Improvements of Some Numerical Radius Inequalities

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Abstract. In this work, we improve and refine some numerical radius inequalities. In particular, for all Hilbert space operators $T$, the famous Kittaneh inequality reads:

$$\frac{1}{4} \|T^*T + TT^*\| \leq w^2(T) \leq \frac{1}{2} \|T^*T + TT^*\|.$$ 

In this work we provide some important refinements for the upper bound of the Kittaneh inequality. Namely, we establish

$$w^2(T) \leq \frac{1}{2} \|T^*T + TT^*\| - \frac{1}{4} \inf_{\|x\|=1} \left( \langle |T|x,x \rangle - \langle |T^*|x,x \rangle \right)^2,$$

which is also refined and improved as

$$w^2(T) \leq \frac{1}{2} \|T^*T + TT^*\| - \frac{1}{2} \inf_{\|x\|=1} \left( \langle |T|x,x \rangle - \langle |T^*|x,x \rangle \right)^2,$$

with the third improvement

$$w^2(T) \leq \frac{1}{4} \| |T| + |T^*|| \|^{2} - \frac{1}{4} \inf_{\|x\|=1} \left( \langle |T|x,x \rangle - \langle |T^*|x,x \rangle \right)^2.$$

Other related results are also obtained.

Key Words and Phrases: mixed Schwarz inequality, numerical radius, Furuta inequality.

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1. Introduction

Let $B(H)$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$ with the identity operator $1_H$ in $B(H)$. A
bounded linear operator $A$ defined on $\mathcal{H}$ is selfadjoint if and only if $\langle Ax, x \rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$. Consider the real vector space $\mathcal{B}(\mathcal{H})_{sa}$ of selfadjoint operators on $\mathcal{H}$ and its positive cone $\mathcal{B}(\mathcal{H})^+$ of positive operators on $\mathcal{H}$. A partial order is naturally equipped on $\mathcal{B}(\mathcal{H})_{sa}$ by defining $A \leq B$ if and only if $B - A \in \mathcal{B}(\mathcal{H})^+$. We write $A > 0$ to mean that $A$ is a strictly positive operator, or equivalently, $A \geq 0$ and $A$ is invertible.

The Schwarz inequality for positive operators reads that if $A$ is a positive operator in $\mathcal{B}(\mathcal{H})$, then
\begin{equation}
|\langle Ax, y \rangle|^2 \leq \langle Ax, x \rangle \langle Ay, y \rangle
\end{equation}
for any vectors $x, y \in \mathcal{H}$.

In 1951, Reid [13] proved an inequality which in some sense was a variant of the Schwarz inequality. In fact, he proved that for all operators $A \in \mathcal{B}(\mathcal{H})$ such that $A$ is positive and $AB$ is selfadjoint the relation
\begin{equation}
|\langle ABx, y \rangle| \leq \|B\| \langle Ax, x \rangle
\end{equation}
holds for all $x \in \mathcal{H}$. In [7], Halmos presented the stronger version of the Reid inequality (2) by replacing $\|B\|$ with $r(B)$.

In 1952, Kato [11] introduced a companion inequality of (1), called the mixed Schwarz inequality, which asserts
\begin{equation}
|\langle Ax, y \rangle|^2 \leq \left| |A|^{2\alpha} x, x \right| \left| |A^*|^{2(1-\alpha)} y, y \right|, \quad 0 \leq \alpha \leq 1.
\end{equation}
for every operator $A \in \mathcal{B}(\mathcal{H})$ and any vectors $x, y \in \mathcal{H}$, where $|A| = (A^*A)^{1/2}$.

In 1988, Kittaneh [10] proved a very interesting extension combining both the Halmos–Reid inequality (2) and the mixed Schwarz inequality (3). His result reads that
\begin{equation}
|\langle ABx, y \rangle| \leq r(B) \|f(|A|) x\| \|g(|A^*|) y\|
\end{equation}
for any vectors $x, y \in \mathcal{H}$, where $A, B \in \mathcal{B}(\mathcal{H})$ are such that $|A|B = B^*|A|$ and $f, g$ are nonnegative continuous functions defined on $[0, \infty)$ satisfying $f(t)g(t) = t$ ($t \geq 0$). For instance, if we set $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ ($0 \leq \alpha \leq 1$) with $B = 1$ in (4), we recapture Kato’s inequality (3). Also, as noticed in [12], if in the inequality (4) $A$ is assumed to be positive, then the condition $AB = B^*A$ is equivalent to saying that $AB$ is self-adjoint. In this case, letting $f(t) = g(t) = t^{1/2}$ and $x = y$, we obtain the generalized Reid inequality (2) as a special case. A non-trivial improvement of (4) was established very recently by the author of this paper in [1]. The Cartesian decomposition form of (4) was also recently proved by Alomari in [2].
In 1994, Furuta [6] proved the following generalization of Kato’s inequality (3):

\[
\left| \langle T \left| T \right|^{\alpha + \beta - 1} x, y \rangle \right|^2 \leq \langle \left| T \right|^{2\alpha} x, x \rangle \langle \left| T \right|^{2\beta} y, y \rangle
\]

(5)

for any \( x, y \in \mathcal{H} \) and \( \alpha, \beta \in [0, 1] \) with \( \alpha + \beta \geq 1 \).

The inequality (5) was generalized for any \( \alpha, \beta \geq 0 \) with \( \alpha + \beta \geq 1 \) by Dragomir in [5]. As noted by Dragomir, the condition \( \alpha, \beta \in [0, 1] \) was assumed by Furuta to fit with the Heinz–Kato inequality, which reads:

\[
|\langle Tx, y \rangle| \leq \|A^\alpha x\| \|B^{1-\alpha} y\|
\]

for any \( x, y \in \mathcal{H} \) and \( \alpha \in [0, 1] \), where \( A \) and \( B \) are positive operators such that \( \|Tx\| \leq \|Ax\| \) and \( \|T^* y\| \leq \|By\| \) for any \( x, y \in \mathcal{H} \).

For a bounded linear operator \( T \) on a Hilbert space \( \mathcal{H} \), the numerical range \( W (T) \) is the image of the unit sphere of \( \mathcal{H} \) under the quadratic form \( x \rightarrow \langle Tx, x \rangle \) associated with the operator. More precisely,

\[
W (T) = \{ \langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1 \}.
\]

Also, the numerical radius is defined to be

\[
w (T) = \sup_{\|x\|=1} \{ |\lambda| : \lambda \in W (T) \} = \sup_{\|x\|=1} |\langle Tx, x \rangle|.
\]

The spectral radius of an operator \( T \) is defined to be

\[
r (T) = \sup \{ |\lambda| : \lambda \in sp (T) \}.
\]

We recall that the usual operator norm of an operator \( T \) is defined to be

\[
\|T\| = \sup \{ \|Tx\| : x \in H, \|x\| = 1 \}.
\]

It is well known that \( w (\cdot) \) defines an operator norm on \( \mathcal{B} (\mathcal{H}) \) which is equivalent to operator norm \( \| \cdot \| \), and for every \( T \in \mathcal{B} (T) \), we have

\[
\frac{1}{2} \|T\| \leq w (T) \leq \|T\|.
\]

(6)

Thus, the usual operator norm and the numerical radius norm are equivalent. The inequalities in (6) are sharp: if \( T^2 = 0 \), then the first inequality becomes an equality, while the second inequality becomes an equality if \( T \) is normal.

In 2003, Kittaneh [10] refined the right-hand side of (6), by proving that

\[
w (T) \leq \frac{1}{2} \left( \|T\| + \|T^2\|^{1/2} \right)
\]

(7)
for any $T \in \mathcal{B} (\mathcal{H})$.

After that in 2005, the same author in [8] proved that

$$
\frac{1}{4} \| A^*A + AA^* \| \leq w^2 (A) \leq \frac{1}{2} \| A^*A + AA^* \|. \quad (8)
$$

The inequality is sharp.

In 2007, Yamazaki [16] improved (8) by proving that

$$
w (T) \leq \frac{1}{2} \left( \| T \| + w \left( \tilde{T} \right) \right) \leq \frac{1}{2} \left( \| T \| + \| T^2 \|^{1/2} \right),
$$
where $\tilde{T} = |T|^{1/2} U |T|^{1/2}$ and $U$ is the unitary operator in the polar decomposition $T$ of the form $T = U |T|$.

In 2008, Dragomir [4] used Buzano inequality to improve (1), by proving that

$$
w^2 (T) \leq \frac{1}{2} (\| T \| + w (T^2)).
$$

This result was also recently generalized by Sattari et al. in [14] and Alomari in [2]. For more recent results about the numerical radius see the recent monograph [3].

In this work, we improve and refine some numerical radius inequalities. In particular, for all Hilbert space operators $T$, the famous Kittaneh inequality reads:

$$
\frac{1}{4} \| T^*T + TT^* \| \leq w^2 (T) \leq \frac{1}{2} \| T^*T + TT^* \|.
$$

In this work we provide some important refinements for the upper bound of the Kittaneh inequality. Namely, we establish

$$
w^2 (T) \leq \frac{1}{2} \| T^*T + TT^* \| - \frac{1}{4} \inf_{\| x \|=1} (\langle |T| x, x \rangle - \langle |T^*| x, x \rangle)^2,
$$
which is also refined and improved as

$$
w^2 (T) \leq \frac{1}{2} \| T^*T + TT^* \| - \frac{1}{2} \inf_{\| x \|=1} (\langle |T| x, x \rangle - \langle |T^*| x, x \rangle)^2,
$$

and

$$
w^2 (T) \leq \frac{1}{2} \| T^*T + TT^* \| - \frac{1}{2} \inf_{\| x \|=1} \left( \langle |T|^2 x, x \rangle^{\frac{1}{2}} - \langle |T^*|^2 x, x \rangle^{\frac{1}{2}} \right)^2,
$$

with the third improvement

$$
w^2 (T) \leq \frac{1}{4} \| |T| + |T^*| \|^2 - \frac{1}{4} \inf_{\| x \|=1} (\langle |T| x, x \rangle - \langle |T^*| x, x \rangle)^2.
$$

Other related results are also obtained.


2. Numerical Radius Inequalities

In order to prove our main result we need the following lemmas:

**Lemma 1.** Let $S \in \mathcal{B}(\mathcal{H})$, $S \geq 0$ and $x \in \mathcal{H}$ be a unit vector. Then, the Jensen’s operator inequality

$$\langle Sx, x \rangle^r \leq \langle S^r x, x \rangle, \quad r \geq 1$$

and

$$\langle S^r x, x \rangle \leq \langle Sx, x \rangle^r, \quad r \in [0, 1].$$

Kittaneh and Manasrah [9] obtained the following result which is a refinement of the scalar Young inequality.

**Lemma 2.** Let $a, b \geq 0$, and $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab + \min \left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( a^{\frac{p}{2}} - b^{\frac{q}{2}} \right)^2 \leq \frac{a^p}{p} + \frac{b^q}{q}.$$  

(11)

Recently, Sheikhhosseini et al. [15] have obtained the following generalization of (11).

**Lemma 3.** If $a, b > 0$, and $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then for $m = 1, 2, 3, \ldots$, 

$$\left( a^{\frac{p}{2}} b^{\frac{q}{2}} \right)^m + r_0^m \left( a^{\frac{m}{2}} - b^{\frac{m}{2}} \right)^2 \leq \left( \frac{a^r}{p} + \frac{b^r}{q} \right)^{\frac{m}{r}}, \quad r \geq 1,$$

(12)

where $r_0 = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}$. In particular, if $p = q = 2$, then

$$\left( a^{\frac{1}{2}} b^{\frac{1}{2}} \right)^m + \frac{1}{2^m} \left( a^{\frac{m}{2}} - b^{\frac{m}{2}} \right)^2 \leq 2^{-m} \left( a^r + b^r \right)^{\frac{m}{r}}.$$

For $m = 1$

$$\left( a^{\frac{1}{2}} b^{\frac{1}{2}} \right)^1 + \frac{1}{2} \left( a^{\frac{1}{2}} - b^{\frac{1}{2}} \right)^2 \leq 2^{-1} \left( a^r + b^r \right)^{\frac{1}{r}}.$$

In what follows, we establish some numerical radius inequalities by providing some refinements of well-known numerical radius inequalities. Let us begin with the following result.
Theorem 1. Let $T \in B(\mathcal{H})$, $\alpha, \beta \geq 0$ be such that $\alpha + \beta \geq 1$. Then

$$w^m \left( T |T|^{\alpha + \beta - 1} \right) \leq \frac{1}{2^m} \left\| T |T|^{2r\alpha} + |T^*|^{2r\beta} \right\|^m - \frac{1}{2^m} \inf_{\|x\| = 1} \left( \frac{\langle |T|^{2\alpha} x, x \rangle^{\frac{m}{2}}}{2} - \frac{\langle |T^*|^{2\beta} x, x \rangle^{\frac{m}{2}}}{2} \right)^2$$

(13)

Proof. Let $y = x$ in (5). Then for all $m \geq 1$ we have

$$\left( \frac{\langle |T|^{2\alpha} x, x \rangle r + \langle |T^*|^{2\beta} x, x \rangle r}{2} \right)^{\frac{m}{2}} \leq \frac{1}{2^m} \left( \frac{\langle |T|^{2\alpha} x, x \rangle}{2} - \frac{\langle |T^*|^{2\beta} x, x \rangle}{2} \right)^2$$

(by (12))

$$\leq \frac{\langle |T|^{2r\alpha} x, x \rangle + \langle |T^*|^{2r\beta} x, x \rangle}{2}$$

(by Lemma 1)

$$- \frac{1}{2^m} \left( \frac{\langle |T|^{2\alpha} x, x \rangle}{2} - \frac{\langle |T^*|^{2\beta} x, x \rangle}{2} \right)^2.$$ 

Taking the supremum over all unit vectors $x \in \mathcal{H}$, we get the desired result. ▷

Corollary 1. Let $T \in B(\mathcal{H})$, $\alpha, \beta \geq 0$ be such that $\alpha + \beta \geq 1$. Then

$$w^2 \left( T |T|^{\alpha + \beta - 1} \right) \leq \frac{1}{2^2} \left\| T |T|^{2r\alpha} + |T^*|^{2r\beta} \right\|^2 - \frac{1}{4} \inf_{\|x\| = 1} \left( \frac{\langle |T|^{2\alpha} x, x \rangle}{2} - \frac{\langle |T^*|^{2\beta} x, x \rangle}{2} \right)^2$$

(14)

Proof. Setting $m = 1$ in (13) we get the desired result. ▷

Remark 1. Setting $r = 1$ in (14), we get

$$w^2 \left( T |T|^{\alpha + \beta - 1} \right) \leq \frac{1}{4} \left\| T |T|^{2\alpha} + |T^*|^{2\beta} \right\|^2 - \frac{1}{4} \inf_{\|x\| = 1} \left( \frac{\langle |T|^{2\alpha} x, x \rangle}{2} - \frac{\langle |T^*|^{2\beta} x, x \rangle}{2} \right)^2$$

for all $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 1$. 
Choosing \( \alpha = \beta = \frac{1}{2} \), we get

\[
\omega^2(T) \leq \frac{1}{4} \| T \| + \| T^* \| \leq \frac{1}{4} \inf_{\| x \| = 1} \left( \| T \| x, x \| - \langle |T^*| x, x \rangle \right)^2.
\]

However, if we choose \( \alpha = \beta = 1 \), we get

\[
\omega^2(T) \leq \frac{1}{4} \| T \|^2 + \| T^* \|^2 - \frac{1}{4} \inf_{\| x \| = 1} \left( \| T \|^2 x, x \| - \langle |T^*|^2 x, x \rangle \right)^2,
\]

or it can be rewritten as

\[
\omega^2(T) \leq \frac{1}{4} \| T^* T + T T^* \|^2 - \frac{1}{4} \inf_{\| x \| = 1} \langle \langle |T^*| T - T T^* \rangle x, x \rangle^2.
\]

A generalization of the above results could be embodied as follows:

**Theorem 2.** Let \( T \in \mathcal{B}(\mathcal{H}) \), \( \alpha, \beta \geq 0 \) be such that \( \alpha + \beta \geq 1 \). Then

\[
\omega^{2s}(T | T |^{\alpha+\beta-1}) \leq 2^{-\frac{s}{2}} \left( \| T \|^2 r^s \| T^* \|^{2s} \right)^{\frac{2}{7}} - \frac{1}{4} \inf_{\| x \| = 1} \left( \left( \langle |T|^{2sr} x, x \rangle - \langle |T^*|^{2sr} x, x \rangle \right)^2 \right)
\]

for all \( r, s \geq 1 \).

**Proof.** Let \( y = x \) in (5). By applying Lemma 3 with \( p = q = 2 \) and \( m = 2 \), we get

\[
\left( \langle |T|^{\alpha+\beta-1} x, x \rangle \right)^{2s}
\]

\[
\leq \langle |T|^{2\alpha} x, x \rangle^{s} \langle |T^*|^{2\beta} x, x \rangle^{s} \quad (t^s \text{ increasing})
\]

\[
\leq \langle |T|^{2\alpha} x, x \rangle \langle |T^*|^{2\beta} x, x \rangle \quad (\text{by convexity of } t^s)
\]

\[
\leq 2^{-\frac{s}{2}} \left( \langle |T|^{2sr} x, x \rangle^s + \langle |T^*|^{2sr} x, x \rangle^s \right)^{\frac{2}{7}} \quad (\text{by Lemma 3})
\]

\[
- \frac{1}{4} \left( \langle |T|^{2sr} x, x \rangle - \langle |T^*|^{2sr} x, x \rangle \right)
\]

\[
\leq 2^{-\frac{s}{2}} \left( \langle |T|^{2sr} x, x \rangle + \langle |T^*|^{2sr} x, x \rangle \right)^{\frac{2}{7}} \quad (\text{by Lemma 1})
\]

\[
- \frac{1}{4} \left( \langle |T|^{2sr} x, x \rangle - \langle |T^*|^{2sr} x, x \rangle \right).
\]

Taking the supremum over all unit vectors \( x \in \mathcal{H} \), we get the desired result. ▷
Corollary 2. Let $T \in \mathcal{B}(\mathcal{H})$, $\alpha, \beta \geq 0$ be such that $\alpha + \beta \geq 1$. Then
\[
w^{2s}\left(T|T|^{\alpha + \beta - 1}\right) \leq \frac{1}{4} \left\| |T|^{2\alpha} + |T^*|^{2\beta}\right\|^2 - \frac{1}{4} \sup_{\|x\|=1} \left[ \langle |T|^{2\alpha} x, x \rangle - \langle |T^*|^{2\beta} x, x \rangle \right] \tag{16}
\]
for all $s \geq 1$.

Proof. Setting $r = 1$ in (15). ◀

Remark 2. Setting $\alpha = \beta = \frac{1}{2}$ in (16), we get
\[
w^{2s}(T) \leq \frac{1}{4} \| |T|^s + |T^*|^s \|^2 - \frac{1}{4} \sup_{\|x\|=1} \left[ \langle |T|^s x, x \rangle - \langle |T^*|^s x, x \rangle \right]
\]
for all $s \geq 1$. In particular case, choosing $s = 1$ we get
\[
w^2(T) \leq \frac{1}{4} \| |T| + |T^*| \|^2 - \frac{1}{4} \sup_{\|x\|=1} \left[ \langle |T| x, x \rangle - \langle |T^*| x, x \rangle \right].
\]

Remark 3. Setting $\alpha = \beta = \frac{1}{s}$, $s \geq 1$, we get
\[
w^{2s}\left(T|T|^{\frac{2}{s}-1}\right) \leq \frac{1}{4} \left\| |T|^2 + |T^*|^2\right\|^2 - \frac{1}{4} \sup_{\|x\|=1} \left[ \langle |T|^2 x, x \rangle - \langle |T^*|^2 x, x \rangle \right].
\]
(17)

In particular case, choosing $s = 1$ in (17), we get
\[
w^2(T|T|) \leq \frac{1}{4} \| |T|^2 + |T^*|^2 \|^2 - \frac{1}{4} \sup_{\|x\|=1} \left[ \langle |T|^2 x, x \rangle - \langle |T^*|^2 x, x \rangle \right],
\]
which can be rewritten as
\[
w^2(T|T|) \leq \frac{1}{4} \| T^*T + TT^* \|^2 - \frac{1}{4} \sup_{\|x\|=1} \left[ \langle |T|^2 x, x \rangle - \langle |T^*|^2 x, x \rangle \right],
\]

Remark 4. Setting $\alpha = \beta = \frac{1}{2}$, $s = 1$, $r = 2$, we get
\[
w^2(T) \leq \frac{1}{2} \left\| |T|^2 + |T|^2\right\|^2 - \frac{1}{4} \sup_{\|x\|=1} \left[ \langle |T|^2 x, x \rangle - \langle |T^*|^2 x, x \rangle \right],
\]
or
\[
w^2(T) \leq \frac{1}{2} \| T^*T + TT^* \|^2 - \frac{1}{4} \sup_{\|x\|=1} \left[ \langle |T|^2 x, x \rangle - \langle |T^*|^2 x, x \rangle \right],
\]
(19)
and this refines the upper bound in the Kittaneh inequality (7).
Theorem 3. Let $T \in \mathcal{B}(\mathcal{H})$, $\alpha, \beta \geq 0$ be such that $\alpha + \beta \geq 1$. Then

$$w^{2s} \left( T |T|^{\alpha+\beta-1} \right) \leq \left\| \frac{1}{p} |T|^{2sp\alpha} + \frac{1}{q} |T^*|^{2sq\beta} \right\|$$

$$- r_0 \inf_{\|x\|=1} \left( \langle |T|^{2s\alpha} x, x \rangle^{\frac{p}{2}} - \langle |T^*|^{2s\beta} x, x \rangle^{\frac{q}{2}} \right)^2$$

for all $s \geq 1$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, where $r_0 := \min\left\{ \frac{1}{p}, \frac{1}{q} \right\}$.

In particular case, we have

$$w^{2s} \left( T |T|^{\alpha+\beta-1} \right) \leq \frac{1}{2} \left\| |T|^{4s\alpha} + |T^*|^{4s\beta} \right\|$$

$$- \frac{1}{2} \inf_{\|x\|=1} \left( \langle |T|^{2s\alpha} x, x \rangle - \langle |T^*|^{2s\beta} x, x \rangle \right)^2$$

Proof. Let $s \geq 1$. Setting $y = x$ in (5), we get

$$\left\| T |T|^{\alpha+\beta-1} x, x \right\|^{2s} \leq \left\| T^{2s\alpha} x, x \right\|^{s} \left\| T^*^{2s\beta} x, x \right\|^{s}$$

(by (5))

$$\leq \left\| T^{2s\alpha} x, x \right\| \left\| T^*^{2s\beta} x, x \right\|$$

(by convexity of $t^s$)

$$\leq \frac{1}{p} \left\| T^{2s\alpha} x, x \right\|^{p} + \frac{1}{q} \left\| T^*^{2s\beta} x, x \right\|^{q}$$

(by Lemma 2)

$$- r_0 \left( \left\| T^{2s\alpha} x, x \right\|^{\frac{p}{2}} - \left\| T^*^{2s\beta} x, x \right\|^{\frac{q}{2}} \right)^2$$

(by Lemma 1)

Taking the supremum over all unit vectors $x \in \mathcal{H}$, we get the required result. The particular case follows by setting $p = q = 2$. ▷

Various interesting special cases could be deduced from (13). In what follows, we give some of these cases in remarks.

Remark 5. Setting $\alpha = \beta = \frac{1}{2}$ in (14), we have

$$w^{2s} (T) \leq \frac{1}{2} \left\| T^{2s} + |T^*|^{2s} \right\| - \frac{1}{2} \inf_{\|x\|=1} \left( \langle |T|^s x, x \rangle - \langle |T^*|^s x, x \rangle \right)^2$$

for all $s \geq 1$. In particular, for $s = 1$ we get

$$w^2 (T) \leq \frac{1}{2} \left\| T^2 + |T^*|^2 \right\| - \frac{1}{2} \inf_{\|x\|=1} \left( \langle |T| x, x \rangle - \langle |T^*| x, x \rangle \right)^2,$$
which can be rewritten as

\[ w^2(T) \leq \frac{1}{2} \|T^*T + TT^*\| - \frac{1}{2} \inf_{\|x\|=1} (\|T\| x, x) - (\|T^*\| x, x))^2. \]  

(22)

This refines the upper bound of the refinement of Kittaneh inequality (19). Clearly, (22) is better than (19), which, in turn, is better than (7).

**Remark 6.** Setting \( \alpha = \beta = 1 \) in (20), we have

\[ w^{2s}(T|T|) \leq \left\| \frac{1}{p} |T|^{2p} + \frac{1}{q} |T^*|^{2q} \right\| 
- r_0 \inf_{\|x\|=1} \left( \left\| |T|^{2s} x, x \right\|^{\frac{p}{2}} - \left\| |T^*|^{2s} x, x \right\|^{\frac{q}{2}} \right)^2 \]

for all \( s \geq 1 \) and \( p, q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), where \( r_0 := \min \{\frac{1}{p}, \frac{1}{q}\} \).

In particular case, choosing \( s = 1 \) and \( p = q = 2 \) in the previous inequality, we get

\[ w^2(T|T|) \leq \frac{1}{2} \left\| |T|^4 + |T^*|^4 \right\| - \frac{1}{2} \inf_{\|x\|=1} \left( \left\| |T|^2 x, x \right\| - \left\| |T^*|^2 x, x \right\| \right)^2. \]

Numerical radius inequality for a special type of Hilbert space operators for commutators can be established as follows:

**Theorem 4.** Let \( T, S \in \mathcal{B}(\mathcal{H}) \), \( \alpha, \beta, \gamma, \delta \geq 0 \) be such that \( \alpha + \beta \geq 1 \) and \( \gamma + \delta \geq 1 \). Then

\[ w \left( T|T|^{\alpha+\beta-1} + S|S|^{\gamma+\delta-1} \right) \]

\[ \leq 2^{-\frac{1}{r}} \left\| T^{2\alpha} + |T^*|^{2\beta} \right\|^{\frac{1}{2}} + 2^{-\frac{1}{r}} \left\| S^{2\gamma} + |S^*|^{2\delta} \right\|^{\frac{1}{2}} 
- \frac{1}{2} \inf_{\|x\|=1} \left( \left\| |T|^{2\alpha} x, x \right\|^{\frac{1}{2}} - \left\| |T^*|^{2\beta} x, x \right\|^{\frac{1}{2}} \right)^2 
- \frac{1}{2} \inf_{\|x\|=1} \left( \left\| |S|^{2\gamma} x, x \right\|^{\frac{1}{2}} - \left\| |S^*|^{2\delta} x, x \right\|^{\frac{1}{2}} \right)^2 \]

for all \( r \geq 1 \).

**Proof.** Employing the triangle inequality, we have

\[ \left\| \left( T|T|^{\alpha+\beta-1} + S|S|^{\gamma+\delta-1} \right) x, x \right\| \]
\[
\leq \left| \langle T | T |^{\alpha + \beta - 1} x, x \rangle \right| + \left| \langle S | S |^{\gamma + \delta - 1} x, x \rangle \right|
\]
\[
\leq \langle |T|^{2\alpha} x, x \rangle \frac{1}{2} \left( \langle |T|^{2\beta} x, x \rangle \frac{1}{2} + \langle |S|^{2\gamma} x, x \rangle \frac{1}{2} \right) + \left( \langle |S|^{2\delta} x, x \rangle \frac{1}{2} \right)^{\frac{1}{2}} \quad \text{(by (5))}
\]
\[
\leq 2^{-\frac{1}{2}} \left( \langle |T|^{2\alpha} x, x \rangle^r + \langle |T|^{2\beta} x, x \rangle^r \right)^{\frac{1}{2}} - \frac{1}{2} \left( \langle |T|^{2\alpha} x, x \rangle^r - \langle |T|^{2\beta} x, x \rangle^r \right) \right)^{2}. \quad \text{(by Lemma 3)}
\]
\[
\leq 2^{-\frac{1}{2}} \left( \langle |T|^{2\alpha} x, x \rangle + \langle |T|^{2\beta} x, x \rangle \right)^{\frac{1}{2}} - \frac{1}{2} \left( \langle |T|^{2\alpha} x, x \rangle - \langle |T|^{2\beta} x, x \rangle \right)^{2} \quad \text{(by Lemma 1)}
\]

Taking the supremum over all unit vectors \(x \in \mathcal{H}\), we get the desired result. ▶

**Corollary 3.** Let \(T, S \in \mathcal{B}(\mathcal{H})\), \(\alpha, \beta, \gamma, \delta \geq 0\) be such that \(\alpha + \beta \geq 1\) and \(\gamma + \delta \geq 1\). Then

\[
w \left( |T| |^{\alpha + \beta - 1} + |S| |^{\gamma + \delta - 1} \right) \leq \frac{1}{2} \left\| |T|^{2\alpha} + |T|^{2\beta} + |S|^{2\gamma} + |S|^{2\delta} \right\|
\]
\[
- \frac{1}{2} \inf_{\|x\|=1} \left( \langle |T|^{2\alpha} x, x \rangle \frac{1}{2} - \langle |T|^{2\beta} x, x \rangle \frac{1}{2} \right)^{2} \quad \text{(24)}
\]
\[
- \frac{1}{2} \inf_{\|x\|=1} \left( \langle |S|^{2\gamma} x, x \rangle \frac{1}{2} - \langle |S|^{2\delta} x, x \rangle \frac{1}{2} \right)^{2}.
\]

**Proof.** Setting \(r = 1\) in the proof of Theorem 4, and then taking the supremum over all unit vectors \(x \in \mathcal{H}\), we get the desired result. ▶

**Remark 7.** Setting \(\alpha = \beta = \gamma = \delta = \frac{1}{2}\) in (24), we get

\[
w (T + S) \leq \frac{1}{2} \left\| |T| + |T| + |S| + |S| \right\| - \frac{1}{2} \inf_{\|x\|=1} \left( \langle |T| x, x \rangle \frac{1}{2} - \langle |T| x, x \rangle \frac{1}{2} \right)^{2}
\]
\[-\frac{1}{2} \inf_{\|x\|=1} \left( \langle |S| x, x \rangle \frac{1}{2} - \langle |S^*| x, x \rangle \frac{1}{2} \right)^2 \]

In particular, taking \( S = T \) we get

\[ w(T) \leq \frac{1}{2} \|T\| + \|T^*\| + \frac{1}{2} \inf_{\|x\|=1} \left( \langle |T| x, x \rangle \frac{1}{2} - \langle |T^*| x, x \rangle \frac{1}{2} \right)^2 . \]

**Remark 8.** Setting \( \alpha = \beta = \gamma = \delta = 1 \) in (24), we get

\[ w(T|T| + S|S|) \leq \frac{1}{2} \|T\|^2 + |T^*|^2 + |S|^2 + |S^*|^2 \]
\[-\frac{1}{2} \inf_{\|x\|=1} \left( \langle |T|^2 x, x \rangle \frac{1}{2} - \langle |T^*|^2 x, x \rangle \frac{1}{2} \right)^2 \]
\[-\frac{1}{2} \inf_{\|x\|=1} \left( \langle |S|^2 x, x \rangle \frac{1}{2} - \langle |S^*|^2 x, x \rangle \frac{1}{2} \right)^2 \]

In particular, taking \( S = T \), we get

\[ w(T|T|) \leq \frac{1}{2} \|T|T| + |T^*|^2 \| - \frac{1}{2} \inf_{\|x\|=1} \left( \langle |T|^2 x, x \rangle \frac{1}{2} - \langle |T^*|^2 x, x \rangle \frac{1}{2} \right)^2 \]
\[= \frac{1}{2} \|T^*T + TT^*\| - \frac{1}{2} \inf_{\|x\|=1} \left( \langle |T|^2 x, x \rangle \frac{1}{2} - \langle |T^*|^2 x, x \rangle \frac{1}{2} \right)^2 \]

**References**


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