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Non Global Solutions For a Class of Klein-Gordon Equations

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Abstract. Under sufficient conditions on the data, solutions to a class of Klein-Gordon equations with exponential type non-linearity and arbitrary positive energy blow-up in finite time.

Key Words and Phrases: nonlinear wave equation, energy, blow-up.

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1. Introduction

This paper studies the nonexistence of global solutions to the semi-linear wave equation

$$\begin{cases} \ddot{u} - \Delta u + u = f_a(u) := e^u - 1 + au\,; \\ (u, \dot{u})_{|t=0} = (u_0, u_1)\,, \end{cases} \tag{1}$$

where u is a real valued function of the variable $(t, x) \in [0, T) \times \Omega$ for some smooth domain $\Omega \subset \mathbb{R}^2$ and $-2 < a < -1$. The above wave problem has various applications in the area of nonlinear optics, plasma physics, fluid mechanics [12].

Before going further, let us recall few historic facts about well-posedness in the energy space of (1) with exponential source term. In two space dimensions, the critical non-linearity for the wave equation is of exponential growth. Indeed, Nakamura and Ozawa proved global well-posedness and scattering for small Cauchy data [8, 9]. Later on, A. Atallah [1] obtained a local existence result for radially symmetric data with a compact support. Ibrahim-Majdoub-Masmoudi [3] obtained global well-posedness in the energy unit ball. Similar results exist in the case of a bounded domain [4]. Recently, Struwe [13, 14] proved unconditional existence of global solution with regular datum.

In [6], the author proved unconditional global well-posedness and linearization of the defocusing semi-linear wave equation (1) for $a = 1$. The same result was proved in [7] for a class of exponential type non-linearities.

The question of existence of non global solutions to nonlinear wave equations was treated by many authors [2, 5, 12, 10]. There exist few results about the blow-up properties for the local solution to Klein-Gordon equations with arbitrary positive energy [15, 16, 11].

It is the aim of this manuscript to extend the work [11], by giving sufficient conditions on the data which give finite time blowing up solutions to (1). Comparing with [15, 16], the key assumption $xf(x) \geq (2+\varepsilon)F(x)$, where F is the primitive of f vanishing on zero, is not satisfied in this note.

The rest of the paper is organized as follows. The next section contains some technical tools needed in the sequel and the main results. The two last sections are dedicated to proving the main results.

In the rest of this note, for simplicity, denote $\|\,.\,\|_{L^2(\Omega)}$ by $\|\,.\,\|$ and $\|\,.\,\|_{H^1(\Omega)}$ by $\|\,.\,\|_{H^1}$.

2. Preliminaries and main results

2.1. Preliminaries

Here and hereafter, for $v, w \in H^1(\Omega)$ and $u \in C([0,T], H^1(\Omega))$, define the quantities

$$F_a := \int_0^{\cdot} f_a(s)\, ds;$$
$$G(v) := \|v\|^2 \quad \text{and} \quad I(v) := \|v\|_{H^1}^2 - \int_\Omega v f_a(v)\, dx;$$
$$E(v,w) := \frac{1}{2}\Big(\|v\|_{H^1} + \|w\|^2\Big) - \int_\Omega F_a(v)\, dx;$$
$$G(t) := G(u(t)), \quad I(t) := I(u(t)), \quad E(t) := E(u(t), \dot{u}(t)).$$

Throughout this manuscript, we consider the infinite set

$$\mathcal{A}_a := \Big\{\varepsilon \in (0,1) \text{ such that } a < -\frac{1}{2}\Big(1 + \frac{e^{\varepsilon-1}}{\varepsilon}\Big)\Big\}, \quad -2 < a < -1. \tag{2}$$

The Cauchy problem (1) was studied in [6, 7], where a local well-posedness result was proved in the energy space.

Proposition 1. *Let* $(u_0, u_1) \in (H^1 \times L^2)(\mathbb{R}^2)$. *Then the semi-linear wave problem* (1) *has a unique maximal solution* u *in the class*

$$C_{T^*}(H^1(\mathbb{R}^2)) \cap C^1_{T^*}(L^2(\mathbb{R}^2)).$$

Moreover, the solution satisfies conservation of the energy $E(t) = E(0)$ *for any* $t \in [0, T^*)$.

Remark 1. *If* Ω *is a smooth bounded domain, then the problem* (1) *with Dirichlet conditions is locally well-posed in the energy space [4].*

The following property about the non-linearity will be useful in the sequel [11].

Lemma 1. *Take* $-2 < a < -1$ *and* $\varepsilon \in \mathcal{A}_a$. *Then,*

1/ $b := b(a, \varepsilon) := \inf_{x\in\mathbb{R}} \Big(x f_a(x) - (2+\varepsilon) F_a(x) \Big) \in \mathbb{R}$*;*

2/ the next inequality holds

$$x f_a(x) \geq (2+\varepsilon) F_a(x) + (a+1)\frac{\varepsilon}{2} x^2, \quad \textit{for any} \quad x \in \mathbb{R}. \tag{3}$$

Finally, recall a standard result about non global solutions to an ordinary differential inequality.

Lemma 2. *Let* $\epsilon > 0$. *There is no real function* $H \in C^2(\mathbb{R}_+)$ *satisfying*

$$H(0) > 0, \ H'(0) > 0 \quad \textit{and} \quad HH'' - (1+\epsilon)(H')^2 \geq 0 \quad \textit{on} \quad \mathbb{R}_+.$$

Proof. Assume by contradiction the existence of such a function. Then $(H^{-(1+\epsilon)}H')' \geq 0$ and

$$\frac{H'}{H^{1+\epsilon}} \geq \frac{H'(0)}{H(0)} > 0.$$

This is a Riccati inequality with blow-up time $T < \frac{1}{\epsilon}\frac{H(0)}{H'(0)}$. This contradiction achieves the proof. ◄

2.2. Main results

This subsection contains two theorems about non global solutions to (1) under sufficient conditions on the data.

Theorem 1. *Let $E_0 > 0$, $-2 < a < -1$ and $\varepsilon \in \mathcal{A}_a$. If $(u_0, u_1) \in (H^1 \times L^2)(\mathbb{R}^2)$ satisfies*

$$E(0) = E_0, \quad I(0) < 0 \quad \text{and} \quad G'(0) > \frac{2(2+\varepsilon)}{\varepsilon(2+a)} E_0,$$

then the maximal solution $u \in C_{T^*}(H^1(\mathbb{R}^2)) \cap C^1_{T^*}(L^2(\mathbb{R}^2))$ to (1) blows up in a finite time. Precisely,

$$T^* < \infty \quad \text{and} \quad \limsup_{t \to T^*} \|u(t)\| = \infty.$$

Now, let us consider non global solutions to (1) in the case of a bounded domain.

Theorem 2. *Let $E_0 > 0$, $-2 < a < -1$, $\varepsilon \in \mathcal{A}_a$. Take $\Omega \subset \mathbb{R}^2$ a smooth bounded domain satisfying*

$$|b| > -(2+\varepsilon)\frac{(1+a)}{(2+a)}\frac{E_0}{|\Omega|}. \tag{4}$$

If $(u_0, u_1) \in (H^1_0 \times L^2)(\Omega)$ satisfies

$$E(0) = E_0, \quad I(0) < 0 \quad \text{and} \quad G'(0) > \frac{2}{\varepsilon}\Big((2+\varepsilon)E_0 + |b||\Omega|\Big),$$

then the maximal solution $u \in C_{T^*}(H^1_0(\Omega)) \cap C^1_{T^*}(L^2(\Omega))$ to (1) with Dirichlet conditions blows up in a finite time. Precisely,

$$T^* < \infty \quad \text{and} \quad \limsup_{t \to T^*} \|u(t)\| = \infty.$$

3. Proof of Theorem 1

The proof is based on the following auxiliary result.

Lemma 3. *Let $E_0 > 0$, $-2 < a < -1$, $\varepsilon \in \mathcal{A}_a$ and $(u_0, u_1) \in (H^1 \times L^2)(\mathbb{R}^2)$ satisfying*

$$E(0) = E_0, \quad I(0) < 0 \quad \text{and} \quad G'(0) > \frac{2(2+\varepsilon)}{\varepsilon(2+a)} E_0.$$

Then the maximal solution $u \in C_{T^*}(H^1) \cap C^1_{T^*}(L^2)$ to (1) satisfies

$$I < 0 \quad \text{and} \quad G' > \frac{2(2+\varepsilon)}{\varepsilon(2+a)} E \quad \text{on} \quad [0, T^*).$$

Proof. Compute using the equation (1), $G' = 2\int_{\mathbb{R}^2} u(x)\dot{u}(x)\,dx$ and $\frac{1}{2}G'' = \|\dot{u}\|^2 - I \geq -I$. Assume that I is not always negative and define

$$t := \min\Big\{s \in (0,T^*) \quad \text{such that} \quad I(s) = 0\Big\}.$$

Then G' is increasing on $[0,t]$ and

$$G' \geq G'(0) > \frac{2(2+\varepsilon)}{\varepsilon(2+a)}E_0 \quad \text{on} \quad [0,t]. \tag{5}$$

Now, since $I(t) = 0$, by (3) we have

$$\begin{aligned}
2E &= \|u(t)\|_{H^1}^2 + \|\dot{u}(t)\|^2 - 2\int_{\mathbb{R}^2} F_a(u(t,x))\,dx \\
&\geq \|u(t)\|_{H^1}^2 + \|\dot{u}(t)\|^2 - \frac{2}{2+\varepsilon}\int_{\mathbb{R}^2}\Big(u(t,x)f_a(u(t,x))dx - \frac{\varepsilon}{2}(1+a)u^2(t,x)\Big)\,dx \\
&\geq \|u(t)\|_{H^1}^2 + \|\dot{u}(t)\|^2 - \frac{2}{2+\varepsilon}(\|u(t)\|_{H^1}^2 - \frac{\varepsilon}{2}(1+a)\|u(t)\|^2).
\end{aligned}$$

Then, thanks to Cauchy-Schwarz inequality, we get

$$\begin{aligned}
2E &\geq \|\dot{u}(t)\|^2 + \frac{\varepsilon}{2+\varepsilon}\|\nabla u(t)\|^2 + \frac{(2+a)\varepsilon}{2+\varepsilon}\|u(t)\|^2 \\
&\geq \frac{\varepsilon(2+a)}{2+\varepsilon}\Big(\|u(t)\|^2 + \|\dot{u}(t)\|^2 + \|\nabla u(t)\|^2\Big) \\
&\geq \frac{\varepsilon(2+a)}{2+\varepsilon}G'(t).
\end{aligned}$$

This contradicts (5) and finishes the proof. ◀

Now, return to the proof of Theorem 1. By contradiction, assume that u is global. Compute, using Cauchy-Schwarz inequality

$$(G')^2 = 4\|u\dot{u}\|_1^2 \leq 4\|u\|^2\|\dot{u}\|^2 \leq 4G\|\dot{u}\|^2.$$

For $\lambda \in \mathbb{R}$, define the real function

$$\begin{aligned}
h_\lambda &:= GG'' - \frac{3+\lambda}{4}(G')^2 \\
&\geq G\Big(G'' - (3+\lambda)\|\dot{u}\|^2\Big) \\
&\geq -G\Big(2I + (1+\lambda)\|\dot{u}\|^2\Big).
\end{aligned}$$

Taking account of (3), write

$$\begin{aligned} 2E &= \|u\|_{H^1}^2 + \|\dot{u}\|^2 - 2\int_{\mathbb{R}^2} F_a(u)\,dx \\ &\geq \|u\|_{H^1}^2 + \|\dot{u}\|^2 - \frac{2}{2+\varepsilon}\int_{\mathbb{R}^2}\Big(uf_a(u)dx - \frac{\varepsilon}{2}(1+a)u^2\Big)\,dx \\ &\geq \|u\|_{H^1}^2 + \|\dot{u}\|^2 - \frac{2}{2+\varepsilon}\Big(\|u\|_{H^1}^2 - I - \frac{\varepsilon}{2}(1+a)\|u\|^2\Big). \end{aligned}$$

Then,

$$2(2+\varepsilon)E \geq (2+\varepsilon)\|\dot{u}\|^2 + \varepsilon\|\nabla u\|^2 + \varepsilon(2+a)\|u\|^2 + 2I.$$

Thus,

$$\begin{aligned} h_\lambda &\geq -G\Big(2I + (1+\lambda)\|\dot{u}\|^2\Big) \\ &\geq G\Big(-2(2+\varepsilon)E + (2+\varepsilon)\|\dot{u}\|^2 + \varepsilon\|\nabla u\|^2 + \varepsilon(2+a)\|u\|^2 - (1+\lambda)\|\dot{u}\|^2\Big) \\ &\geq G\Big(-2(2+\varepsilon)E + (1+\varepsilon-\lambda)\|\dot{u}\|^2 + \varepsilon\|\nabla u\|^2 + \varepsilon(2+a)\|u\|^2\Big). \end{aligned} \tag{6}$$

Using the previous lemma, we have $G' > \frac{2(2+\varepsilon)}{\varepsilon(2+a)}E$, so

$$\begin{aligned} h_{1-\varepsilon(1+a)} &\geq G\Big((1+\varepsilon-(1-\varepsilon(1+a)))\|\dot{u}\|^2 + \varepsilon(2+a)\|u\|^2 - 2(2+\varepsilon)E\Big) \\ &\geq \varepsilon(2+a)G\Big(\|\dot{u}\|^2 + \|u\|^2 - G'\Big) \\ &> 0. \end{aligned}$$

Finally,

$$GG'' - (1 - \frac{\varepsilon(1+a)}{4})(G')^2 > 0.$$

Theorem 1 is proved thanks to Proposition 1 and the fact that $a < -1$.

4. Proof of Theorem 2

The proof is based on the following intermediate result.

Lemma 4. *Let $E_0 > 0$, $-2 < a < -1$, $\varepsilon \in \mathcal{A}_a$ and $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain satisfying* (4). *If $(u_0, u_1) \in (H_0^1 \times L^2)(\Omega)$ satisfies*

$$E(0) = E_0, \quad I(0) < 0 \quad \text{and} \quad G'(0) > \frac{2}{\varepsilon}\Big((2+\varepsilon)E_0 + |b||\Omega|\Big),$$

then the maximal solution $u \in C_{T^*}(H_0^1(\Omega)) \cap C_{T^*}^1(L^2(\Omega))$ to (1) *satisfies*

$$I < 0 \quad \text{and} \quad G' > \frac{2}{\varepsilon}\Big((2+\varepsilon)E + |b||\Omega|\Big) \quad \text{on} \quad [0, T^*).$$

Proof. Using the equation (1), write $G' = 2\int_\Omega u(x)\dot u(x)\,dx$ and $\frac{1}{2}G'' = \|\dot u\|^2 - I > -I$. Assume that I is not always negative and take

$$t := \min\Big\{ s \in (0,T^*) \quad \text{such that} \quad I(s) = 0 \Big\}.$$

Then, G' is increasing on $[0,t]$ and

$$G' \geq G'(0) > \frac{2}{\varepsilon}\Big((2+\varepsilon)E + |b||\Omega|\Big) \quad \text{on} \quad [0,t]. \tag{7}$$

Now, since $I(t) = 0$, by (3) we have

$$\begin{aligned} 2E &= \|u(t)\|_{H^1}^2 + \|\dot u(t)\|^2 - 2\int_{\mathbb{R}^2} F_a(u)\,dx \\ &\geq \|u(t)\|_{H^1}^2 + \|\dot u(t)\|^2 - \frac{2}{2+\varepsilon}\int_{\mathbb{R}^2}\Big(uf_a(u)dx - \frac{\varepsilon}{2}(1+a)u(t)^2\Big)\,dx \\ &\geq \|u(t)\|_{H^1}^2 + \|\dot u(t)\|^2 - \frac{2}{2+\varepsilon}(\|u(t)\|_{H^1}^2 - \frac{\varepsilon}{2}(1+a)\|u(t)\|^2). \end{aligned}$$

By Cauchy-Schwarz inequality, we get

$$\begin{aligned} 2E &\geq \|\dot u(t)\|^2 + \frac{\varepsilon}{2+\varepsilon}\|\nabla u(t)\|^2 + \frac{(2+a)\varepsilon}{2+\varepsilon}\|u(t)\|^2 \\ &\geq \frac{\varepsilon(2+a)}{2+\varepsilon}\Big(\|u(t)\|^2 + \|\dot u(t)\|^2 + \|\nabla u(t)\|^2\Big) \\ &\geq \frac{\varepsilon(2+a)}{2+\varepsilon}G'(t). \end{aligned}$$

Thus,

$$\frac{2}{\varepsilon}\Big((2+\varepsilon)E + |b||\Omega|\Big) < G'(t) \leq \frac{2(2+\varepsilon)}{\varepsilon(2+a)}E_0.$$

This contradicts (4). So $I < 0$ on $[0,T^*)$. The proof is finished. ◀

Now, return to the proof of Theorem 2. By contradiction, assume that u is global. Arguing as in the previous section and using (6) with Lemma 4 and (4), yields

$$\begin{aligned} h_{1-\varepsilon(1+a)} &\geq G\Big((1+\varepsilon-(1-\varepsilon(1+a)))\|\dot u\|^2 + \varepsilon(2+a)\|u\|^2 - 2(2+\varepsilon)E\Big) \\ &\geq G\Big(\varepsilon(2+a)(\|\dot u\|^2 + \|u\|^2) - 2(2+\varepsilon)E\Big) \\ &\geq G\Big(2(1+a)(2+\varepsilon)E + (2+a)|b\Omega|\Big) \\ &> 0. \end{aligned}$$

Finally,

$$GG'' - (1 - \frac{\varepsilon(1+a)}{4})(G')^2 > 0.$$

Theorem 2 is proved thanks to Proposition 1 and the fact that $a < -1$.

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