^{1/2}J_\nu(x\rho_k) : k \in \mathbb{N})\) in the space \(L^2((0; 1))\) if \(\nu \geq -1/2\) and \((\rho_k : k \in \mathbb{N})\) is a sequence of distinct nonzero complex numbers. In [10], some sufficient conditions for the basis property of this system in \(L^2((0; 1))\) have been found. In [11], a criterion for the completeness and minimality of more general systems \((\Theta_{k,\nu,p} : k \in \mathbb{N})\), \(\Theta_{k,\nu,p}(x) := x^{-p-1}\sqrt{\pi}p^{1/2}J_\nu(x\rho_k)\), in the space \(L^2(((0; 1)); x^{2\nu+p}dx)\) has been established, where \(J_\nu\) is the Bessel function of the first kind of index \(\nu \geq 1/2\), \(p \in \mathbb{R}\) and \((\rho_k : k \in \mathbb{N})\) is a sequence of distinct nonzero complex numbers. For \(\nu = 3/2\) and \(p = 1\) such criterion is also available in [9]. Those results are formulated in terms of sequences of zeros of functions from some classes of entire functions.

In this paper, using the methods of [7, §§ 3.3, 4.1], [11], we establish some new necessary and sufficient conditions for the completeness of systems \((\Theta_{k,\nu,p} : k \in \mathbb{N})\) in \(L^2(((0; 1)); x^{2\nu+p}dx)\) depending on the properties of the sequence \((\rho_k : k \in \mathbb{N})\) of distinct nonzero complex numbers (see Theorems 1–3). This complements the results of papers [9]–[11].

2. Preliminaries

Let \(\log^+ x = \max(0; \log x)\) for \(x > 0\). By \(C_1, C_2, \ldots\) we denote arbitrary positive constants. To prove our main results we need the following auxiliary lemmas.
Lemma 1. (see [7, p. 4263]) Let $F$ be an entire function of exponential type $\sigma \leq 1$ for which the integral
$$
\int_{-\infty}^{+\infty} \frac{\log^+ |F(x)|}{\lambda + x^2} \, dx
$$
exists and let $(z_k : k \in \mathbb{N})$ be a sequence of nonzero roots of the function $F(z)$. Then
$$
\sum_{k \in \mathbb{N}} \left| \text{Im} \frac{1}{z_k} \right| < +\infty.
$$

Lemma 2. (see [7, p. 4304]) Let $(\lambda_n : n \in \mathbb{N})$ be a sequence of distinct nonzero complex numbers such that
$$
\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|} < +\infty.
$$
Then the infinite product
$$
G(z) = z^m \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{\lambda_n^2} \right), \quad m \in \mathbb{Z}_+,
$$
defines an entire function of minimal type of order 1 with $|G(x)| \leq \exp(\theta(|x|))$, $x \in \mathbb{R}$, where $\theta(x)$ is a positive increasing function satisfying
$$
\int_{0}^{+\infty} \frac{\theta(x)}{x^2} \, dx < +\infty.
$$

Lemma 3. (see [7, p. 4303]) Let $\omega(x)$ be a positive nondecreasing function on $[0; +\infty)$ satisfying $\int_{0}^{+\infty} x^{-2} \omega(x) \, dx < +\infty$. Then, for all $a > 0$, there exists an entire function
$$
F(z) = \int_{-a}^{a} e^{izt} \, d\sigma(t)
$$
satisfying $|F(x)| \leq \exp(-\omega(|x|))$, $|x| > x_0$, where $\sigma(t)$ is a function of bounded variation on the segment $[-a; a]$.

Lemma 4. (see [3], [11]) Let $\nu \geq -1/2$. A function $f$ has the representation
$$
f(z) = z^{-\nu} \int_{0}^{1} \sqrt{t} J_\nu(zt) \gamma(t) \, dt
$$
with some function $\gamma \in L^2(0; 1)$ if and only if it is an even entire function of exponential type $\sigma \leq 1$ such that $z^{\nu+1/2} f(z) \in L^2(0; +\infty)$. 
Lemma 5. (see [11]) Let $\nu \geq 1/2$ and $p \in \mathbb{R}$. An entire function $\Omega$ has the representation

$$\Omega(z) = z^{-\nu} \int_0^1 \sqrt{t} J_\nu(tz) t^{p-1} h(t) \, dt$$

(1)

with $h \in L^2((0; 1); x^2 p \, dx)$ if and only if it is an even entire function of exponential type $\sigma \leq 1$ such that $z^{-\nu+1/2}(z^{2\nu} \Omega(z))' \in L^2(0; +\infty)$. In this case,

$$h(t) = t^{-p} \int_0^{+\infty} \sqrt{t^2 z} J_{\nu-1}(tz) z^{-\nu+1/2}(z^{2\nu} \Omega(z))' \, dz.$$

Lemma 6. (see [11]) Let $\nu \geq 1/2$, $p \in \mathbb{R}$ and $(\rho_k : k \in \mathbb{N})$ be a sequence of nonzero complex numbers such that $\rho_k^2 \neq \rho_n^2$ if $k \neq n$. For a system $(\Theta_{k,\nu,p} : k \in \mathbb{N})$ to be incomplete in the space $L^2((0; 1); x^2 p \, dx)$ it is necessary and sufficient that the sequence $(\rho_k : k \in \mathbb{Z} \setminus \{0\})$, $\rho_{-k} := -\rho_k$, $k \in \mathbb{N}$, be a subsequence of zeros of some nonzero even entire function $\Omega$ of exponential type $\sigma \leq 1$ satisfying $z^{-\nu+1/2}(z^{2\nu} \Omega(z))' \in L^2(0; +\infty)$.

Lemma 7. (see [11]) Let $\nu \geq 1/2$, $p \in \mathbb{R}$ and an entire function $\Omega$ be defined by the formula (1). Then for all $z = x + iy = re^{i\varphi} \in \mathbb{C}$, we have

$$|\Omega(z)| \leq C_1 (1 + |z|)^{-\nu} \exp(|\Im z|).$$

3. Main results

Theorem 1. Let $\nu \geq 1/2$, $p \in \mathbb{R}$ and $(\rho_k : k \in \mathbb{N})$ be a sequence of distinct nonzero complex numbers such that $|\Im \rho_k| \geq \delta |\rho_k|$ for all $k \in \mathbb{N}$ and some $\delta > 0$. For a system $(\Theta_{k,\nu,p} : k \in \mathbb{N})$ to be complete in $L^2((0; 1); x^2 p \, dx)$ it is necessary and sufficient that

$$\sum_{k=1}^{\infty} \frac{1}{|\rho_k|} = +\infty.$$  

(2)

Proof. Necessity. Suppose, to the contrary, that the system $(\Theta_{k,\nu,p} : k \in \mathbb{N})$ is not complete in $L^2((0; 1); x^2 p \, dx)$. Then, by Lemma 6, there exists a nonzero even entire function $\Omega$ of exponential type $\sigma \leq 1$ which vanishes at the points $\rho_k$ and satisfies $z^{-\nu+1/2}(z^{2\nu} \Omega(z))' \in L^2(0; +\infty)$. By Lemma 5, the function $\Omega$ is of the kind (1). Due to Lemma 7, we have $|\Omega(x)| \leq C_1 (1 + |x|)^{-\nu}$ for $\nu \geq 1/2$ and all $x \in \mathbb{R}$. This implies

$$\int_{-\infty}^{+\infty} \log^+ \frac{|\Omega(x)|}{1 + x^2} \, dx < +\infty.$$
Therefore, by Lemma 1, we get
\[ \sum_{k \in \mathbb{N}} \left| \text{Im} \frac{1}{\rho_k} \right| < +\infty. \]

Since \( |\text{Im} \rho_k| \geq \delta |\rho_k| \), \( \delta > 0 \), for all \( k \in \mathbb{N} \), and
\[ \left| \text{Im} \frac{1}{\rho_k} \right| = \frac{|\text{Im} \rho_k|}{|\rho_k|^2} \geq \frac{\delta}{|\rho_k|}, \]
we have \( \sum_{k=1}^{\infty} \frac{1}{|\rho_k|} < +\infty \). This contradicts condition (2).

**Sufficiency.** Let the condition (2) not be fulfilled. Let us prove that the system \( (\Theta_{k,\nu,p} : k \in \mathbb{N}) \) is not complete in \( L^2((0; 1); x^{2p}dx) \). Let
\[ G(z) = \prod_{k=1}^{\infty} \left( 1 - \frac{z^2}{\rho_k^2} \right). \]

Since \( \sum_{k=1}^{\infty} \frac{1}{|\rho_k|} < +\infty \), then by Lemma 2, the function \( G \) is an even entire function of minimal type of order 1 and \( |G(x)| \leq \exp(\theta(|x|)), x \in \mathbb{R} \), where \( \theta : [0; +\infty) \to [0; +\infty) \) is an increasing function satisfying \( \int_0^{+\infty} t^{-2} \theta(t) dt < +\infty \). Further,
\[ \int_0^{+\infty} t^{-2}(1+\nu+1/2) \log(1+t) dt < +\infty. \]

Furthermore, according to Lemma 3, there exists an even entire function \( F \) of exponential type \( \sigma \leq 1 \) satisfying
\[ |F(x)| \leq \exp(-\theta(|x|) - (1+\nu+1/2) \log(1+|x|)), \quad |x| > x_0. \]

Consider the function \( f(z) = G(z)F(z) \). The function \( f \) is an even entire function of exponential type \( \sigma \leq 1 \) vanishing at the points \( \rho_k \) and satisfying the estimate
\[ |x^{\nu+1/2}f(x)| \leq |x|^{\nu+1/2} \exp(- (1+\nu+1/2) \log(1+|x|)) \leq (1+|x|)^{-1}, \quad x \in \mathbb{R}. \]

Hence \( z^{\nu+1/2}f(z) \in L^2((0; 1); t^{2p}dt) \) and by Lemma 4 the function \( f \) can be represented in the form \( f(z) = z^{-\nu} \int_0^1 \sqrt{t}J_\nu(zt)\gamma(t) dt \) with some function \( \gamma \in L^2((0; 1); t^{2p}dt) \). Since \( h(t) := t^{1-p}\gamma(t) \in L^2((0; 1); t^{2p}dt) \), we have the representation \( f(z) = z^{-\nu} \int_0^1 \sqrt{t}J_\nu(tz)h(t) dt \) with \( h \in L^2((0; 1); x^{2p}dx) \). Thus, by Lemmas 5 and 6, the system \( (\Theta_{k,\nu,p} : k \in \mathbb{N}) \) is incomplete in \( L^2((0; 1); x^{2p}dx) \). The theorem is proved.
Let \( n(t) \) be the number of points of the sequence \((\rho_k : k \in \mathbb{N}) \subset \mathbb{C}\) satisfying the inequality \(|\rho_k| \leq t\), i.e., \( n(t) := \sum_{|\rho_k| \leq t} 1 \), and let

\[
N(r) := \int_0^r \frac{n(t)}{t} \, dt, \quad r > 0.
\]

**Theorem 2.** Let \( \nu \geq 1/2, p \in \mathbb{R} \) and \((\rho_k : k \in \mathbb{N})\) be an arbitrary sequence of distinct nonzero complex numbers. If

\[
\lim_{r \to +\infty} \left( N(r) - \frac{2r}{\pi} + \nu \log(1 + r) \right) = +\infty,
\]

then the system \((\Theta_{k,\nu,p} : k \in \mathbb{N})\) is complete in \( L^2((0; 1); x^{2p} \, dx) \).

*Proof.* It suffices to assume the incompleteness of the system \((\Theta_{k,\nu,p} : k \in \mathbb{N})\) and prove that

\[
\lim_{r \to +\infty} \left( N(r) - \frac{2r}{\pi} + \nu \log(1 + r) \right) < +\infty. \tag{3}
\]

By Lemmas 5 and 6, there exists a nonzero even entire function \( \Omega \) of the form (1) of exponential type \( \sigma \leq 1 \) such that \( z^{-\nu + 1/2} (z^{2\nu} \Omega(z))' \in L^2(0; +\infty) \) and for which the sequence \((\rho_k : k \in \mathbb{N})\) is a subsequence of zeros. We may suppose \( \Omega(0) = 1 \). Then, consecutively applying the Jensen formula (see [6], [7, p. 4316]) and Lemma 7, we obtain

\[
N(r) \leq \frac{1}{2\pi} \int_0^{2\pi} \log |\Omega(re^{i\varphi})| \, d\varphi \leq C_2 + \frac{1}{2\pi} \int_0^{2\pi} (r|\sin \varphi| - \nu \log(1 + r)) \, d\varphi = \frac{2r}{\pi} - \nu \log(1 + r) + C_2, \quad r > 0,
\]

whence it follows (3). The theorem is proved. \( \blacksquare \)

**Theorem 3.** Let \( \nu \geq 1/2, p \in \mathbb{R} \) and \((\rho_k : k \in \mathbb{N})\) be an arbitrary sequence of distinct nonzero complex numbers. Let \(|\rho_k| \leq \Delta k + \beta\) for \( 0 < \Delta \leq \pi/2, -\Delta < \beta < \Delta(\nu - 1/2) \) and all sufficiently large \( k \in \mathbb{N} \). Then the system \((\Theta_{k,\nu,p} : k \in \mathbb{N})\) is complete in \( L^2((0; 1); x^{2p} \, dx) \).

*Proof.* Let \( \mu_k = \Delta k + \beta, k \in \mathbb{N}, n_1(t) = \sum_{\mu_k \leq t} 1 \) and \( N_1(r) = \int_0^r t^{-1} n_1(t) \, dt \).

Then \( n(t) \geq n_1(t), N(r) \geq N_1(r) \) and \( n_1(t) = m \) for \( \Delta m + \beta \leq t < \Delta(m + 1) + \beta \).
(n_1(t) = 0 on (0; \mu_1)). Let r \in [\mu_s; \mu_{s+1})$. Then $s = \frac{r}{\Delta} + O(1)$ as $r \to +\infty$. Therefore, using formula \( \log(1 + x) = x - \frac{x^2}{2} + O(x^3), x \to 0 \), we obtain

\[
N_1(r) = \sum_{k=1}^{s-1} \int_{\mu_k}^{\mu_{k+1}} \frac{n_1(t)}{t} dt + \int_{\mu_s}^{r} \frac{n_1(t)}{t} dt = \sum_{k=1}^{s-1} k \log \frac{\mu_{k+1}}{\mu_k} + s \log \frac{r}{\mu_s} = s - \sum_{k=1}^{s-1} k \log \left(1 + \frac{\Delta}{\Delta k + \beta}\right) + O(1) = \frac{(\Delta k + \beta) - \beta}{\Delta k + \beta} \left(\frac{1}{\Delta k + \beta} - \frac{\Delta}{2(\Delta k + \beta)^2}\right) + O(1) = \frac{r}{\Delta} - \left(\frac{1}{2} + \frac{\beta}{\Delta}\right) \log r + O(1) \text{ as } r \to +\infty.
\]

Hence, for $0 < \Delta \leq \pi/2$ and $-\Delta < \beta < \Delta(\nu - 1/2)$, we get

\[
\lim_{r \to +\infty} \left(N(r) - \frac{2r}{\pi} + \nu \log(1 + r)\right) \geq \lim_{r \to +\infty} \left(N_1(r) - \frac{2r}{\pi} + \nu \log(1 + r)\right) \geq \lim_{r \to +\infty} \left(\frac{r}{\Delta} - \left(\frac{1}{2} + \frac{\beta}{\Delta}\right) \log r - \frac{2r}{\pi} + \nu \log r + O(1)\right) = +\infty.
\]

Thus, according to Theorem 2, we obtain the required proposition. The proof of theorem is completed.

\[\Box\]

**References**


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Received 30 May 2017
Accepted 11 May 2020