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On Perturbation of the Schrödinger Operator with a Localized Complex-Valued Potential

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Abstract. The Schrödinger operator on the axis with a localized potential, which is the sum of a small potential and a potential with a contracting carrier (narrow potential), which can grow unlimitedly as the carrier is compressed, is considered. Potentials depend on two small consistent parameters. One of the parameters describes the length of the carrier of a narrow potential and the value of a small potential, the reciprocal of the second corresponds to the potential values. A sufficient condition is obtained under which an eigenvalue appears from the edge of the continuous spectrum and its asymptotic behavior is constructed. A sufficient condition is also given under which the eigenvalue does not appear.

Key Words and Phrases: Schrödinger operator, perturbation, asymptotics.

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1. Introduction

As it is well known, under a perturbation of a quantum waveguide, eigenvalues can appear from the edge of the continuous spectrum. In particular, in [1], [2], for slightly curved waveguides, the existence of such an eigenvalue was proved and, in [3], the asymptotic behavior of this eigenvalue was explored. In [3], the conditions under which an eigenvalue appears from the border of the continuous spectrum under a perturbation of a quantum waveguide by a small potential were presented and its asymptotic behavior was explored. In [4]–[6], local deformations of waveguides were considered. In [4], it was proved that, if the mean value of a perturbation is positive, then an eigenvalue appears from the border of the continuous spectrum and, if this value is negative, then there is no eigenvalue of this kind. Moreover, in the two-dimensional case, the asymptotic behavior of the eigenvalue thus appearing was studied, and, as proved in [5], such an eigenvalue can both appear and not appear in the critical case in which the mean value of the perturbation of the boundary is zero. For the case where the eigenvalue still appears, its asymptotic behavior was studied in [6]. The results of [1]–[5] and the most of those of [6] were obtained using the Birman-Schwinger method for self-adjoint operators. In [7], the two-dimensional Schrödinger operator with magnetic and electric potentials was considered.

In [14], a rather arbitrary small localized perturbation of waveguides with various types of boundary conditions was considered. In [10], [11], based on the methods proposed in [8] and [9], respectively, two-parameter perturbations of the Schrödinger and Hill operators on the axis were considered. Disturbance conditions under which eigenvalues arise from the edge of the continuous spectrum values were obtained. In [12], based on the approach proposed in [13], the results in [10] were generalized to a wider class of parameters (in the case where the perturbation is the sum of two narrow potentials).

In this paper, we study the eigenvalues of the Schrödinger operator on the axis with a localized potential, which is the sum of a small potential and potential with a contracting support that can grow unlimitedly as its support contracts. Potentials depend on two small parameters. One of the parameters describes the length of the carrier for a narrow potential and the value of a small potential, the reciprocal of the second corresponds to the potential values. In contrast to [10], weaker restrictions are imposed on the ratio of parameters (the product of one parameter by the reciprocal of the second parameter tends to zero).

The structure of the work is as follows. In the next, second section, a theorem which constitutes the main result of the paper, is formulated. In the third section some auxiliary statements are formulated and proved. The fourth section gives a proof of the main theorem. In the final, fifth section, qualitative statements about spectrum structure of the considered operator are given.

2. Statement of the problem and formulation of results

Let $V_1(x)$ and $V_2(x)$ be complex-valued functions from $C_0^\infty(\mathbb{R})$, x_1 be an arbitrary number, and the parameters $0 < \mu, \varepsilon \ll 1$ satisfy the relation

$$\varepsilon\mu^{-1} = o(1). \tag{1}$$

In this paper we consider the operator

$$\mathcal{H}_{\varepsilon,\mu} := -\frac{d^2}{dx^2} + \mu^{-1}\left(V_1\left(\frac{x-x_1}{\varepsilon}\right) + \varepsilon V_2(x)\right) \tag{2}$$

in $L_2(\mathbb{R})$ on the domain $W_2^2(\mathbb{R})$.

It is well known (see, for example, [15, Chapter V]) that the operator $\mathcal{H}_0 := -\frac{d^2}{dx^2}$ in $L_2(\mathbb{R})$ on domain $W_2^2(\mathbb{R})$ is self-adjoint, its discrete spectrum is empty, and the continuous spectrum coincides with the semi-axis $[0, +\infty)$.

Suppose that the segment $Q = [a, b]$ is such that $x_1 \in Q$ and $\mathrm{supp}V_j(x) \subset Q$, $j = 1, 2$. We denote

$$\langle g \rangle := \int_{\mathbb{R}} g(t)dt.$$

Our main result is the following theorem.

Theorem 1. *Suppose that the condition (1) holds true. If*

$$\mathrm{Re}\left(\langle V_1 \rangle + \langle V_2 \rangle\right) > 0, \tag{3}$$

then there exists the unique eigenvalue $\lambda_{\varepsilon,\mu}$ of operator $\mathcal{H}_{\varepsilon,\mu}$ tending to zero as $\varepsilon \to 0$. This eigenvalue is simple and its asymptotics has the form

$$\lambda_{\varepsilon,\mu} = -\frac{1}{4}\left(\varepsilon\mu^{-1}\right)^2 \left(\langle V_1 \rangle + \langle V_2 \rangle\right)^2 + O\left(\varepsilon^3\mu^{-3}\right). \tag{4}$$

If

$$\mathrm{Re}\left(\langle V_1 \rangle + \langle V_2 \rangle\right) < 0, \tag{5}$$

then operator $\mathcal{H}_{\varepsilon,\mu}$ has no eigenvalues converging to zero as $\varepsilon \to 0$.

3. Auxiliary statements

It is easy to see that the function

$$W_j(\xi) = \frac{1}{2}\int_{\mathbb{R}} |\xi - t| V_j(t)dt,$$

solves the equation

$$W_j''(\xi) = V_j(\xi), \quad j = 1, 2. \tag{6}$$

In what follows we assume that $x_1 \notin \mathrm{supp}V_2(x)$. Then for sufficiently small ε we have $\mathrm{supp}V_1(\frac{x-x_1}{\varepsilon}) \cap \mathrm{supp}V_2(x) = \varnothing$. That is why there exist fixed intervals $Q_1 \subset Q$ and $Q_2 \subset Q$ such that $\mathrm{supp}V_1(\frac{x-x_1}{\varepsilon}) \subset Q_1$, $\mathrm{supp}V_2(x) \subset Q_2$ and $Q_1 \cap Q_2 = \varnothing$. We choose the cut-off functions $\chi_j(x)$ satisfying the following conditions: functions $\chi_1(x), \chi_2(x)$ equal to unity for $x \in Q_j$ and zero at $x \notin Q_j$ respectively. Therefore, they satisfy the condition $\chi_1(x)\chi_2(x) \equiv 0$.

We follow the approach proposed in [13]. We put

$$\varphi_{\varepsilon,\mu}(x) := 1 + \varepsilon^2\mu^{-1}\chi_1(x)W_1\left(\frac{x - x_1}{\varepsilon}\right) + \varepsilon\mu^{-1}\chi_2(x)W_2(x). \tag{7}$$

We denote by $U_{\varepsilon,\mu}$ the operator of multiplication by function $\varphi_{\varepsilon,\mu}(x)$:

$$U_{\varepsilon,\mu}[v] := \varphi_{\varepsilon,\mu}(x)v. \tag{8}$$

Operator $U_{\varepsilon,\mu}$ performs one-to-one correspondence of $L_2(\mathbb{R})$ onto itself. Hence, the eigenvalues of $\mathcal{H}_{\varepsilon,\mu}$ coincide with the eigenvalues of operator $U^{-1}_{\varepsilon,\mu}\mathcal{H}_{\varepsilon,\mu}U_{\varepsilon,\mu}$ (see [13]).

Lemma 1. *Suppose that the condition (1) holds true. Then operator $U_{\varepsilon,\mu}$ satisfies the estimates*

$$\left|U^{-1}_{\varepsilon,\mu}[1]\right| \leqslant C_1, \quad x \in Q, \tag{9}$$

$$U^{-1}_{\varepsilon,\mu}[1] = 1 + O(\varepsilon\mu^{-1}), \quad x \in Q, \tag{10}$$

where the constant C_1 is independent of ε, μ.

Proof. Estimate (9) follows immediately from the definition of functions χ_j, W_j and (1), (7), (8). It follows from (8) that

$$U^{-1}_{\varepsilon,\mu}[1] = \frac{1}{\varphi_{\varepsilon,\mu}(x)} := \widetilde{\varphi}_{\varepsilon,\mu}(x).$$

Further,

$$\widetilde{\varphi}_{\varepsilon,\mu}(x) = \widetilde{\varphi}_{\varepsilon,\mu}(0) + \widetilde{\varphi}'_{\varepsilon,\mu}(c)x, \quad 0 < c < x, \quad x \in Q,$$

where

$$\widetilde{\varphi}'_{\varepsilon,\mu}(x) = \frac{-\varepsilon\mu^{-1}}{\varphi^2_{\varepsilon,\mu}(x)}\Big(\chi_1(x)W_1'(\xi_1) + \chi_2(x)W_2'(x) + \varepsilon\chi_1'(x)W_1(\xi_1) + \chi_2'(x)W_2(x)\Big),$$

$\xi_1 = (x - x_1)\varepsilon^{-1}$.

The definition of function $\widetilde{\varphi}_{\varepsilon,\mu}(x)$ yields

$$\begin{aligned}
\widetilde{\varphi}_{\varepsilon,\mu}(0) &= \frac{1}{1+q}, \\
\widetilde{\varphi}'_{\varepsilon,\mu}(c) &= -\frac{\varepsilon\mu^{-1}}{\varphi^2_{\varepsilon,\mu}(c)}\Big(\chi_1(c)W_1'\left(\frac{c-x_1}{\varepsilon}\right) + \varepsilon\chi_1'(c)W_1\left(\frac{c-x_1}{\varepsilon}\right) \\
&\quad + \chi_2(c)W_2'(c) + \chi_2'(c)W_2(c)\Big),
\end{aligned} \tag{11}$$

where

$$q = \varepsilon^2\mu^{-1}\chi_1(0)W_1\left(\frac{-x_1}{\varepsilon}\right) + \varepsilon\mu^{-1}\chi_2(0)W_2(0).$$

By (1), (7) and the definition of the functions χ_j, W_j, $j = 1, 2$ we obtain the estimates

$$|q| < C_2\varepsilon\mu^{-1}, \qquad |\varphi_{\varepsilon,\mu}^{-2}(c)| \leqslant C_3,$$

where C_2, C_3 are independent of ε, μ. Consequently, the last estimates and relations (11) imply

$$\begin{aligned}\widetilde{\varphi}_{\varepsilon,\mu}(0) =& 1 + O(q) = 1 + O(\varepsilon\mu^{-1}),\\ \widetilde{\varphi}'_{\varepsilon,\mu}(c) =& O(\varepsilon\mu^{-1}).\end{aligned}$$

The latter estimates imply (10). Lemma is completely proved. ◄

Lemma 2. *Suppose that the condition (1) is satisfied. Then the representation*

$$U_{\varepsilon,\mu}^{-1}\mathcal{H}_{\varepsilon,\mu}U_{\varepsilon,\mu} = \mathcal{H}_0 + \varepsilon\mu^{-1}\mathcal{L}_{\varepsilon,\mu} \tag{12}$$

holds true, where $\mathcal{L}_{\varepsilon,\mu}$ is a second order differential operator with bounded compactly supported coefficients satisfying the estimate

$$\|\mathcal{L}_{\varepsilon,\mu}u\|_{L_2(\mathbb{R})} \leqslant C_4\|u\|_{W_2^2(Q)}, \tag{13}$$

with C_4 independent of ε, μ.

Proof. By (2) and (8) we get

$$\begin{aligned}\mathcal{H}_{\varepsilon,\mu}U_{\varepsilon,\mu} =& \big(\mathcal{H}_0 + \mu^{-1}\left(V_1(\xi_1) + \varepsilon V_2(x)\right)\big)\Big[1 + \varepsilon^2\mu^{-1}\chi_1(x)W_1(\xi_1)\\ &+ \varepsilon\mu^{-1}\chi_2(x)W_2(x)\Big]\\ =& \mathcal{H}_0 + \mu^{-1}\left(V_1(\xi_1) + \varepsilon V_2(x)\right) + \varepsilon^2\mu^{-1}\chi_1(x)\mathcal{H}_0[W_1(\xi_1)]\\ &+ \varepsilon\mu^{-1}\chi_2(x)\mathcal{H}_0[W_2(x)]\\ &+ \varepsilon^2\mu^{-1}\Big(-\chi_1''(x)W_1(\xi_1) - \chi_1(x)W_1(\xi_1)\frac{d^2}{dx^2}\\ &- 2\varepsilon^{-1}\chi_1'(x)W_1'(\xi_1) - 2\chi_1'(x)W_1(\xi_1)\frac{d}{dx}\\ &- 2\varepsilon^{-1}\chi_1(x)W_1'(\xi_1)\frac{d}{dx}\Big) + \varepsilon\mu^{-1}\Big(-\chi_2''(x)W_2(x)\\ &- \chi_2(x)W_2(x)\frac{d^2}{dx^2}\\ &- 2\chi_2'(x)W_2'(x) - 2\chi_2'(x)W_2(x)\frac{d}{dx} - 2\chi_2(x)W_2'(x)\frac{d}{dx}\Big)\\ &+ \varepsilon^2\mu^{-2}\Big(\chi_1(x)V_1(\xi_1)W_1(\xi_1) + \chi_2(x)V_2(x)W_2(x)\\ &+ V_1(\xi_1)\chi_2(x)W_2(x) + V_2(x)\chi_1(x)W_1(\xi_1)\Big).\end{aligned}$$

By virtue of equations (6), the definitions of the operator $\mathcal{H}_0$ and the functions χ_j, $j = 1, 2$ from the last equalities we obtain

$$\begin{aligned}
&\mu^{-1}\left(V_1(\xi_1) + \varepsilon V_2(x)\right) + \varepsilon^2\mu^{-1}\chi_1(x)\mathcal{H}_0[W_1(\xi_1)] + \varepsilon\mu^{-1}\chi_2(x)\mathcal{H}_0[W_2(x)] = \\
&\mu^{-1}\left(V_1(\xi_1) + \varepsilon V_2(x)\right) + \varepsilon^2\mu^{-1}\chi_1(x)\left(-\varepsilon^{-2}W_1''(\xi_1)\right) + \varepsilon\mu^{-1}\chi_2(x)\left(-W_2''(x)\right) = \\
&\mu^{-1}\left(V_1(\xi_1) + \varepsilon V_2(x)\right) - \mu^{-1}\left(V_1(\xi_1) + \varepsilon V_2(x)\right) = 0.
\end{aligned}$$

It follows from the definition of χ_j that

$$V_1(\xi_1)\chi_2(x)W_2(x) \equiv 0, \qquad V_2(x)\chi_1(x)W_1(\xi_1) \equiv 0.$$

In view of these identities and from (7), (8) we obtain

$$\begin{aligned}
\mathcal{H}_{\varepsilon,\mu}U_{\varepsilon,\mu} =& U_{\varepsilon,\mu}[1]\mathcal{H}_0 + \varepsilon\mu^{-1}\Big(- \varepsilon\chi_1''(x)W_1(\xi_1) - 2\chi_1'(x)W_1'(\xi_1) \\
& - 2\varepsilon\chi_1'(x)W_1(\xi_1)\frac{d}{dx} - 2\chi_1(x)W_1'(\xi_1)\frac{d}{dx} - \chi_2''(x)W_2(x) \\
& - 2\chi_2'(x)W_2'(x) - 2\chi_2'(x)W_2(x)\frac{d}{dx} - 2\chi_2(x)W_2'(x)\frac{d}{dx} \\
& + \varepsilon\mu^{-1}V_1(\xi_1)W_1(\xi_1) + \varepsilon\mu^{-1}V_2(x)W_2(x)\Big).
\end{aligned}$$

From (8) and the last equality, the representation (12) follows, where

$$\begin{aligned}
\mathcal{L}_{\varepsilon,\mu} =& U^{-1}_{\varepsilon,\mu}[1]\Big(- \varepsilon\chi_1''(x)W_1(\xi_1) - 2\chi_1'(x)W_1'(\xi_1) - 2\varepsilon\chi_1'(x)W_1(\xi_1)\frac{d}{dx} \\
& - 2\chi_1(x)W_1'(\xi_1)\frac{d}{dx} - \chi_2''(x)W_2(x) - 2\chi_2'(x)W_2'(x) \\
& - 2\chi_2'(x)W_2(x)\frac{d}{dx} - 2\chi_2(x)W_2'(x)\frac{d}{dx}\Big) \\
& + U^{-1}_{\varepsilon,\mu}[1]\varepsilon\mu^{-1}\Big(V_1(\xi_1)W_1(\xi_1) + V_2(x)W_2(x)\Big).
\end{aligned} \tag{14}$$

Further, we show that the operator $\mathcal{L}_{\varepsilon,\mu}$ satisfies estimate (13). It follows from (9) that

$$\begin{aligned}
\|\mathcal{L}_{\varepsilon,\mu}u\|_{L_2(\mathbb{R})} \leqslant& \varepsilon C_1\Big(\Big\|\chi_1''(x)W_1(\xi_1)u\Big\|_{L_2(\mathbb{R})} + 2\Big\|\chi_1'(x)W_1'(\xi_1)u\Big\|_{L_2(\mathbb{R})} \\
& + 2\varepsilon\Big\|\chi_1'(x)W_1(\xi_1)\frac{du}{dx}\Big\|_{L_2(\mathbb{R})} + 2\Big\|\chi_1(x)W_1'(\xi_1)\frac{du}{dx}\Big\|_{L_2(\mathbb{R})} \\
& + \Big\|\chi_2''(x)W_2(x)u\Big\|_{L_2(\mathbb{R})} + 2\Big\|\chi_2'(x)W_2'(x)u\Big\|_{L_2(\mathbb{R})} \\
& + 2\Big\|\chi_2'(x)W_2(x)\frac{du}{dx}\Big\|_{L_2(\mathbb{R})} + 2\Big\|\chi_2(x)W_2'(x)\frac{du}{dx}\Big\|_{L_2(\mathbb{R})}\Big) \\
& + \varepsilon\mu^{-1}C_1\left(\Big\|V_1(\xi_1)W_1(\xi_1)u\Big\|_{L_2(\mathbb{R})} + \Big\|V_2(x)W_2(x)u\Big\|_{L_2(\mathbb{R})}\right).
\end{aligned} \tag{15}$$

Next, we estimate each term on the right-hand side of (15). By the definition of the functions χ_j, $j = 1, 2$ we have

$$\left\|\chi_1''(x)W_1(\xi_1)u\right\|_{L_2(\mathbb{R})} = \left\|\chi_1''(x)W_1(\xi_1)u\right\|_{L_2(Q)} \leqslant \left\|\chi_1''(x)W_1(\xi_1)\right\|_{C(Q)}\|u\|_{L_2(Q)} \leqslant C_5\|u\|_{W_2^2(Q)},$$

$$\left\|\chi_1'(x)W_1'(\xi_1)u\right\|_{L_2(\mathbb{R})} = \left\|\chi_1'(x)W_1'(\xi_1)u\right\|_{L_2(Q)} \leqslant \left\|\chi_1'(x)W_1'(\xi_1)\right\|_{C(Q)}\|u\|_{L_2(Q)} \leqslant C_6\|u\|_{W_2^2(Q)},$$

$$\left\|\chi_1(x)W_1'(\xi_1)\frac{du}{dx}\right\|_{L_2(\mathbb{R})} = \left\|\chi_1(x)W_1'(\xi_1)\frac{du}{dx}\right\|_{L_2(Q)} \leqslant \left\|\chi_1(x)W_1'(\xi_1)\right\|_{C(Q)}\left\|\frac{du}{dx}\right\|_{L_2(Q)} \leqslant C_7\|u\|_{W_2^2(Q)},$$

$$\left\|\chi_1'(x)W_1(\xi_1)\frac{du}{dx}\right\|_{L_2(\mathbb{R})} = \left\|\chi_1'(x)W_1(\xi_1)\frac{du}{dx}\right\|_{L_2(Q)} \leqslant \left\|\chi_1'(x)W_1(\xi_1)\right\|_{C(Q)}\left\|\frac{du}{dx}\right\|_{L_2(Q)} \leqslant C_8\|u\|_{W_2^2(Q)},$$

$$\left\|\chi_2''(x)W_2(x)u\right\|_{L_2(\mathbb{R})} = \left\|\chi_2''(x)W_2(x)u\right\|_{L_2(Q)} \leqslant \left\|\chi_2''(x)W_2(x)\right\|_{C(Q)}\|u\|_{L_2(Q)} \leqslant C_9\|u\|_{W_2^2(Q)},$$

$$\left\|\chi_2'(x)W_2'(x)u\right\|_{L_2(\mathbb{R})} = \left\|\chi_2'(x)W_2'(x)u\right\|_{L_2(Q)} \leqslant \left\|\chi_2'(x)W_2'(x)\right\|_{C(Q)}\|u\|_{L_2(Q)} \leqslant C_{10}\|u\|_{W_2^2(Q)},$$

$$\left\|\chi_2'(x)W_2(x)\frac{du}{dx}\right\|_{L_2(\mathbb{R})} = \left\|\chi_2'(x)W_2(x)\frac{du}{dx}\right\|_{L_2(Q)} \leqslant \left\|\chi_2'(x)W_2(x)\right\|_{C(Q)}\left\|\frac{du}{dx}\right\|_{L_2(Q)} \leqslant C_{11}\|u\|_{W_2^2(Q)},$$

$$\left\|\chi_2(x)W_2'(x)\frac{du}{dx}\right\|_{L_2(\mathbb{R})} = \left\|\chi_2(x)W_2'(x)\frac{du}{dx}\right\|_{L_2(Q)} \leqslant \left\|\chi_2(x)W_2'(x)\right\|_{C(Q)}\left\|\frac{du}{dx}\right\|_{L_2(Q)} \leqslant C_{12}\|u\|_{W_2^2(Q)},$$

$$\begin{aligned}\Big\|V_1(\xi_1)W_1(\xi_1)u\Big\|_{L_2(\mathbb{R})} &= \Big\|V_1(\xi_1)W_1(\xi_1)u\Big\|_{L_2(Q)} \leqslant \Big\|V_1(\xi_1)W_1(\xi_1)\Big\|_{C(Q)}\|u\|_{L_2(Q)}\\ &\leqslant C_{13}\|u\|_{W_2^2(Q)},\end{aligned}$$

$$\begin{aligned}\Big\|V_2(x)W_2(x)u\Big\|_{L_2(\mathbb{R})} &= \Big\|V_2(x)W_2(x)u\Big\|_{L_2(Q)} \leqslant \Big\|V_2(x)W_2(x)\Big\|_{C(Q)}\|u\|_{L_2(Q)}\\ &\leqslant C_{14}\|u\|_{W_2^2(Q)},\end{aligned}$$

where the constants C_j, $j = 5,\dots,14$ are independent of ε,μ. Estimate (13) follows from the last estimates, (15) and (1). Lemma is proved completely. ◄

4. Proof of the theorem

We introduce the notations

$$\begin{aligned}&m^{(1)}_{\varepsilon,\mu} := \int_{\mathbb{R}} \mathcal{L}_{\varepsilon,\mu}[1]dx, \qquad m^{(2)}_{\varepsilon,\mu} := \int_{\mathbb{R}} \mathcal{L}_{\varepsilon,\mu}\left[\int_{\mathbb{R}} |x-t|\mathcal{L}_{\varepsilon,\mu}[1]dt\right]dx,\\ &k_{\varepsilon,\mu} := \frac{\varepsilon\mu^{-1}}{2}m^{(1)}_{\varepsilon,\mu} + \frac{(\varepsilon\mu^{-1})^2}{2}m^{(2)}_{\varepsilon,\mu}.\end{aligned}\tag{16}$$

Since operator $\mathcal{L}_{\varepsilon,\mu}$ satisfies inequality (13), Theorem 1 in [8] implies that once

$$k_{\varepsilon,\mu} = \varepsilon\mu^{-1}c_1 + (\varepsilon\mu^{-1})^2 c_2 + O\left((\varepsilon\mu^{-1})^3\right), \quad c_1, c_2 = \mathrm{const},\tag{17}$$

a sufficient condition for the existence of an eigenvalue converging to zero as $\varepsilon,\mu \to 0$ of the operator $(\mathcal{H}_0 - \varepsilon\mu^{-1}\mathcal{L}_{\varepsilon,\mu})$ is the inequality

$$\mathrm{Re}(c_1 + \varepsilon\mu^{-1}c_2) < 0,\tag{18}$$

while a sufficient condition for the absence of such eigenvalue is the inequality

$$\mathrm{Re}(c_1 + \varepsilon\mu^{-1}c_2) > 0.\tag{19}$$

If (18) is satisfied, then the operator $(\mathcal{H}_0 - \varepsilon\mu^{-1}\mathcal{L}_{\varepsilon,\mu})$ has the unique eigenvalue converging to zero. This eigenvalue is simple and has the asymptotics

$$\lambda_{\varepsilon,\mu} = -\left(\varepsilon\mu^{-1}c_1 + (\varepsilon\mu^{-1})^2 c_2\right)^2 + O\left(c_1(\varepsilon\mu^{-1})^4 + (\varepsilon\mu^{-1})^5\right).\tag{20}$$

It follows from (14) that

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{L}_{\varepsilon,\mu}[1]dx = \int_{\mathbb{R}} &U^{-1}_{\varepsilon,\mu}[1]\Big(-\varepsilon\chi_1''(x)W_1(\xi_1) - 2\chi_j'(x)W_1'(\xi_1)\\ &- \chi_2''(x)W_2(x) - 2\chi_2'(x)W_2'(x)\Big)dx\\ &+ \varepsilon\mu^{-1}\int_{\mathbb{R}} U^{-1}_{\varepsilon,\mu}[1]\Big(V_1(\xi_1)W_1(\xi_1) + V_2(x)W_2(x)\Big)dx.\end{aligned}\tag{21}$$

Changing the variable, we obtain the estimate

$$\begin{aligned}\int_{\mathbb{R}} V_1\left(\frac{x-x_1}{\varepsilon}\right) W_1\left(\frac{x-x_1}{\varepsilon}\right) dx &= \varepsilon \int_{\mathbb{R}} V_1(t)W_1(t)dt \\ &= \varepsilon \int_Q V_1(t)W_1(t)dt = O(\varepsilon), \\ \int_{\mathbb{R}} V_2\,(x)\,W_2\,(x)\,dx = \int_Q V_2\,(x)\,W_2\,(x)\,dx &= O(1).\end{aligned}$$

The last estimates, by virtue of (1) and (10), imply the estimate

$$\varepsilon\mu^{-1}\int_{\mathbb{R}} U^{-1}_{\varepsilon,\mu}[1]\Big(V_1(\xi_1)W_1(\xi_1)+V_2(x)W_2(x)\Big)dx = O(\varepsilon\mu^{-1}).$$

From (21) and the last estimate we obtain

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{L}_{\varepsilon,\mu}[1]dx = \int_{\mathbb{R}} U^{-1}_{\varepsilon,\mu}[1]\Big(&-\varepsilon\chi_1''(x)W_1(\xi_1)-2\chi_j'(x)W_1'(\xi_1) \\ &-\chi_2''(x)W_2(x)-2\chi_2'(x)W_2'(x)\Big)dx + O(\varepsilon\mu^{-1}).\end{aligned}$$

From the last equality and (6) it follows

$$\begin{aligned}&\int_{\mathbb{R}} \left(-\varepsilon\chi_1''(x)W_1(\xi_1)-2\chi_1'(x)W_1'(\xi_1)\right)dx = -\int_{\mathbb{R}}\chi_1'(x)W_1'(\xi_1)dx \\ &= \varepsilon^{-1}\int_{\mathbb{R}}\chi_1(x)W_1''(\xi_1)dx = \varepsilon^{-1}\int_{\mathbb{R}}V_1(\xi_1)dx = \int_{\mathbb{R}}V_1(t)dt,\end{aligned} \tag{22}$$

$$\begin{aligned}\int_{\mathbb{R}} \left(-\chi_2''(x)W_1(x)-2\chi_2'(x)W_2'(x)\right)dx &= -\int_{\mathbb{R}}\chi_2'(x)W_2'(x)dx \\ &= \int_{\mathbb{R}}\chi_2(x)W_2''(x)dx = \int_{\mathbb{R}}V_2(x)dx.\end{aligned} \tag{23}$$

From (10), (21), (22) and (23) it follows that

$$\int_{\mathbb{R}} \mathcal{L}_{\varepsilon,\mu}[1]dx = \langle V_1\rangle + \langle V_2\rangle + O(\varepsilon\mu^{-1}). \tag{24}$$

Let us prove the estimate

$$m^{(2)}_{\varepsilon,\mu} = O(1). \tag{25}$$

By (16) we get

$$m^{(2)}_{\varepsilon,\mu}=\int_{\mathbb{R}}\mathcal{L}_{\varepsilon,\mu}\left[\int_{\mathbb{R}}|x-t|\mathcal{L}_{\varepsilon,\mu}[1]dt\right]dx=\int_{\mathbb{R}}\mathcal{L}_{\varepsilon,\mu}[f(x)]dx,$$

where

$$f(x):=\int_{\mathbb{R}}|x-t|\mathcal{L}_{\varepsilon,\mu}[1]dt. \tag{26}$$

The definition of operator $\mathcal{L}_{\varepsilon,\mu}$ for $x\in Q$ and (24) yield

$$\left|\int_{\mathbb{R}}|x-t|\mathcal{L}_{\varepsilon,\mu}[1]dt\right|=\left|\int_{Q}|x-t|\mathcal{L}_{\varepsilon,\mu}[1]dt\right|\leqslant C_Q\left|\int_{Q}\mathcal{L}_{\varepsilon,\mu}[1]dt\right|\leqslant C_{15},$$

where $C_Q=\max\limits_{x,t\in Q}|x-t|$, therefore

$$|f(x)|\leqslant C_{15},\quad x\in Q, \tag{27}$$

where C_{15} is independent of ε,μ.

Estimate (15) yields

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\mathcal{L}_{\varepsilon,\mu}[f(x)]dx\right|\leqslant C_1\Big(\varepsilon\int_{\mathbb{R}}\Big|W_1(\xi_1)\chi_1''(x)f(x)\Big|dx\\
&+2\int_{\mathbb{R}}\Big|\chi_1'(x)W_1'(\xi_1)f(x)\Big|dx+2\varepsilon\int_{\mathbb{R}}\Big|W_1(\xi_1)\chi_1'(x)f'(x)\Big|dx\\
&+2\int_{\mathbb{R}}\Big|\chi_1(x)W_1'(\xi_j)f'(x)\Big|dx+\int_{\mathbb{R}}\Big|W_2(x)\chi_2''(x)f(x)\Big|dx\\
&+2\int_{\mathbb{R}}\Big|\chi_2'(x)W_2'(x)f(x)\Big|dx+2\int_{\mathbb{R}}\Big|W_2(x)\chi_2'(x)f'(x)\Big|dx\\
&+2\int_{\mathbb{R}}\Big|\chi_2(x)W_2'(x)f'(x)\Big|dx\Big)+\\
&+\varepsilon\mu^{-1}C_1\left(\int_{\mathbb{R}}\Big|V_1(\xi_1)W_1(\xi_1)f(x)\Big|dx+\int_{\mathbb{R}}\Big|V_2(x)W_2(x)f(x)\Big|dx\right).
\end{aligned} \tag{28}$$

Since functions χ_j and V_j are compactly supported, by (27) we obtain the estimates

$$\begin{aligned}
&\int_{\mathbb{R}} \Big|W_1(\xi_1)\chi_1''(x)f(x)\Big|dx = \int_Q \Big|W_1(\xi_1)\chi_1''(x)f(x)\Big|dx \leqslant C_{16},\\
&\int_{\mathbb{R}} \Big|\chi_1'(x)W_1'(\xi_1)f(x)\Big|dx = \int_Q \Big|\chi_1'(x)W_1'(\xi_1)f(x)\Big|dx \leqslant C_{17},\\
&\int_{\mathbb{R}} \Big|V_1(\xi_1)W_1(\xi_1)f(x)\Big|dx = \int_Q \Big|V_1(\xi_1)W_1(\xi_1)f(x)\Big|dx \leqslant C_{18},\\
&\int_{\mathbb{R}} \Big|W_2(x)\chi_2''(x)f(x)\Big|dx = \int_Q \Big|W_2(x)\chi_2''(x)f(x)\Big|dx \leqslant C_{19},\\
&\int_{\mathbb{R}} \Big|\chi_2'(x)W_2'(x)f(x)\Big|dx = \int_Q \Big|\chi_2'(x)W_2'(x)f(x)\Big|dx \leqslant C_{20},\\
&\int_{\mathbb{R}} \Big|V_2(x)W_2(x)f(x)\Big|dx = \int_Q \Big|V_2(\xi_1)W_2(x)f(x)\Big|dx \leqslant C_{21},
\end{aligned} \tag{29}$$

where the constants C_j, $j = 16, \dots, 21$ are independent of ε, μ.

It follows from (26) that

$$f'(x) = \int_{-\infty}^{x} \mathcal{L}_{\varepsilon,\mu}[1]dt + \int_{+\infty}^{x} \mathcal{L}_{\varepsilon,\mu}[1]dt. \tag{30}$$

We denote

$$\begin{aligned}
f_1(x) :=& \chi_1(x)W_1'(\xi_1), \quad f_2(x) := \chi_1'(x)W_1(\xi_1),\\
f_3(x) :=& \chi_2(x)W_2'(x), \quad f_4(x) := \chi_2'(x)W_2(x).
\end{aligned}$$

By (30) the inequality

$$\int_{\mathbb{R}} \Big|f_j(x)f'(x)\Big|dx \leqslant \int_{\mathbb{R}} \Big|f_j(x)\int_{-\infty}^{x} \mathcal{L}_{\varepsilon,\mu}[1]dt\Big|dx + \int_{\mathbb{R}} \Big|f_j(x)\int_{+\infty}^{x} \mathcal{L}_{\varepsilon,\mu}[1]dt\Big|dx$$

holds true, where $j = 1, 2, 3, 4$. Changing the order of integration in the second term, we rewrite the latter inequality in the form

$$\int_{\mathbb{R}} \big|f_j(x)f'(x)\big|\,dx \leqslant 2\int_{\mathbb{R}} \left|f_j(x)\int_{-\infty}^{x} \mathcal{L}_{\varepsilon,\mu}[1]dt\right|dx. \tag{31}$$

By (24) we have

$$\begin{aligned}
&\int_{\mathbb{R}} \left|f_j(x)\int_{-\infty}^{x} \mathcal{L}_{\varepsilon,\mu}[1]dt\right|dx = \int_Q \left|f_j(x)\int_{-\infty}^{x} \mathcal{L}_{\varepsilon,\mu}[1]dt\right|dx\\
&\leqslant \int_Q |f_j(x)|\left|\int_{-\infty}^{x} \mathcal{L}_{\varepsilon,\mu}[1]dt\right|dx \leqslant \int_Q |f_j(x)|\left|\int_Q \mathcal{L}_{\varepsilon,\mu}[1]dt\right|dx\\
&\leqslant C_{22}^j \left|\int_Q \mathcal{L}_{\varepsilon,\mu}[1]dt\right| \leqslant C_{23}^j,
\end{aligned}$$

where the constants C_{23}^j, $j = 1, \dots, 4$ are independent of ε, μ.

This inequality and (31) imply that

$$\int_{\mathbb{R}} \left| f_j(x) f'(x) \right| dx \leqslant 2C_{23}^j.$$

The last inequality and (28), (29) imply the estimate (25).

From (16), (24), (25) it follows

$$k_{\varepsilon,\mu} = \frac{\varepsilon\mu^{-1}}{2} \left(\langle V_1 \rangle + \langle V_2 \rangle \right) + O\left(\varepsilon^2 \mu^{-2}\right). \tag{32}$$

From (17), (32) it follows

$$c_1 = \frac{\langle V_1 \rangle + \langle V_2 \rangle}{2}, \qquad c_2 = O(1). \tag{33}$$

It follows from (18) and (19) that the sufficient condition for the existence of an eigenvalue converging to zero as $\varepsilon, \mu \to 0$ for the operator $(\mathcal{H}_0 + \varepsilon\mu^{-1}\mathcal{L}_{\varepsilon,\mu})$ is the inequality

$$\mathrm{Re}(c_1 + \varepsilon\mu^{-1} c_2) > 0, \tag{34}$$

while the sufficient condition for the absence of such eigenvalue is the inequality

$$\mathrm{Re}(c_1 + \varepsilon\mu^{-1} c_2) < 0. \tag{35}$$

It follows from (1) and (33) that for sufficiently small ε, μ, the sign of $\mathrm{Re}(c_1 + \varepsilon\mu^{-1}c_2)$ coincides with the sign of $\mathrm{Re}(c_1)$. Consequently, inequalities (3) and (5) follow from (34), (35) and (33). Asymptotics (4) is implied by (20) and (33). The theorem is proved.

5. Concluding remarks

It follows from Lemma 2.3 of [14] that the following statement is true.

Theorem 2. *If condition (1) is satisfied, then the continuous spectrum of the operator $\mathcal{H}_{\varepsilon,\mu}$ coincides with the continuous spectrum of the operator $\mathcal{H}_0$.*

It follows from Theorem 1 in [8] that apart from the eigenvalue converging to zero, all other eigenvalues of operator $\mathcal{H}_{\varepsilon,\mu}$ (if they exist) tend to infinity as $\varepsilon, \mu \to 0$.

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