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On the Basicity of Linear Phase Systems of Sines and Cosines in Sobolev-Morrey Spaces

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Abstract. In this work, systems of sines $1 \cup \{\sin(n-\beta)t\}_{n\geq 1}$ and cosines

$$1 \cup \{\cos(n-\beta)t\}_{n\geq 1}$$

are considered, where β is a complex parameter. The subspace $MW_{p,\alpha}(0,\pi)$ of the Sobolev-Morrey space $W^1_{p,\alpha}(0,\pi)$ in which continuous functions are dense is considered. Criterion for the completeness, minimality and basicity of these systems with respect to the parameter β in the subspace $MW_{p,\alpha}(0,\pi)$, $1 < p < +\infty$, $0 < \alpha < 1$ is found.

Key Words and Phrases: Sobolev-Morrey spaces, trigonometric systems, completeness, minimality, basicity.

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1. Introduction

Morrey spaces were introduced by Morrey [1] in the study of partial differential equations and appeared to be quite useful in the research of local behavior of solutions to elliptic partial differential equations, a priori estimates and other topics in the theory of PDE. More precisely, it is a useful tool in the qualitative theory of elliptic differential equations [2, 3]. Furthermore, it provides a large class of examples of mild solutions to the Navier-Stokes system [4]. In the context of fluid dynamics, Morrey spaces have been used to model flow when vorticity is a singular measure supported on some set in R^3 [5]. There is a lot of research dealing with fundamental problems in these spaces concerning differential equations, potential theory, maximal and singular operator theory and approximation theory (c.f. [6] and the references above).

Recently, there has been an increasing interest non-standart function spaces in the context of various specific problems of modern mathematics. For more information on related issues, see monographs [7-12]. The desire to study the differential equations describing the above mentioned problems in non-standart Sobolev spaces required the study of the basis properties of the systems

$$\sin(n-\beta)t, \quad n \in N, \tag{1}$$

and

$$\cos(n-\beta)t, \quad n \in Z_+ \quad (Z_+ = \{0\} \cup N), \tag{2}$$

in the corresponding non-standart Lebesgue spaces of functions. In [13-22], the basis properties of the systems

$$\left\{e^{i(n-\beta\, sign\, n)t}\right\}_{n\in Z}, \tag{3}$$

$$1 \cup \left\{e^{i(n-\beta\, sign\, n)t}\right\}_{n\neq 0}, \tag{4}$$

were established in Lebesgue spaces of functions with variable summability exponent and in Morrey type spaces. B.T. Bilalov [13] established criteria for the basis properties of the system (3) in Morrey type spaces. In [22], similar results were obtained for the system (4) in the same spaces. Basicity in Morrey type spaces of a system of eigenfunctions of some discontinuous differential operators has been studied in [23], [24].

Later, in [25], criteria for the basis properties of the system of sines (1) and cosines (2) in Morrey-type spaces have been found. Basis properties of the exponential system (3) in Sobolev-Morrey spaces have been treated in paper [26].

In this paper, we introduce a method for investigating the basis properties of the systems $1 \cup \{\sin(n-\beta)t\}_{n\geq 1}$ and $1 \cup \{\cos(n-\beta)t\}_{n\geq 0}$ where β is a complex parameter, in Sobolev-Morrey spaces.

2. Needful information

We will use the following standard notations. N will be the set of natural numbers; Z will be the set of integers and $[x]$ will denote the integer part of number x.

Let's define the Morrey space $L^{p,\alpha}(a,b)$. It is a Banach spaces of all measurable functions on (a,b) with the finite norm

$$\|f\|_{L^{p,\alpha}(a,b)} = \sup_{I\subset(a,b)} \left(|I|^{\alpha-1}\int_I |f(t)|^p dt\right)^{1/p},$$

where sup is taken over all intervals $I \subset (a,b)$. For $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ the following continuous embedding holds: $L^{p,\alpha_1} \subset L^{p,\alpha_2}$. It is easy to see that

$L^{p,1}(a,b) = L_p(a,b)$ and $L^{p,0}(a,b) = L_\infty(a,b)$ are true. Moreover $L^{p,\alpha}(a,b) \subset L_1(a,b)$, $\forall p > 1$, $\forall \alpha \in [0,1]$. It is known that $L^{p,\alpha}(a,b)$, $1 \le p < +\infty$, $\alpha \in (0,1)$, is not separable and $C[a,b]$ is not dense in it. Let

$$\widetilde{L}^{p,\alpha}(a,b) = \left\{ f \in L^{p,\alpha}(a,b) : \|f(\cdot+\delta) - f(\cdot)\|_{L^{p,\alpha}(a,b)} \to 0 \right\}$$

The closure $\widetilde{L}^{p,\alpha}$ in $L^{p,\alpha}$ will be denoted by $M^{p,\alpha}(a,b)$, i.e. $M^{p,\alpha} = \overline{\widetilde{L}^{p,\alpha}}$. As shown in [13], $M^{p,\alpha}(a,b)$, for $1 \le p < +\infty$, $0 \le \alpha < 1$, is a separable Banach space and $C_0^\infty(a,b)$ (infinity differentiable and finitely supported functions on (a,b)) is dense in it. When defining the space $M^{p,\alpha}(a,b)$, the function $f(\cdot)$ is assumed to be extended outside the interval (a,b) by zero.

3. Morrey-Sobolev space

Let $0 \le \alpha \le 1$, $p \ge 1$. By $W^1_{p,\alpha}(a,b)$ we denote the space of functions which belong, together with their derivatives of first order, to the space $L^{p,\alpha}(a,b)$ equipped with the norm

$$\|f\|_{W^1_{p,\alpha}} = \|f\|_{L^{p,\alpha}} + \|f'\|_{L^{p,\alpha}}. \tag{5}$$

Denote by $\widetilde{W}^1_{p,\alpha}$ the linear subspace of $W^1_{p,\alpha}$ consisting of functions whose first order derivatives are continuous with respect to the shift operator. By $MW^1_{p,\alpha}$ we denote the closure of this space with respect to the norm (5).

By $\mathscr{M}_{p,\alpha}$ we denote the direct sum of $M^{p,a}$ and C (C is the complex plane)

$$\mathscr{M}_{p,\alpha} = M^{p,\alpha} \oplus C.$$

Let us define the norm in $\mathscr{M}_{p,\alpha}$ in the following way:

$$\|\hat{u}\|_{\mathscr{M}_{p,\alpha}} = \|u\|_{L^{p,\alpha}} + |\lambda|, \forall \hat{u} = (u;\lambda) \in \mathscr{M}_{p,\alpha}.$$

The following lemma is true.

Lemma 1. *The operator* $(A\hat{u})(t) = \lambda + \int_a^t u(\tau)\,d\tau$ *is an isomorphism from* $\mathscr{M}_{p,\alpha}$ *to* $MW^1_{p,\alpha}$.

Proof. First let us show that $v(t) = (A\hat{u})t \in W^1_{p,\alpha}$. Indeed, since $L^{p,\alpha} \subset L_p \subset L_1$, we have

$$\|v(t)\|_{L^{p,\alpha}} = \left\| \lambda + \int_a^t u(\tau)\,d\tau \right\|_{L^{p,\alpha}} \le \|\lambda\|_{L_{p,\alpha}} + \left\| \int_a^t u(\tau)\,d\tau \right\|_{L^{p,\alpha}} \le$$

$$\le (b-a)^{\frac{\alpha}{p}} |\lambda| + \sup_{I \subset (a,b)} \left\{ \frac{1}{|I|^{1-\alpha}} \int_I \left| \int_a^t u(\tau)\,d\tau \right|^p dt \right\}^{1/p} \le$$

$$\leq (b-a)^{\frac{\alpha}{p}} |\lambda| + \sup_{I \subset (a,b)} \left\{ \frac{1}{|I|^{1-\alpha}} \int_I \left(\int_a^b |u(\tau)| \, d\tau \right)^p dt \right\}^{1/p} =$$

$$= (b-a)^{\frac{\alpha}{p}} |\lambda| + (b-a)^{\frac{\alpha}{p}} \|u\|_{L_1(a,b)} < +\infty. \qquad (6)$$

Also, since $v'(t) = u(t) \in L^{p,\alpha}$, we have $v(t) \in W^1_{p,\alpha}$.

By using the absolute continuity of the Lebesgue integral when $t \in (a,b)$, uniformly with respect to t, we have $\int_t^{t+\delta} u(\tau) d\tau \to 0$, as $\delta \to 0$, if $u \in L^{p,\alpha}$.

Now we show that $v(t) \in MW^1_{p,\alpha}$. From $u \in M^{p,\alpha}$ it follows

$$\|v(\cdot+\delta) - v(\cdot)\|_{W^1_{p,\alpha}} = \|v(\cdot+\delta) - v(\cdot)\|_{L^{p,\alpha}} + \left\| v'(\cdot+\delta) - v'(\cdot) \right\|_{L^{p,\alpha}} =$$

$$= \left\| \int_{\cdot}^{\cdot+\delta} u(\tau) \, d\tau \right\|_{L^{p,\alpha}} + \|u(\cdot+\delta) - u(\cdot)\|_{L^{p,\alpha}} \to 0, \quad \delta \to 0.$$

Let us show that A is a bounded operator. We have

$$\|A(\hat{u})\|_{W^1_{p,\alpha}} = \left\| \lambda + \int_a^t u(\tau) \, d\tau \right\|_{L^{p,\alpha}} + \|u(\tau)\|_{L^{p,\alpha}}.$$

Taking into account (6)

$$\|A(\hat{u})\|_{W^1_{p,\alpha}} \leq (b-a)^{\frac{\alpha}{p}} |\lambda| + (b-a)^{\frac{\alpha}{p}} \|u\|_{L_1(a,b)} + \|u\|_{L^{p,\alpha}}.$$

As

$$\|u\|_{L_1} \leq C_1 \|u\|_{L_p} \leq C_2 \|u\|_{L^{p,\alpha}},$$

we get the validity of the following inequality:

$$\|A(\hat{u})\|_{W^1_{p,\alpha}} \leq C \left(|\lambda| + \|u\|_{L^{p,\alpha}} \right) = C \|\hat{u}\|_{\mathscr{M}_{p,\alpha}}, C = const.$$

Let us show that $\ker A = \{0\}$. Let $A\hat{u} = 0$, i.e. $\lambda + \int_a^t u(\tau) \, d\tau = 0$. If we differentiate both sides, we get $u(t) = 0$, a.e.. Thus $\lambda = 0$. We have $\hat{u} = 0$. For $\forall v \in MW^1_{p,\alpha}$ taking $\hat{v} = (v'; v(a))$ we have $\hat{v} \in L_{p,\alpha}$ and $A(\hat{v}) = v$. It means that $R_A = MW^1_{p,\alpha}$, where R_A is a range of the operator A . It follows from Banach's inverse operator theorem that the inverse of A is a continuous operator. The lemma is proved. ◀

The following theorem is true.

Theorem 1. *Let* $2Re\beta + \frac{\alpha}{p} \notin Z$, $\beta \neq 1$, $1 < p < +\infty$, $0 < \alpha < 1$. *System* $1 \cup \{\sin(n-\beta)t\}_{n \geq 1}$ *forms a basis for* $MW^1_{p,\alpha}(0,\pi)$ *if and only if* $[Re\beta + \frac{\alpha}{2p} - \frac{1}{2}] = 0$. *Moreover, for* $[Re\beta + \frac{\alpha}{2p} - \frac{1}{2}] < 0$, *it is not complete, but minimal; and for* $[Re\beta + \frac{\alpha}{2p} - \frac{1}{2}] > 0$ *it is complete, but not minimal in* $MW^1_{p,\alpha}(0,\pi)$.

Proof. It is known that the system $\{\cos(n-\beta)t\}_{n\geq 1}$ is a basis in the spaces $M^{p,\alpha}(0,\pi)$, if $[Re\beta+\frac{\alpha}{2p}-\frac{1}{2}]=0$ [25].

Let us prove that the system $\{\widehat{u}_n\}_{n\geq 0}$ forms a basis for $\mathscr{M}^{p,\alpha}(0,\pi)$, where

$$\widehat{u}_0=\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \widehat{u}_n=\begin{pmatrix} (n-\beta)\cos(n-\beta)t \\ 0 \end{pmatrix}, \quad n\geq 1.$$

Let as show that for $\forall \widehat{u}\in\mathscr{M}^{p,\alpha}$ there exists a unique decomposition

$$\widehat{u}=\sum_{n=0}^{\infty} C_n\widehat{u}_n. \tag{7}$$

This decomposition is equivalent to the one

$$u(t)=\sum_{n=1}^{\infty} C_n(n-\beta)\cos(n-\beta)t \tag{8}$$

and the equality $\lambda=C_0$, where $\widehat{u}=\begin{pmatrix} u(t) \\ \lambda \end{pmatrix}$.

Following [25], we obtain the existence of decomposition (8) which is unique. Therefore, the decomposition (7) also exists and is unique. I.e. the system $\{\widehat{u}_n\}_{n\geq 0}$ forms a basis for $\mathscr{M}^{p,\alpha}(0,\pi)$. We can easily calculate that for the operator

$$A\widehat{u}=\lambda+\int_0^t u(\tau)d\tau$$

the relations

$$A(\widehat{u}_0)=1, \quad A(\widehat{u}_n)=\sin(n-\beta)t,$$

are true if A is an isomorphism. Then the system $1\cup\{\sin(n-\beta)t\}_{n\geq 1}$ forms a basis for $MW^1_{p,\alpha}(0,\pi)$, if $[Re\beta+\frac{\alpha}{2p}-\frac{1}{2}]=0$.

Consider the case where $Re\beta+\frac{\alpha}{2p}-\frac{1}{2}<0$. Let, for example, the inequalities

$$-1<Re\beta+\frac{\alpha}{2p}-\frac{1}{2}<0 \;\Leftrightarrow\; 0<Re(\beta+1)+\frac{\alpha}{2p}-\frac{1}{2}<1$$

hold. Consider the system $1\cup\{\sin(n-\beta)\}_{n\geq 0}$ and transform it to

$$1\cup\{\sin(n-(\beta+1)+1)\}_{n\geq 0}\equiv 1\cup\{\sin(n-\beta_1)t\}_{n\geq 1},$$

where $\beta_1=\beta+1$. Consequently, the inequality

$$0<Re\beta_1+\frac{\alpha}{2p}-\frac{1}{2}<1,$$

holds and as a result the system $1\cup\{\sin(n-\beta)t\}_{n\geq 0}$ forms a basis, and therefore the system $1\cup\{\sin(n-\beta)t\}_{n\geq 1}$ is minimal, but not complete in $MW^1_{p,\alpha}(0,\pi)$. Continuing this process, we see that for $Re\beta+\frac{\alpha}{2p}-\frac{1}{2}<0$ the system $1\cup\{\sin(n-\beta)t\}_{n\geq 1}$ is minimal, but not complete in $MW^1_{p,\alpha}(0,\pi)$.

With similar reasoning, it is proved that for $Re\beta+\frac{\alpha}{2p}-\frac{1}{2}>1$, the system $1\cup\{\sin(n-\beta)t\}_{n\geq 1}$ is complete, but not minimal in $MW^1_{p,\alpha}(0,\pi)$. The theorem is proved. ◄

Remark 1. *If $\beta=1$, then it can be shown in a similar way that $1\cup\{t\}\cup\{\sin nt\}_{n\geq 1}$ forms a basis for $MW^1_p(0,\pi)$.*

Theorem 2. *Let $Re\beta+\frac{\alpha}{2p}\notin Z$, $1<p<+\infty$, $0<\alpha<1$. The system $1\cup\{\cos(n-\beta)t\}_{n\geq 1}$ forms a basis for $MW^1_{p,\alpha}(0,\pi)$ if and only if $[Re\beta+\frac{\alpha}{2p}]=0$. Moreover, for $[Re\beta+\frac{\alpha}{2p}]<0$ it is not complete, but minimal; and for $[Re\beta+\frac{\alpha}{2p}]>0$ it is complete, but not minimal in $MW^1_{p,\alpha}(0,\pi)$.*

Proof. It is known that the system $\{\sin(n-\beta)t\}_{n\geq 1}$ forms a basis for $M^{p,\alpha}(0,\pi)$, if $[Re\beta+\frac{\alpha}{2p}]=0$ [25]. We will prove that the system $\{\widehat{u}_n\}_{n\geq 0}$ forms a basis for $\mathscr{M}^{p,\alpha}(0,\pi)$, where

$$\widehat{u}_0=\begin{pmatrix}0\\1\end{pmatrix},\quad \widehat{u}_n=\begin{pmatrix}-(n-\beta)sin(n-\beta)t\\1\end{pmatrix},\quad n\geq 1.$$

Let us show that for $\forall\widehat{u}\in\mathscr{M}^{p,\alpha}$ there exists a unique decomposition

$$\widehat{u}=\sum_{n=0}^{\infty}C_n\widehat{u}_n, \tag{9}$$

where $\widehat{u}=\begin{pmatrix}u(t)\\\lambda\end{pmatrix}$.

This decomposition is equivalent to the following two decompositions:

$$u(t)=-\sum_{n=1}^{\infty}(n-\beta)C_n\sin(n-\beta)t, \tag{10}$$

$$\lambda=\sum_{n=0}^{\infty}C_n. \tag{11}$$

Following [25], we obtain the unique existence of (10) which belongs to the space $M^{p,\alpha}$. Since $L^{p,\alpha}\subset L_p$, by the main result of [27], if $1<p\leq 2$, $Re\beta+\frac{1}{2p}<1$, then the Hausdorff-Young inequality holds for the system $\{\sin(n-\beta)t\}_{n\geq 1}$. That is for $1 < p \leq 2$, we have

$$\left(\sum_{n=1}^{\infty} |C_n(n-\beta)|^q\right)^{1/q} \leq M\|u\|_{L_p},$$

where $\frac{1}{p}+\frac{1}{q}=1$. Applying Hölder's inequality, we obtain

$$\sum_{n=1}^{\infty}|C_n| = \sum_{n=1}^{\infty}\frac{1}{|n-\beta|}\cdot|C_n(n-\beta)| \leq$$

$$\leq \left(\sum_{n=1}^{\infty}\frac{1}{|n-\beta|^p}\right)^{1/p}\cdot\left(\sum_{n=1}^{\infty}|C_n(n-\beta)|^q\right)^{1/q} < +\infty.$$

Let us note that, when $n \geq 1$, we have $n-\beta \neq 0$. Therefore, with the condition $Re\beta + \frac{\alpha}{2p} < 1$ the inequality $Re\beta < 1$ is valid.

For $p > 2$, since $L^{p,\alpha} \subset L_p \subset L_2$, we have

$$\left(\sum_{n=1}^{\infty} |C_n(n-\beta)|^2\right)^{1/2} \leq M\|u\|_{L_2},$$

and

$$\sum_{n=1}^{\infty}|C_n| = \sum_{n=1}^{\infty}\frac{1}{|n-\beta|}\cdot|C_n(n-\beta)| \leq$$

$$\leq \left(\sum_{n=1}^{\infty}\frac{1}{|n-\beta|^2}\right)^{1/2}\cdot\left(\sum_{n=1}^{\infty}|C_n(n-\beta)|^2\right)^{1/2} < +\infty.$$

Consider the case $Re\beta + \frac{1}{2p} \geq 1$. Since we have also $p > 1$, $0 < \alpha < 1$, $\left[Re\beta + \frac{\alpha}{2p}\right] = 0$, we get $Re\beta + \frac{1}{2p} < 1 + \frac{1}{2} = \frac{3}{2}$. Then $Re\beta_2 + \frac{1}{2p} < 1$ for $\beta_2 = \beta - 1$ and it follows from [27] that Hausdorff-Young inequality is valid for the system $\{\sin(n-\beta_2)t\}_{n\geq 1}$ in $L_p(0,\pi)$ $(1 < p \leq 2)$.

Taking into account that $\{\sin(n-\beta_2)t\}_{n\geq 1} \equiv \{\sin(k-\beta)t\}_{k\geq 2}$, (10) can be written in the following form:

$$\widetilde{u}(t) = \sum_{n=1}^{\infty} -(n-\beta_2)C_{n+1}\sin(n-\beta_2)t,$$

where $\widetilde{u}(t) = u(t) + (1-\beta)C_1\sin(1-\beta)t$.

Then the absolute convergence of the series $\sum\limits_{n=1}^{\infty} C_n$ is shown in a similar way as in the case $Re\beta + \frac{1}{2p} < 1$.

Thus, the conditions $\left[Re\beta + \frac{\alpha}{2p}\right] = 0$ provide the convergence of the series $\sum\limits_{n=1}^{\infty} C_n$. Therefore, in the series (11), the coefficient C_0 is uniquely defined. So, we have shown the existence and uniqueness of the expansion (9) for all $\widehat{u} \in \mathscr{M}^{p,\alpha}$.

Thus, the system $\{\widehat{u}_n\}_{n\geq 0}$ forms a basis for $\mathscr{M}^{p,\alpha}$. We can easily show that, for the operator

$$A\widehat{u} = \lambda + \int\limits_0^t u(\tau)d\tau,$$

the following relations are true: $A(\widehat{u}_0) = 1$, $A(\widehat{u}_n) = \cos(n-\beta)t$, $n \geq 1$.

Since A is an isomorphism, the system $1 \cup \{cos(n-\beta)t\}_{n\geq 1}$ forms a basis for $MW^1_{p,\alpha}(0,\pi)$, if $[Re\beta + \frac{\alpha}{2p}] = 0$. Consider the case where $Re\beta + \frac{\alpha}{2p} < 0$. Let, for example, the inequalities $-1 < Re\beta + \frac{\alpha}{2p} < 0 \Leftrightarrow 0 < Re(\beta+1) + \frac{\alpha}{2p} < 1$ hold. Consider the system $1 \cup \{cos(n-\beta)t\}_{n\geq 0}$ and transform it to $1 \cup \{\cos(n-(\beta+1)+1\}_{n\geq 0} \equiv 1 \cup \{\cos(n-\beta_1)t\}_{n\geq 1}$, where $\beta_1 = \beta + 1$. Consequently, the inequalities $0 < Re\beta_1 + \frac{\alpha}{2p} < 1$ hold and, as a result, the system $1 \cup \{\cos(n-\beta)t\}_{n\geq 0}$ forms a basis, and therefore the system $1 \cup \{\cos(n-\beta)t\}_{n\geq 1}$ is minimal, but not complete in $MW^1_{p,\alpha}(0,\pi)$.

Continuing this process, we see that for $Re\beta + \frac{\alpha}{2\beta} < 1$ the system $1 \cup \{\cos(n-\beta)t\}_{n\geq 0}$ is minimal, but not complete in $MW^1_{p,\alpha}$.

By a similar reasoning, it is proved that for $Re\beta + \frac{\alpha}{2p} > 1$, the system $1 \cup \{\cos(n-\beta)t\}_{n\geq 1}$ is complete, but not minimal in $MW^1_{p,\alpha}(0,\pi)$. So Theorem 2 is also proved. ◄

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