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On Fredholm Integral Equations of the First Kind with Nonlinear Deviation

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Abstract. The problems of solvability and construction of solutions of a nonlinear Fredholm functional-integral equation of the first kind with degenerate kernel and nonlinear deviation are considered. Using the method of regularization in combination with the method of degenerate kernel, an implicit functional equation with nonlinear deviation is obtained. Since Fredholm functional-integral equation of the first kind is ill-posed, the boundary conditions to ensure the uniqueness of the solution is used. In order to use the method of successive approximations and prove the unique solvability, the obtained implicit functional equation is transformed to the nonlinear Volterra type functional-integral equation of the second kind. The unique solvability of the problem is proved.

Key Words and Phrases: integral equation of first kind, nonlinear functional equation, degenerate kernel, nonlinear deviation, boundary conditions, regularization, unique solvability.

2010 Mathematics Subject Classifications: 34A08, 26A33

1. Formulation of the problem

Integral equations are mathematical models of many physical processes and the operations in technical systems. In applications of differential and integral equations the approximation methods play an important role. Different methods are used for the approximation solution of differential, integral and integro-differential equations (see, e.g., [1, 2, 3, 4, 5, 6]). The work [7] is dedicated to the study of nonlinear Volterra integral equations with weakly singular kernels by generalized Jacobi-Galerkin method.

Different analytical and numerical methods are used in the theory of nonlinear Fredholm integral equations of the first kind. The Adomian decomposition method (see [8]) has been efficiently used in the theory of linear and nonlinear

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problems for differential, integral and integro-differential equations. This method provides the solution as an infinite series in which each term can be easily determined. It was further developed in [9, 10, 11, 12, 13]. Hadamard [14] introduced a mathematical term of well-posed problem. Well-posed models of physical phenomena should have the following three properties: a solution exists; a solution is unique; continuous dependence of the solution on the data. The method of regularization was established independently by Phillips [15] and Tikhonov [16]. It was used before in treating linear Fredholm integral equations of the first kind. The method of regularization consists of replacing ill-posed problem by well-posed problem.

In this paper, we will study the problems of unique solvability and construction of solutions of a nonlinear Fredholm functional-integral equation of the first kind with degenerate kernel and nonlinear deviation. When a kernel of integral equation is degenerate, it is easy to replace this integral equation by another kind of integral equation, which is convenient for solving. The integral and integro-differential equations with degenerate kernels were considered by many authors (see, e.g., [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]).

In our paper, using the regularization method in combination with the method of degenerate kernel, we obtain a functional-implicit equation with nonlinear deviation. Fredholm functional-integral equation of the first kind is ill-posed. So, we use boundary conditions to ensure the uniqueness of the solution. In order to use the method of successive approximations, we transform the implicit functional equation to the nonlinear Volterra type functional-integral equation of the second kind. In our case, this Volterra type functional-integral equation of the second kind is ill-posed, too. We prove the unique solvability of this problem by the given boundary conditions.

On the segment $[0; T]$ the nonlinear Fredholm integral equation of the first kind is considered

$$\lambda \int_0^T K(t, s)\, F(s, u(s), u[\delta(s, u(s))])\, ds = f(t) \tag{1}$$

under the following conditions:

$$\left[\begin{array}{l} u(0) = \varphi_0 = \text{const}, \\ u(t) = \varphi_1(t), \ t \in [-h_1; 0], \\ u(t) = \varphi_2(t), \ t \in [T; T + h_2], \end{array}\right. \tag{2}$$

where $0 < T$ is a given real number, λ is a nonzero compatibility parameter, $F(t, u, v) \in C([0; T] \times X \times X)$, $\delta(t, u) \in C([0; T] \times X)$, $-h_1 < \delta(t, u) <$

$T + h_2,\ 0 < h_i = \text{const},\ i = 1, 2,\ \varphi_1(t) \in C[-h_1; 0],\ \varphi_2(t) \in C[T; T + h_2]$, $K(t, s) = \sum_{i=1}^{k} a_i(t)\, b_i(s),\ 0 \neq a_i(t),\ b_i(s) \in C[0; T],\ X$ is a closed set in the set of real numbers. Here it is assumed that each of the systems of functions $a_i(t)$, $i = \overline{1, k}$, and $b_i(s),\ i = \overline{1, k}$, is linearly independent, $\varphi_1(0) = \varphi_0,\ \varphi_2(T) = u(T)$.

2. Method of regularization and method of degenerate kernel

Taking into account the degeneracy of the kernel, equation (1) can be rewritten in the following form:

$$\lambda \int_0^T \sum_{i=1}^{k} a_i(t)\, b_i(s)\, F(s, u(s), u[\delta(s, u(s))])\, ds = f(t). \tag{3}$$

Using the notation

$$\vartheta(t) = F(t, u(t), u[\delta(t, u(t))]) \tag{4}$$

and introducing new unknown function $\vartheta_\varepsilon(t)$, we obtain from (3) the approximation of the Fredholm integral equation of the second kind with a small parameter

$$\varepsilon \vartheta_\varepsilon(t) = f(t) - \lambda \int_0^T \sum_{i=1}^{k} a_i(t)\, b_i(s)\, \vartheta_\varepsilon(s)\, ds, \tag{5}$$

where

$$\lim_{\varepsilon \to 0} \vartheta_\varepsilon(t) = \vartheta(t), \tag{6}$$

$0 < \varepsilon$ is a small parameter.

Using the new notation

$$\alpha_i = \int_0^T b_i(s)\, \vartheta_\varepsilon(s)\, ds, \tag{7}$$

the integral equation (5) can be rewritten as follows:

$$\vartheta_\varepsilon(t) = \frac{1}{\varepsilon}\left[f(t) - \lambda \sum_{i=1}^{k} a_i(t)\, \alpha_i \right]. \tag{8}$$

Substituting (8) into (7), we obtain the system of linear equations (SLE)

$$\alpha_i + \lambda \sum_{j=1}^{k} \alpha_j\, A_{ij} = B_i, \quad i = \overline{1, k}, \tag{9}$$

where

$$A_{ij} = \frac{1}{\varepsilon}\int\limits_0^T b_i(s)\, a_j(s)\, d\,s,\ \ B_i = \frac{1}{\varepsilon}\int\limits_0^T b_i(s)\, f(s)\, d\,s. \tag{10}$$

Consider the following determinants:

$$\Delta(\lambda) = \begin{vmatrix} 1+\lambda A_{11} & A_{12} & \ldots & A_{1k} \\ A_{21} & 1+\lambda A_{22} & \ldots & A_{2k} \\ \ldots & \ldots & \ldots & \ldots \\ A_{k1}^m & A_{k2} & \ldots & 1+\lambda A_{kk} \end{vmatrix} \neq 0, \tag{11}$$

$$\Delta_i(\lambda) = \begin{vmatrix} 1+\lambda A_{11} & \ldots & A_{1(i-1)} & B_1 & A_{1(i+1)} & \cdots & A_{1k} \\ A_{21} & \ldots & A_{2(i-1)} & B_2 & A_{2(i+1)} & \cdots & A_{1k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{k1} & \ldots & A_{k(i-1)} & B_k & A_{k(i+1)} & \cdots & 1+\lambda A_{kk} \end{vmatrix},\ i=\overline{1,\,k}.$$

SLE (9) is uniquely solvable for any finite right-hand sides, if the nondegeneracy condition (11) of the Fredholm determinant is fulfilled. The determinant $\Delta(\lambda)$ in (11) is a polynomial with respect to λ of degree not greater than k. The equation $\Delta(\lambda) = 0$ has at most k different real roots. We denote them by $\mu_l\,(l=\overline{1,\,p},\ 1\le p\le k)$. Then $\lambda = \mu_l$ are called irregular values of the spectral parameter λ. Other values of the spectral parameter $\lambda \neq \mu_l$ are called regular. The solutions of SLE (9) for regular values of parameter λ are written as

$$\alpha_i = \frac{\Delta_i(\lambda)}{\Delta(\lambda)}, \quad i=\overline{1,\,k}. \tag{12}$$

Substituting (12) into (8), we obtain

$$\vartheta_\varepsilon(t) = \frac{1}{\varepsilon}\left[f(t) - \lambda\sum_{i=1}^{k} a_i(t)\,\frac{\Delta_i(\lambda)}{\Delta(\lambda)}\right]. \tag{13}$$

By virtue of formula (10), we suppose that

$$f(t) = \lambda\sum_{i=1}^{k} a_i(t)\, c_i, \quad c_i - \lambda\frac{\Delta_i(\lambda)}{\Delta(\lambda)} = \varepsilon C_i, \tag{14}$$

where $c_i,\, C_i = \text{const}, \quad i=\overline{1,\,k}$.

We recall that λ is a compatibility parameter between free term function $f(t)$ and the kernel of integral equation (1). So, we choose one of the regular values of the parameter λ such that the first of conditions (14) is fulfilled. Then, taking into account (6), from (13) we obtain

$$\vartheta(t) = \lambda \sum_{i=1}^{k} C_i \, a_i(t). \tag{15}$$

Now the function $\vartheta(t)$ is known and defined by the formula (15). So, we solve the implicit functional equation (4). We rewrite this implicit equation as

$$0 = G\,(t,\, u\,(t),\, u\,[\delta\,(t,\, u\,(t))]\}) \tag{16}$$

with given conditions (2), where $G = F - \vartheta$.

We consider two examples to illustrate the convenience of the above method.

Example 1. *We consider the following simple integral equation of the first kind without deviation:*

$$\lambda \int_0^1 t\,s\,(e^s + 2u\,(s))^2\, ds = 18t. \tag{17}$$

First, we use the regularization method. So, we denote $\vartheta\,(t) = \left(e^t + 2u\,(t)\right)^2$. *Then we have a second kind Fredholm integral equation*

$$\varepsilon\vartheta_\varepsilon(t) = 18t - \lambda \int_0^1 t\,s\,\vartheta_\varepsilon(s)\, ds,$$

where the formula (6) *is true. So, we have*

$$\vartheta_\varepsilon(t) = \frac{1}{\varepsilon}\left[18t - \lambda\, t \int_0^1 s\,\vartheta_\varepsilon(s)\, ds\right]. \tag{18}$$

Now we use degenerate method by denoting

$$\alpha = \int_0^1 s\,\vartheta_\varepsilon(s)\, ds. \tag{19}$$

Taking into account (19)*, from* (18) *we come to the following relation:*

$$\vartheta_\varepsilon(t) = \frac{t}{\varepsilon}\left[18 - \lambda\,\alpha\right]. \tag{20}$$

We substitute the relation (20) *into* (19)*. Then we obtain algebraic equation*

$$\alpha = \frac{18 - \lambda\,\alpha}{3\varepsilon},$$

which has a solution

$$\alpha = \frac{18}{3\varepsilon + \lambda}. \tag{21}$$

Substituting this solution (21) *into representation* (20)*, we derive*

$$\vartheta_\varepsilon(t) = \frac{18t}{\varepsilon}\left[1 - \lambda\frac{1}{3\varepsilon + \lambda}\right] = \frac{54t}{3\varepsilon + \lambda}. \tag{22}$$

Passing to the limit in (22) *as* $\varepsilon \to 0$*, we have* $\vartheta\,(t) = \frac{54t}{\lambda}$*. So, we obtain the implicit functional equation*

$$\left(e^t + 2u^2(t)\right)^2 = \frac{54t}{\lambda}.$$

It is not difficult to see that this implicit equation for $\lambda = \frac{8}{e^2+1}$ *has the solutions*

$$u\,(t) = \frac{\pm 3\sqrt{3\,(e^2+1)\,t} - 2e^t}{4}. \tag{23}$$

It is obvious that these solutions satisfy the given integral equation (17) *for* $\lambda = \frac{8}{e^2+1}$*. This integral equation* (17) *for this value* $\lambda = \frac{8}{e^2+1}$ *has another solution:* $u(t) = e^t$*.*

Example 2. *As a second example, we consider the simple integral equation*

$$\lambda\int_0^1 \left(2t^2 + 1\right) s\,u^3(s)\,ds = 4t^2 + 2 \tag{24}$$

with initial condition $u\,(0) = \sqrt[3]{\frac{1}{5}}$*. We denote* $\vartheta\,(t) = u^3\,(t)$*. Then we have a second kind Fredholm integral equation*

$$\varepsilon\vartheta_\varepsilon(t) = 4t^2 + 2 - \lambda\int_0^1 \left(2t^2 + 1\right)\, s\,\vartheta_\varepsilon(s)\,ds.$$

Hence, we have

$$\vartheta_\varepsilon(t) = \frac{1}{\varepsilon}\left[4t^2 + 2 - \lambda\left(2t^2 + 1\right)\int_0^1 s\,\vartheta_\varepsilon(s)\,ds\right]. \tag{25}$$

Denoting

$$\alpha = \int_0^1 s\,\vartheta_\varepsilon(s)\,ds, \tag{26}$$

from (25) *we come to the following relation*

$$\vartheta_\varepsilon(t) = \frac{1}{\varepsilon}\left(2t^2+1\right)(2-\lambda\alpha). \tag{27}$$

Substituting the relation (27) *into* (26), *we obtain the algebraic equation*

$$\alpha = \frac{2-\lambda\,\alpha}{\varepsilon},$$

the solution of which is

$$\alpha = \frac{2}{\varepsilon+\lambda}. \tag{28}$$

Substituting the solution (28) *into* (27), *we derive*

$$\vartheta_\varepsilon(t) = \frac{1}{\varepsilon}\left(2t^2+1\right)\left(2-\lambda\frac{2}{\varepsilon+\lambda}\right) = \frac{2\left(2t^2+1\right)}{\varepsilon+\lambda}. \tag{29}$$

Passing to the limit in (29) *as* $\varepsilon \to 0$, *we have*

$$\vartheta(t) = \lim_{\varepsilon\to 0}\vartheta_\varepsilon(t) = \lim_{\varepsilon\to 0}\frac{2\left(2t^2+1\right)}{\varepsilon+\lambda} = \frac{2\left(2t^2+1\right)}{\lambda}.$$

So, we obtain the implicit functional equation

$$u^3(t) = \frac{2\left(2t^2+1\right)}{\lambda}.$$

This implicit equation for $\lambda = 10$ *has the solution*

$$u(t) = \sqrt[3]{\frac{2t^2+1}{5}}. \tag{30}$$

It is obvious that this solution satisfies the given integral equation (24). *This integral equation for* $\lambda = 10$ *has another solution:* $u(t) = t$. *But, the unique solution of this integral equation* (24) *with initial condition* $u\,(0) = \sqrt[3]{\frac{1}{5}}$ *is* (30).

3. Transform into nonlinear Volterra type integral equation

To solve the implicit functional equation (16), we use the method of successive approximations in combination with the method of compression mapping. However, it is impossible to apply the method of successive approximations directly to the equation (16) with nonlinear deviation. Therefore, we propose the following method.

On the segment $[\,0;\,T\,]$ we take arbitrary positive definite and continuous function $K_0(t)$. We introduce the notation

$$\psi(t,\, s) = \int\limits_s^t K_0(\theta)\, d\theta, \;\; \psi(t,\, 0) = \psi(t), \;\; t \in [0;\, T].$$

It is obvious that $\psi(t,\, s) = \psi(t) - \psi(s)$. By the solution of equation (1) we mean a continuous function $u(t)$ on the segment $[0;\, T]$ that satisfies equation (1) with the given conditions (2) and the Lipschitz condition

$$\|\, u\,(t) - u\,(s)\,\| \le L_0\, |\, t - s\,|\,, \tag{31}$$

where $0 < L_0 = \text{const}$, $\|\, u\,(t)\,\| = \max\limits_{0 \le t \le T} |\, u\,(t)\,|$.

We write the implicit equation (16) as

$$u\,(t) + \int\limits_0^t K_0(s)\, u\,(s)\, ds = u\,(t) + \int\limits_0^t K_0(s)\, u\,(s)\, ds$$

$$+ G\,(t,\, u\,(t),\, u\,[\delta\,(t,\, u\,(t))])\,, \;\; t \in [0;\, T].$$

Hence, using resolvent of the kernel $[-K_0(s)]$, we obtain

$$u\,(t) = u\,(t) + \int\limits_0^t K_0(s)\, u\,(s)\, ds + G\,(t,\, u\,(t),\, u\,[\delta\,(t,\, u\,(t))])$$

$$+ \int\limits_0^t K_0(s)\, \exp\,\{-\psi(t,\, s)\}\, \left\{ -\, u\,(s) + \int\limits_0^s K_0(\theta)\, u\,(\theta)\, d\theta \right.$$

$$\left. - G\,(s,\, u\,(s),\, u\,[\delta\,(s,\, u\,(s))]) \right\} d\, s, \;\; t \in [0;\, T]. \tag{32}$$

Applyng Dirichlet's formula to (32) (see [28]), we derive the following Volterra type nonlinear functional-integral equation of the second kind:

$$u(t) = \Im(t;\, u) \equiv \int\limits_0^t H(t,\, s)\, u(s)\, ds$$

$$+ [u(t) + G(t,\, u(t),\, u[\delta(t,\, u(t))])] \cdot \exp\{-\psi(t)\}$$

$$+ \int\limits_0^t K_0(s)\, \exp\{-\psi(t,\, s)\}\, \{u(t) - u(s) + G(t,\, u(t),\, u[\delta(t,\, u(t))])$$

$$-G(s,\, u(s),\, u[\delta(s,\, u(s))])\}\, ds, \quad t \in [0;\, T], \tag{33}$$

where

$$H(t,\, s) = K_0(s)\, \exp\{-\psi(t,\, s)\} - \int\limits_s^t K_0(\theta)\, \exp\{-\psi(t,\, \theta)\}\, K_0(\theta,\, s)\, d\theta. \tag{34}$$

Remark 1. *We conditionally call the nonlinear functional-integral equation* (33) *a Volterra type nonlinear functional-integral equation of the second kind. As this integral equation* (33) *is ill-posed* [29], *we will study it for the given conditions* (2). *In addition, we consider the conditions* (2) *as* $u(t-0) = u(t+0)$ *at the points* $t = 0$ *and* $t = T$.

Let the conditions (11) and (14) be fulfilled. Then, instead of the Fredholm functional-integral equation of the first kind (1), we will study the Volterra type functional-integral equation of the second kind (33) with conditions (2).

Theorem 1. *Let the condition* (31) *be fulfilled and*

1). $\|\, G(t,\, u(t),\, v(t)\})\,\| \leq M_0, \quad 0 < M_0 = \text{const}$;

2). $|\, G(t,\, u_1(t),\, v_1(t)) - G(t,\, u_2(t),\, v_2(t))\,|$

$$\leq L_1(t)\, (|\, u_1(t) - u_2(t)\,| + |\, v_1(t) - v_2(t)\,|)\,, \; 0 < L_1(t) \in C[0;T];$$

3). $|\delta(t, u_1(t)) - \delta(t, u_2(t))| \leq L_2(t)\, |u_1(t) - u_2(t)|\,, \; 0 < L_2(t) \in C[0;T]$;

4). $\rho < 1$, *where*

$$\rho = \max_{0 \leq t \leq T} \left[\|\, K_0(t,\, s)\,\| \int\limits_0^t Q,(t,\, s)\, ds + [1 + L_1(t)\,(2 + L_0 L_2(t))]\, Q(t,\, 0) \right],$$

$$Q\,(t,\,s) = \exp\,\{-\psi\,(t,\,s)\} + 2\int\limits_s^t K_0(\theta)\,\exp\,\{-\psi\,(t,\,\theta)\}\,d\theta.$$

Then the functional-integral equation (33) *with conditions* (2) *has a unique solution on the segment* $[0;\,T]$.

Proof. We suppose that Picard iteration process for functional-integral equation (33) is given by

$$\left\{\begin{array}{l}\left[\begin{array}{l} u_0(t) = \varphi_1(t),\ t \in [-h_1;\,0],\\ u_0(t) = \varphi_0,\ \ t \in [0;\,T],\\ u_0(t) = \varphi_2(t),\ t \in [T;\,T+h_2],\end{array}\right.\\ \left[\begin{array}{l} u_{n+1}(t) = \varphi_1(t),\ t \in [-h_1;\,0],\\ u_{n+1}(t) = \Im(t;\,u_n),\ \ n \in \mathbb{N},\ \ t \in [0;\,T],\\ u_{n+1}(t) = \varphi_2(t),\ t \in [T;\,T+h_2].\end{array}\right.\end{array}\right.$$

First, let's estimate the function $H(t,\,s)$ given by formula (34):

$$|\,H(t,\,s)\,| \leq \|\,K_0(t,\,s)\,\|\,\exp\,\{-\psi(t,\,s)\} + \int\limits_s^t \|\,K_0(t,\,\theta)\,\|\,K_0(\theta)\,\exp\,\{-\psi(t,\,\theta)\}\,d\theta$$

$$\leq \|\,K_0(t,\,s)\,\| \cdot Q\,(t,\,s), \tag{35}$$

where

$$Q\,(t,\,s) = \exp\,\{-\psi(t,\,s)\} + 2\int\limits_s^t K_0(\theta)\,\exp\,\{-\psi(t,\,\theta)\}\,d\theta.$$

It is obvious that the following estimate is true:

$$\|\,u_0(t)\,\| \leq \max\left\{|\,\varphi_0\,|\,;\ \max_{-h_1\leq t\leq 0}|\,\varphi_1(t)\,|\,;\ \max_{T\leq t\leq T+h_2}|\,\varphi_2(t)\,|\right\} = \Delta_0 < \infty. \tag{36}$$

By virtue of conditions of theorem and Picard process, using estimates (35) and (36), for the first approximation $u_1(t)$ we obtain the estimate

$$|\,u_1(t)\,| \leq \int\limits_0^t \|\,H(t,\,s)\,\| \cdot \|\,u_0(s)\,\|\,ds$$

$$+\,[\|\,u_0(t)\,\| + \|\,G\,(t,\,u_0(t),\,u_0\,[\delta\,(t,\,u_0(t))])\,\|] \cdot \exp\,\{-\psi(t)\}$$

$$+\int\limits_0^t K_0(s)\,\exp\,\{-\psi(t,\,s)\}$$

$$\times\,[\|\,u_0(t)-u_0(s)\,\|+2\,\|\,G\,(t,\,u_0(t),\,u_0[\delta\,(t,\,u_0(t))])\,\|]\;ds$$

$$\le\Delta_0\int\limits_0^t\|\,K_0(t,\,s)\,\|\cdot Q\,(t,\,s)\,ds+(\Delta_0+M_0)\cdot\exp\{-\psi(t)\}$$

$$+\int\limits_0^t\|\,K_0(t,\,s)\,\|\cdot\exp\,\{-\psi(t,\,s)\}\,(L_0\,|\,t-s|+2M_0)\;ds$$

$$\le\Delta_0\,\|\,K_0(t,\,s)\,\|\int\limits_0^t Q\,(t,\,s)\,ds+\Delta_1 Q\,(t,\,0),\tag{37}$$

where

$$\Delta_1=\max\,\{\Delta_0+M_0;\;\;\|\,K_0(t,\,s)\,\|\cdot(L_0T+2M_0)\}\,.$$

By virtue of the conditions of theorem, similar to (37), for arbitrary difference $u_{n+1}(t)-u_n(t)$ of approximations we have

$$|\,u_{n+1}(t)-u_n(t)\,|\le\int\limits_0^t\|\,H(t,\,s)\,\|\cdot\|\,u_n(s)-u_{n-1}(s)\,\|\;ds$$

$$+\,[\,\|\,u_n(t)-u_{n-1}(t)\,\|+L_1(t)\,(\|\,u_n(t)-u_{n-1}(t)\,\|$$

$$+\,\|\,u_n\;[\delta\;(t,\,u_n(t))]-u_{n-1}\;[\delta\;(t,\,u_{n-1}(t))]\;\|)]\cdot\exp\{-\psi(t)\}$$

$$+2\int\limits_0^t K_0(s)\,\exp\,\{-\psi(t,\,s)\}\,[\|\,u_n(s)-u_{n-1}(s)\,\|+L_1(s)\,(\|\,u_n(s)-u_{n-1}(s)\,\|$$

$$+\,\|\,u_n\;[\delta\;(s,\,u_n(s))]-u_{n-1}\;[\delta\;(s,\,u_{n-1}(s))]\;\|)]\;ds.\tag{38}$$

To estimate the norm $\|\,u_n\;[\delta\;(t,\,u_n(t))]-u_{n-1}\;[\delta\;(t,\,u_{n-1}(t))]\,\|$ in (38) we use condition (31) and third condition of the theorem. Then we have

$$\|\,u_n\;[\delta\;(t,\,u_n(t))]-u_{n-1}\;[\delta\;(t,\,u_{n-1}(t))]\,\|$$

$$\le\|\,u_n\;[\delta\;(t,\,u_n(t))]-u_{n-1}\;[\delta\;(t,\,u_n(t))]\,\|$$

$$+\,\|\,u_{n-1}\;[\delta\;(t,\,u_n(t))]-u_{n-1}\;[\delta\;(t,\,u_{n-1}(t))]\,\|$$

$$\leq \| u_n(t) - u_{n-1}(t) \| + L_0 \, | \, \delta \, (t, \, u_n(t)) - \delta \, (t, \, u_{n-1}(t)) \, |$$
$$\leq \| u_n(t) - u_{n-1}(t) \| + L_0 L_2(t) \, \| u_n(t) - u_{n-1}(t) \|$$
$$= (1 + L_0 L_2(t)) \, \| u_n(t) - u_{n-1}(t) \| \, . \tag{39}$$

Substituting (39) into (38), we obtain

$$| \, u_{n+1}(t) - u_n(t) \, | \leq \int\limits_0^t \| K_0(t, \, s) \| \cdot Q \, (t, \, s) \cdot \| u_n(s) - u_{n-1}(s) \| \, ds$$
$$+ \left[1 + L_1(t) \, (2 + L_0 L_2(t)) \right] \cdot \| u_n(t) - u_{n-1}(t) \| \cdot \exp \{ -\psi(t) \}$$
$$+ \int\limits_0^t \| K_0(t, \, s) \| \cdot \exp \, \{ -\psi(t, \, s) \} \left[1 + L_1(s) \, (2 + L_0 L_2(s)) \right] \cdot \| u_n(s) - u_{n-1}(s) \| \, ds.$$

Hence, it follows that

$$\| u_{n+1}(t) - u_n(t) \| \leq \rho \cdot \| u_n(t) - u_{n-1}(t) \| \, , \tag{40}$$

where

$$\rho = \max_{0 \leq t \leq T} \left[\| K_0(t, \, s) \| \int\limits_0^t Q \, (t, \, s) \, ds + \left[1 + L_1(t) \, (2 + L_0 L_2(t)) \right] \, Q \, (t, \, 0) \right] .$$

When choosing the function $K_0(t)$ we take into account that

$$\psi(t, \, s) = \int\limits_s^t K_0(\theta) \, d\theta \gg 1, \;\; t \in [0; \, T].$$

Hence, we obtain $\exp \{ -\psi(t) \} \ll 1$. So, the functions $H(t, s)$ and $Q(t, s)$ are small. Then we can choose the functions $L_1(t), \;\; L_2(t)$ such that

$$\rho = \max_{0 \leq t \leq T} \left[\| K_0(t, \, s) \| \int\limits_0^t Q \, (t, \, s) \, ds + \left[1 + L_1(t) \, (2 + L_0 L_2(t)) \right] \, Q \, (t, \, 0) \right] < 1.$$

According to the last condition of the theorem, we have $\rho < 1$. We consider the solution of the integral equation (33) in the space of continuous functions $C \, [0; \, T]$, satisfying condition (31). So, it follows from the estimate (40) that the integral operator on the right-hand side of (33) with conditions (2) is compression mapping. So, the estimates (36), (37) and (40) imply that the integral equation (33) with conditions (2) has a unique solution on the segment $[0; \, T]$. The theorem is proved. ◄

4. Conclusion

In this paper, we studied the problems of unique solvability and construction of solutions of a nonlinear Fredholm functional-integral equation (1) of the first kind with degenerate kernel and nonlinear deviation. This Fredholm functional-integral equation is ill-posed. So, we use boundary conditions (2) to ensure the uniqueness of the solution. In order to use the method of successive approximations we reduce the implicit functional equation (16) to the nonlinear Volterra type functional-integral equation of the second kind. Here we used the regularization method in combination with the method of degenerate kernel. So, the specificity of this paper is that the Fredholm functional-integral equation is replaced by the Volterra type functional-integral equation (33).

We conditionally called the nonlinear functional-integral equation (33) a Volterra type nonlinear functional-integral equation of the second kind. As this integral equation (33) is ill-posed, too, we studied it for the given conditions (2). In addition, in the conditions (2) we suppose that $u\,(t-0) = u\,(t+0)$ at the points $t = 0$ and $t = T$.

Let the conditions (11) and (14) be fulfilled. Then, instead of the Fredholm functional-integral equation of the first kind (1) we studied the Volterra type functional-integral equation of the second kind (33) with conditions (2). The theorem of unique solvability of the problem (1), (2) was proved.

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