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Solution in the Small and Interior Schauder-type Estimate for the m-th Order Elliptic Operator in Morrey-Sobolev Spaces

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Abstract. A higher order elliptic operator with non-smooth coefficients in Morrey-Sobolev spaces on a bounded domain in R^n is considered. These spaces are nonseparable, and infinite differentiable functions are not dense in them. For this reason, the classical methods of establishing interior (and other) estimates with respect to these operators, the possibility of extending functions (with a bounded norm), determining the trace and results associated with this concept, etc. are not applicable in Morrey-Sobolev spaces, and to establish these facts a different research approach should be chosen. This paper focuses on these issues. An extension theorem is proved, the trace of a function in a Morrey-Sobolev space on a $(n-1)$ dimensional smooth surface is defined, a theorem on the existence of a strong solution in the small is proved, an interior Schauder-type estimate in Morrey-Sobolev spaces is established. A constructive characterization of the space of traces of functions from the Morrey-Sobolev space is given, which differs from the characterization given earlier by S. Campanato. [42] who offered a different characterization of the space of traces based on different considerations. It should be noted that the approach proposed in this work differs from the classical approach to determining the trace.
Key Words and Phrases: elliptic operator, Morrey-Sobolev spaces, solution in the small, Schauder-type estimates.

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1. Introduction

Let L be an elliptic differential operator of $m-$th order

$$(Lu)\ (x) = \sum_{|p|\leq m} a_p\,(x)\ \partial^p u\,(x)\ \ ,\quad x \in \Omega\,, \tag{1}$$

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with real coefficients $a_p \in L_\infty(\Omega)$, $\forall p: \ |p| \le m$. The theory of equations $Lu = f$ in classical spaces (in Holder and Sobolev spaces) is quite well developed (mainly for $m = 2$) and covered in well-known monographs (e.g., [1, 3, 4, 5, 10, 11, 12, 13, 14, 19]). Besides, since Morrey's work [15] appeared, some issues have begun to be studied in so-called non-standard function spaces including Morrey spaces, grand-Lebesgue spaces, Lebesgue spaces with variable summability index, etc. (for more details on related issues we refer the reader to monographs [8, 9, 16, 17, 18, 19, 20] and articles [21, 22, 23, 24, 25, 26, 27, 28, 29, 30] dedicated to approximation problems in these spaces). The successes of harmonic analysis in non-standard spaces made it possible to consider solvability problems for differential equations in these spaces (see, e.g., [6, 7, 8, 9, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41]). The thing that distinguishes the study of differential equations in these spaces is that they are not separable, and infinitely differentiable functions are not dense in them. This circumstance creates specific difficulties in the study of solvability of differential equations. It becomes possible to consider problems in non-separable and separable formulations. In separable formulations, it is possible to study the corresponding problems according to the classical scheme. In this scheme, the establishment of many facts such as the concepts of trace and trace space, the corresponding estimates, compactness theorems with respect to the corresponding Sobolev spaces, extension theorems, etc., are based on properties like completeness of infinitely differentiable functions in Sobolev spaces, continuity of shift operator in these spaces, possibility of approximating the functions in these spaces by their means, etc. The results obtained in [6, 7, 8, 9, 31] are related to this direction. In non-separable case, the above properties are not true, and therefore, to establish such facts, other methods should be used which do not involve these properties. The works [31, 32, 33, 34, 35, 36, 37, 38, 39, 10, 41] are dedicated to this case. It should be noted that in these works the concept of the trace of functions from Morrey-Sobolev spaces is not well defined, there is no extension theorem and results related to these circumstances. Moreover, in most of these works (except for [41]) the equation (1) is considered in the case where $m = 2$ and the lower-order terms are absent. In this paper, we try to fill these gaps.

A higher order elliptic operator with non-smooth coefficients in Morrey-Sobolev spaces on a bounded domain in R^n is considered in this work. These spaces are non-separable, and infinite differentiable functions are not dense in them. For this reason, the classical methods of establishing interior and other estimates with respect to these operators, the possibility of extending functions (with a bounded norm), determining the trace and obtaining results associated with this concept, etc. are not applicable in Morrey-Sobolev spaces, and to establish these facts a different research approach should be chosen. This paper focuses on these issues.

An extension theorem is proved, the trace of a function in a Morrey-Sobolev space on an $(n-1)$-dimensional smooth surface is defined, a theorem on the existence of a strong solution in the small is proved, an interior Schauder-type estimate in Morrey-Sobolev spaces is established. A constructive characterization of trace space of functions from the Morrey-Sobolev space is given, which differs from the one provided earlier by S.Campanato [42] who offered a different characterization of trace space based on different considerations. It should also be noted that the approach to determination of trace proposed in this work differs from the classical one.

2. Needful information

2.1. Notations

We will use the following standard notations. Z_+ will be the set of non-negative integers. $B_r(x_0) = \{x \in R^n : |x - x_0| < r\}$ will denote the open ball in R^n centered at x_0, where $|x| = \sqrt{x_1^2 + ... + x_n^2}$, $x = (x_1, ..., x_n)$. $\Omega_r(x_0) = \Omega \bigcap B_r(x_0)$, $B_r = B_r(0)$, $\Omega_r = \Omega_r(0)$. $mes(M)$ will stand for the Lebesgue measure of the set M; $\partial\Omega$ will be the boundary of the domain Ω; $\bar{\Omega} = \Omega \bigcup \partial\Omega$; $M_1 \Delta M_2$ will denote the symmetric difference between the sets M_1 and M_2; $diam\,\Omega$ will stand for the diameter of the set Ω; $\rho(x; M)$ will be the distance between x and the set M; by $[X; Y]$ we will denote a Banach space of bounded operators acting from X to Y; $[X] = [X; X]$; and $\|T\|_{[X;Y]}$ will denote the norm of the operator T, acting boundedly from X to Y; R_T will be the range of the operator T. $W_p^k(\Omega)$ will denote the classical Sobolev space of differentiable functions of k-th degree; $[\cdot]$ will denote the integer part of the corresponding number. $\Omega_1 \subset\subset \Omega_r$ will mean that $\bar{\Omega}_1 \subset \Omega_2$.

2.2. Elliptic operator of m-th order

Let $\Omega \subset R^n$ be some bounded domain with boundary $\partial\Omega$. We will use the notations of [4]. $\alpha = (\alpha_1, ..., \alpha_n)$ will be the multiindex with the coordinates $\alpha_k \in Z_+$, $\forall k = \overline{1,n}$; $\partial_i = \frac{\partial}{x_i}$ will denote the differentiation operator, $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}...\partial_n^{\alpha_n}$. For every $\xi = (\xi_1, ..., \xi_n)$ we assume $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}...\xi_n^{\alpha_n}$. Let L be an elliptic differential operator of m-th order

$$L = \sum_{|p| \le m} a_p(x)\,\partial^p, \tag{2}$$

where $p = (p_1, ..., p_n)$, $p_k \in Z_+$, $\forall k = \overline{1, n}$, $a_p(\cdot) \in L_\infty(\Omega)$ are real functions, i.e. the characteristic form

$$Q(x, \xi) = \sum_{|p|=m} a_p(x) \xi^p,$$

is definite a.e. for $x \in \Omega$.

In the sequel we will assume that the ellipticity condition holds. Consider the elliptic operator L_0:

$$L_0 = \sum_{|p|=m} a_p^0 \partial^p, \tag{3}$$

with the constant coefficients a_p^0.

In what follows, by solution of the equation $Lu = f$ we mean a strong solution (see [4]), i.e. u belongs to the corresponding space and satisfies a.e. the equality $Lu = f$.

Let's consider the elliptic operator (2) and assign a "tangential operator"

$$L_{x_0} = \sum_{|p|=m} a_p(x_0) \partial^p. \tag{4}$$

to it at every point $x_0 \in \Omega$. Denote by $J_{x_0}(\cdot)$ the fundamental solution of the equation $L_{x_0}\varphi = 0$ in accordance with the results of monograph [4] (see Chapter V, p. 221). The function $J_{x_0}(\cdot)$ is called a parametrics for the equation $L\varphi = 0$ with a singularity at the point x_0. Let

$$S_{x_0}\varphi = \psi(x) = \int J_{x_0}(x - y) \varphi(y) dy,$$

and

$$T_{x_0} = S_{x_0}(L_{x_0} - L). \tag{5}$$

In establishing the existence of the solution of the equation $Lu = f$, the significant role is played by the following

Lemma 1. *[4] Let $\varphi \in W_p^2(\Omega)$, $1 \le p < +infty$ and φ have a compact support in Ω. Then*

$$\varphi = T_{x_0}\varphi + S_{x_0}L\varphi,$$

and if

$$\varphi = T_{x_0}\varphi + S_{x_0}f,$$

then $L\varphi = f$.

We will use the following concept.

Definition 1. *We will say that the operator L has the property P_{x_0}) if its coefficients satisfy the conditions: i) $a_p \in L_\infty(B_r(x_0))$, $\forall |p| \leq m$, for some $r > 0$; ii) $\exists r > 0$: for $|p| = m$ the coefficient $a_p(\cdot)$ coincides a.e. in $B_r(x_0)$ with some function bounded and continuous at the point x_0.*

In substantiating many of the subsequent arguments, the following well-known fact plays an exceptional role (see, for instance, [5, p. 39]).

Theorem 1. *Let the functions u_k, $k \in N$ have generalized derivatives of the same form $\vartheta_k = \partial^p u_k$ in a finite domain $\Omega \subset R^n$. If both sequences $\{u_k\}$ and $\{\vartheta_k\}$ converge in the metric $L_1(\Omega)$ to the limits $u(x)$ and $\vartheta(x)$, respectively, then in the domain Ω the function $\vartheta(x)$ is a generalized derivative of $u(x)$ of the same form, i.e. $\vartheta = \partial^p u$.*

2.3. Morrey, Morrey-Sobolev spaces

Let us define the spaces under consideration. Let $\Omega \subset R^n$ be some domain. Morrey space $L_{q,\lambda}(\Omega)$, $1 \leq q < +\infty$, $0 < \lambda < n$, is a Banach space of (Lebesgue) measurable functions on Ω with the norm

$$\|f\|_{L_{q,\lambda}(\Omega)} = \sup_{r>0} \left(\frac{1}{r^\lambda} \int_{\Omega_r} |f|^q \, dx \right)^{\frac{1}{q}},$$

where sup is taken over all balls $B_r \subset R^n$. Similarly we define the Morrey-Sobolev space $W^m_{q,\lambda}(\Omega)$ with the norm

$$\|f\|_{W^m_{q,\lambda}(\Omega)} = \sum_{|p| \leq m} \|\partial^p f\|_{L_{q,\lambda}(\Omega)} .$$

We will also consider the space $N^m_{q,\lambda}(\Omega)$ with the norm

$$\|f\|_{N^m_{q,\lambda}(\Omega)} = \sum_{|p| \leq m} d_\Omega^{|p| - \frac{n}{q}} \|\partial^p f\|_{L_{q,\lambda}(\Omega)} ,$$

where d_Ω is a diameter of the domain Ω. It is quite obvious that the norms of the spaces $W^m_{q,\lambda}(\Omega)$ and $N^m_{q,\lambda}(\Omega)$ are equivalent and therefore the stocks of functions of these spaces are the same.

3. Extension Theorem

Consider the question of the extendibility of the functions from $W^k_{p,\lambda}(\Omega)$ outside Ω for smooth domains. First, consider the case, where Ω is a parallelepiped

$K_a^+ = K_a \bigcap \{ y_n > 0\}$ and $\Omega' = K_a = \left\{ |y_i| < a\,,\ i = \overline{1, n} \right\}$ is a cube with edge $2a > 0$. So, let $f \in W^k_{q,\lambda}\,(K_a^+)$, $1 \le q < +\infty$, $0 < \lambda < n$. Define

$$F\,(y) = \begin{cases} f\,(y)\,, & y \in K_a^+\,, \\ \sum_{i=1}^{k+1} A_i f\left(y';\, -\frac{y_n}{i}\right)\,, & y \in K_a^-\,, \end{cases}$$

where $y' = (y_1, ..., y_{n-1}\,)$, $K_a^- = K_a \bigcap \{ y_n < 0\}$ and $\{A_1, ..., A_{k+1}\}$ is the solution of the linear algebraic system

$$\sum_{i=1}^{k+1} \left(-\frac{1}{i}\right)^s A_i = 1\,, \quad s = \overline{0, k}.$$

It is clear that $f \in W^k_q\,(K_a^+)$. Then, according to the facts from the classical theory, we obtain $F \in W^k_q\,(K_a)$, and for $\forall p \in Z^n_+ \,:\ \ |p| \le k$, the derivative $\partial^p F$ for $y \in K_a^-$ is calculated by the formula

$$\partial^p F\,(y) = \sum_{i=1}^{k+1} A_i \left(-\frac{1}{i}\right)^{p_n} \partial^p f\,\left(y';\, -\frac{y_n}{i}\right)\,,$$

where $p = (p_1, ..., p_n)$. Applying Hölder's inequality, we have

$$|\,\partial^p F\,(y)|^q \le K^{\frac{q}{q'}} \max_{1 \le i \le k+1} |A_i|^q \sum_{i=1}^{k+1} \left| \partial^p f\,\left(y';\, -\frac{y_n}{i}\right) \right|^q,$$

where $K = \sum_{i=1}^{k+1} i$. Consequently

$$|\,\partial^p F\,(y)\,|^q = c \sum_{i=1}^{k+1} \left|\, \partial^p f\,\left(y';\, -\frac{y_n}{i}\right) \right|^{\,q}, \tag{6}$$

where the constant $c > 0$ is independent of f. Let $B_r \subset R^n$ be an arbitrary ball of the radius $r > 0$. For any set $M \subset R^n$, assume $M_r = M \bigcap B_r$. Integrating (6) with respect to $y \in (K_a^-)_r$ we have

$$\int_{(K_a^-)_r} |\,\partial^p F\,(y)|^q\, dy \le c \sum_{i=1}^{k+1} \int_{(K_a^-)_r} \left| \partial^p f\,\left(y';\, -\frac{y_n}{i}\right) \right|^q dy =$$

$$= c \sum_{i=1}^{k+1} i \int_{(K_a^+)_r{}' \bigcap \{y_n < \frac{a}{i}\}} |\,\partial^p f\,\,(y)\,|^q\, dy \le ck \int_{(K_a^+)_r} |\partial^p f\,(y)|^q\, dy,$$

where ${(K_a^+)_r}' = K_a^+ \bigcap {B_r}'$, ${B_r}'$ is the image of the ball ${B_r}'$ under the mapping $(y'; y_n) \to (y'; -y_n)$. It is quite obvious that ${B_r}'$ is also a ball of radius r. Thus,

$$\frac{1}{r^\lambda} \int_{(K_a^-)_r} |\partial^p F(y)|^q \, dy \le c \frac{1}{r^\lambda} \int_{(K_a^+)_r} |\partial^p f(y)|^q \, dy \le c \, \|\partial^p f\|_{L_{q,\lambda}(K_a^+)} ,$$

where the constant $c > 0$ is independent of r and f. Hence it directly follows that

$$\| \partial^p F\|_{L_{q,\lambda}(K_a)} \le c \, \| \partial^p f\|_{L_{q,\lambda}(K_a^+)} ,$$

where the constant $c > 0$ is independent of $f \in W^k_{q,\lambda}(K_a^+)$. So, the following lemma is true.

Lemma 2. *For* $\forall f \in W^k_{q,\lambda}(K_a^+)$, $1 \le q < +\infty$, $0 < \lambda < n$, *there exists its extension* $F \in W^k_{q,\lambda}(K_a)$, *which satisfies the estimate*

$$\| F\|_{W^k_{q,\lambda}(K_a)} \le c \, \| f\|_{W^k_{q,\lambda}(K_a^+)} ,$$

where the constant $c > 0$ *is independent of* f.

Introduce the following

Definition 2. *Let* $\Omega_x; \Omega_y \subset R^n$ *be a bounded domain and* $\Psi : \Omega_x \to \Omega_y$ *be some mapping. We say that a mapping* Ψ *has property* $\mathscr{B})$, *if there exist constants* $c_1; c_2 > 0$ *such that for* $\forall B_r : r > 0$, *there exist balls* ${B_r}'; {B_r}'' \subset R^n$ *of radius* r *for which*

$$\Omega_{yc_1r} \subset \Psi(\Omega_{xr}) \subset \Omega_{yc_2r},$$

holds, where $\Omega_{xr} = \Omega_x \bigcap B_r$; $\Omega_{yc_kr} = \Omega_y \bigcap B'_{c_kr}$; $k = 1, 2$.

Denote by $\Phi(y) = (\varphi_1(y); ...; \varphi_n(y))$ and $\Psi(x) = (\psi_1(x); ...; \psi_n(x))$ the vector functions. In the sequel, we will essentially use the following lemmas.

Lemma 3. *Let* $\Omega_x; \Omega_y \subset R^n$ *be bounded domains,* $y = \Psi : \Omega_x \to \Omega_y$ *be one-to-one mapping from the domain* Ω_x *into* Ω_y, $x = \Phi : \Omega_y \to \Omega_x$ *be an inverse mapping and* $\Psi \in C^{(1)}\left(\overline{\Omega}_x\right)$; $\Phi \in C^{(1)}\left(\overline{\Omega}_y\right)$. *Then the mappings* Ψ *and* Φ *have property* $\mathscr{B})$.

Proof. Let $B_r \subset R^n$ be an arbitrary ball and $\Omega_{xr} = \Omega_x \bigcap B_r$. Take $\forall x_1; x_2 \in \Omega_{xr}$. Since $\Psi \in C^{(1)}\left(\overline{\Omega}_x\right)$, it is well known that (see e.g., [2, p. 48]) $\exists \, c > 0$ (obviously the constant c is independent of $r > 0$), for which the inequality

$$|\Psi(x_1) - \Psi(x_2)| \le c \, |x_1 - x_2| < cr, \forall \, x_1; x_2 \in \Omega_{xr},$$

holds. This immediately implies that there exists an absolute constant $c_1 > 0$ such that for $\forall\, r > 0$ there is a ball $B_r \subset R^n$ with

$$\Psi\left(\Omega_{xr}\right) \subset c_1 B_r \subset \Omega_{yc_1 r}.$$

It is clear that the same is true for the inverse mapping Φ:

$$\Phi\left(\Omega_{yr}\right) \subset \Omega_{xc_2 r}, \forall\, r > 0. \tag{7}$$

From (7) it directly follows

$$\Omega_{yr} \subset \Psi\,\left(\Omega_{xc_2 r}\right), \forall\, r > 0.$$

Replacing $c_2\, r$ with r again we obtain

$$\Omega_{ycr} \subset \Psi\,\left(\Omega_{xr}\right), \forall\, r > 0,$$

where $c > 0$ is an absolute constant.

Lemma is proved. ◄

Using this lemma, we establish the validity of the following

Lemma 4. *Let Ω_x; $\Omega_y \subset R^n$ be bounded domains, $y = \Psi : \Omega_x \to \Omega_y$ be one-to-one mapping from the domain Ω_x into Ω_y, $x = \Phi : \Omega_y \to \Omega_x$ be an inverse mapping and $\Psi \in C^{(k)}\left(\overline{\Omega}_x\right)$;$\Phi \in C^{(k)}\left(\overline{\Omega}_y\right)$. Then from $\forall\, f \in W^k_{q,\lambda}\left(\Omega_y\right)$, $1 \le q < +\infty$, $0 < \lambda < 1$, it follows $g\left(x\right) = f\left(\Psi\left(x\right)\right) \in W^k_{q,\lambda}\left(\Omega_x\right)$ and the linear mapping $f\left(y\right) \to g\left(x\right)$ is continuous. The same is true for the function $f\left(y\right) = g\left(\Phi\left(y\right)\right)$, $g \in W^k_{q,\lambda}\left(\Omega_x\right)$.*

Proof. It suffices to carry out the proof for the case $k = 1$ and let $g\left(x\right) = f\left(\Psi\left(x\right)\right)$, $x \in \Omega_x$, $f \in W^1_{q,\lambda}\left(\Omega_y\right)$. It is clear that $f \in W^1_q\left(\Omega_y\right)$ and then, as is known (see, e.g., [1, p. 188]):

$$\frac{\partial\, g\left(x\right)}{\partial x_i} = \frac{\partial \psi_1}{\partial x_i}\frac{\partial f}{\partial y_1} + ... + \frac{\partial \psi_n}{\partial x_i}\frac{\partial f}{\partial y_n}, i = \overline{1, n}, \text{ a.e. } x \in \Omega_x.$$

This immediately implies

$$\left|\frac{\partial\, g\left(x\right)}{\partial x_i}\right| \le c \sum_{j=1}^{n} \left| \frac{\partial f}{\partial y_j} \Big/_{y_j = \psi_j\left(x\right)} \right|, i = \overline{1, n}, \text{ a.e. } x \in \Omega_x,$$

where $c > 0$ is a constant independent of f. Let $B_r \subset R^n$ be an arbitrary ball of radius $r > 0$. We have

$$I(r) = \frac{1}{r^\lambda} \int_{\Omega_{xr}} \left| \frac{\partial g(x)}{\partial x_i} \right|^q dx \le \sum_{j=1}^{n} \frac{1}{r^\lambda} \int_{\Omega_{xr}} \left| \frac{\partial f}{\partial y_j} \Big/ _{y_j = \psi_j(x)} \right|^q dx =$$

$$= / y = \Psi(x)\,; x = \Phi(y) / =$$

$$= \sum_{j=1}^{n} \frac{1}{r^\lambda} \int_{\Psi(\Omega_{xr})} \left| \frac{\partial f}{\partial y_j} \right|^q |J_\Phi(y)|\, dy \le c \sum_{j=1}^{n} \frac{1}{r^\lambda} \int_{\Psi(\Omega_{xr})} \left| \frac{\partial f}{\partial y_j} \right|^q dy.$$

By Lemma 3, the mappings Ψ and Φ have property $\mathscr{B}$) and, as a result, for $\forall\, r > 0$, there exists a ball $B_r \subset R^n$ such that $\Psi(\Omega_{xr}) \subset \Omega_{yc_1r}$. Thus

$$I(r) \le c \sum_{j=1}^{n} \frac{1}{r^\lambda} \int_{\Omega_{yc_1r}} \left| f'_{y_j} \right|^q dy = \frac{c}{c_1^\lambda} \sum_{j=1}^{n} \frac{1}{r^\lambda} \int_{\Omega_{yr}} \left| f'_{y_j} \right|^q dy \le c \sum_{|p|=1} \|\partial^p f\|_{L_{q,\lambda}(\Omega_y)}.$$

Consequently

$$\sum_{|p|=1} \|\partial^p g\|_{L_{q,\lambda}(\Omega_x)} \le c \sum_{|p|=1} \|\partial^p f\|_{L_{q,\lambda}(\Omega_y)}.$$

In a similar way the estimate

$$\|g\|_{L_{q,\lambda}(\Omega_x)} \le c\, \|f\|_{L_{q,\lambda}(\Omega_y)},$$

is established, where the constant $c > 0$ is independent of f. As a result, we obtain $g \in W^1_{q,\lambda}(\Omega_x)$ and the estimate

$$\|g\|_{W^1_{q,\lambda}(\Omega_x)} \le c\, \|f\|_{W^1_{q,\lambda}(\Omega_y)},$$

is true. The converse of the lemma is established in exactly the same way.

Lemma is proved. ◀

Following the monograph [3, p. 129], we also prove the following

Lemma 5. *Let $\Omega \subset R^n$ be a bounded domain and for $\forall\, \xi \in \partial\Omega$ there exist a ball $B_r(\xi)$ of some radius $r = r(\xi) > 0$ such that for $\forall f \in W^m_{q,\lambda}(\Omega)$, $\exists F_\xi(\cdot)$: $F_\xi/_{\Omega_r(\xi)} = f$ $(\Omega_r(\xi) = \Omega \bigcap B_r(\xi))$ and $F_\xi \in W^m_{q,\lambda}(B_r(\xi))$. Moreover, let the inequality*

$$\|F_\xi\|_{W^m_{q,\lambda}(B_r(\xi))} \le c\, \|f\|_{W^m_{q,\lambda}(\Omega)},$$

be fulfilled, where the constant $c > 0$ is independent of f. Then, for any $\rho > 0$, there exists an extension $F(\cdot)$ of the function f to the domain $\Omega_\rho = \bigcup_{x\in\Omega} B_\rho(x)$

with the following properties: i) $F \in W^m_{q,\lambda}(\Omega_\rho)$ *and* $F(x) = 0, \ \forall x \in \Omega_\rho \backslash \Omega_{\rho/2}$*; ii) there exists a constant* $c > 0$ *independent of* f *(*c *may depend on* Ω *and* ρ*) such that*

$$\|F\|_{W^m_{q,\lambda}(\Omega_\rho)} \leq c \|f\|_{W^m_{q,\lambda}(\Omega)} . \tag{8}$$

Proof. Based on the conditions of the lemma, we assume that for $\forall\, \xi \in \overline{\Omega}$ there exists a ball $B_r(\xi) : \ F_\xi \in W^m_{q,\lambda}(B_r(\xi))$ and $F_\xi /_{\Omega_r(\xi)} = f$. Without loss of generality we can assume that $r = r(\xi) < \rho$. It is obvious that $\overline{\Omega} \subset \bigcup_{\xi \in \overline{\Omega}} B_{r/3}(\xi)$. Then it is clear that there exist balls $B_{r_k/3}(x_k), \ k = \overline{1, N} : \overline{\Omega} \subset \bigcup_{k=\overline{1,N}} B_{r_k/3}(x_k)$. Consider the function $\theta_k \in C^\infty(R^n)$, $k = \overline{1, N}$:

$$\theta_k(x) = \begin{cases} 1, \ x \in B_{r_k/3}(x_k) , \\ 0, \quad x \notin B_{r_k/2}(x_k) . \end{cases}$$

Set $\sigma_k(x) = 1 - \theta_k(x)$, $k = \overline{1, N}$, and let

$$\gamma_1(x) = \sigma_1(x) , \ \ \gamma_k(x) = \sigma_1(x) \cdot ... \cdot \sigma_{k-1}(x)\, \theta_k(x) , k = \overline{2, N}.$$

Define the function $f_k(\cdot) , \ k = \overline{1, N}$, in the following way:

$$f_k(x) = \begin{cases} F_{x_k}(x) , \ x \in B_{r_k}(x_k) , \\ f(x) , \ x \in \Omega , \\ 0, \quad x \notin \Omega . \end{cases} \tag{9}$$

As is shown in [3, p.130],

$$\gamma_1(x) + ... + \gamma_i(x) = 1, \forall x \in \bigcup_{j \leq i} S_{r_j/3}(x_j) . \tag{10}$$

Assume

$$F(x) = \sum_{k=1}^{N} \gamma_k(x) f_k(x) .$$

From (10) it follows that $F/_Q = f$. Note that $\gamma_k(x) = 0 , \ x \in \bigcup_{i<k} B_{r_i/3}(x_i)$ and $\gamma_k(x) = 0$, for $x \notin B_{r_k/2}(x_k)$, $\forall\, k = \overline{1, N}$. Then it is obvious that $\gamma_k f_k \in W^m_{q,\lambda}(\Omega_\rho) , \forall k = \overline{1, N}$; and as a result, $F \in W^m_{q,\lambda}(\Omega_\rho)$. Without loss of generality we can assume that $0 \leq \gamma_k \leq 1$, for $\forall\, k = \overline{1, N}$. We have

$$\|F\|_{W^m_{q,\lambda}(\Omega_\rho)} \leq \sum_{k=1}^{N} \|f_k\|_{W^m_{q,\lambda}(\Omega_\rho)} .$$

From expression (9) for f_k and from the conditions of the lemma we directly obtain

$$\|f_k\|_{W^m_{q,\lambda}(\Omega_\rho)} \leq c_k \|f\|_{W^m_{q,\lambda}(\Omega)} ,$$

where the constant $c_k > 0$ is independent of f. Hence we get the validity of the estimate (8).

Lemma is proved. ◄

With Lemmas 2-5 established, completely similar to the proof of the extension theorem in the monograph [3, p. 130], the following extension theorem for the functions from Morrey-Sobolev spaces can be proved.

Theorem 2. ***(extension)*** *Let $\Omega \subset\subset \Omega' : \Omega ; \Omega' \subset R^n$ be bounded domains and $\partial\Omega \in C^{(m)}$. Then for $\forall f \in W^m_{q,\lambda}(\Omega)$, $1 \leq q < +\infty$, $0 < \lambda < n$, there exists a compactly supported function $F \in W^m_{q,\lambda}(\Omega')$ in Ω' (i.e., there exists a compact $K \subset \Omega' : F = 0$ outside K) : $F/_\Omega = f$ and*

$$\|F\|_{W^m_{q,\lambda}(\Omega')} \leq c \|f\|_{W^m_{q,\lambda}(\Omega)} ,$$

where the constant $c > 0$ is independent of f.

4. The concept of trace. The trace space

Let us define the concept of a trace for functions from Morrey-Sobolev spaces $W^m_{q,\lambda}(\Omega)$. In this case, the concept of a trace cannot be defined following the classical scheme. Therefore, we will apply a different scheme to define this concept. It is sufficient to define it for the functions from $W^1_{q,\lambda}(\Omega)$, $1 \leq q < +\infty$, $0 < \lambda < n$. Let $\Omega \subset R^n$ be a bounded domain with the boundary $\partial\Omega \in C^{(1)}$. Let $S \subset \overline{\Omega} : S \in C^{(1)}$ be some $(n-1)$ dimensional surface and $f \in W^1_{q,\lambda}(\Omega)$ be an arbitrary function. Consequently, $f \in W^1_q(\Omega)$ and it is well known that f has a trace f_S and moreover, the following inequality holds:

$$\|f_S\|_{L_q(S)} \leq c \|f\|_{W^1_q(\Omega)} , \tag{11}$$

where the constant $c > 0$ is independent of f. Denote $f_S = \Gamma_S f$, $\forall f \in W^1_{q,\lambda}(\Omega)$. f_S is called the trace of the function $f \in W^1_{q,\lambda}(\Omega)$ on S. The linear space $\Gamma_S\left(W^1_{q,\lambda}(\Omega)\right)$ is denoted by $W^1_{q,\lambda}(S)$. It is obvious that $W^1_{q,\lambda}(S) \subset L_q(S)$. We also define

$$\overset{0}{W}{}^1_{q,\lambda}(\Omega; S) = \{f \in W^1_{q,\lambda}(\Omega) : f_S = 0 \;\; a.e. \; on \; S$$
$$with \; respect \; to \; the \; measure \; d\sigma\} .$$

We can easily prove the following

Lemma 6. *Let $\Omega \subset R^n$ be a bounded domain and $S \subset \bar{\Omega}$ be $(n-1)$-dimensional surface of class $C^{(1)}$. Then $\overset{0}{W}{}^1_{q,\lambda}(\Omega;\, S)$ is a subspace in $W^1_{q,\lambda}(\Omega)$.*

Proof. Let $\{f_k\} \subset \overset{0}{W}{}^1_{q,\lambda}(\Omega;\, S)$ be some Cauchy sequence, i.e.

$$\|f_{k_1} - f_{k_2}\|_{W^1_{q,\lambda}(\Omega)} \to 0\,, \ \ k_1; k_2 \to \infty.$$

It is clear that

$$\exists\, f \in W^1_{q,\lambda}(\Omega) : \ \|f - f_k\|_{W^1_{q,\lambda}(\Omega)} \to 0\,, \ k \to \infty.$$

Let us show that $f_S = \Gamma_S f = 0$ a.e. on S with respect to the measure $d\sigma$. It is obvious that $f_{kS} = 0\,, \ \forall k \in N$, and $(f - f_k)_S = f_S$ (by definition). Consequently

$$\|f_S\|_{L_q(S)} = \|(f - f_k)_S\|_{L_q(S)} \le c\, \|f - f_k\|_{W^1_q(\Omega)} \le$$

$$\le c\, \|f - f_k\|_{W^1_{q,\lambda}(\Omega)} \to 0\,, \ \ k \to \infty.$$

This immediately implies that $f_S = 0$ a.e. on S (with respect to the measure $d\sigma$). As a result, $f \in \overset{0}{W}{}^1_{q,\lambda}(\Omega;\, S)$ and the closedness of $\overset{0}{W}{}^1_{q,\lambda}(\Omega;\, S)$ is proved.

Lemma is proved. ◀

Let us denote by $\mathscr{F}^1_{q,\lambda}(S)$ the quotient (factor) space

$$\mathscr{F}^1_{q,\lambda}(S) = W^1_{q,\lambda}(\Omega) \Big/ {\overset{0}{W}{}^1_{q,\lambda}(\Omega; S)}\,,$$

with the corresponding factor norm

$$\|F\|_{\mathscr{F}^1_{q,\lambda}(S)} = \inf_{f \in F} \|f\|_{W^1_{q,\lambda}(\Omega)}\,.$$

The following lemma is true.

Lemma 7. *Let $\Omega \subset R^n$ be a bounded domain with boundary $\partial\,\Omega \in C^{(1)}$ and $S \subset \overline{\Omega}:\ S \in C^{(1)}$ be some $(n-1)$-dimensional surface. Then the (linear) spaces $\mathscr{F}^1_{q,\lambda}(S)$ and $W^1_{q,\lambda}(S)$ are isomorphic.*

Proof. Take an arbitrary factor class $F \in \mathscr{F}^1_{q,\lambda}(S)$ and let $\forall\, f_1;\ f_2 \in F$. Put $f = f_1 - f_2$. It is evident that $f \in \overset{0}{W}{}^1_{q,\lambda}(\Omega;\, S)$ and consequently, $f_S = 0 \ \Rightarrow f_{1S} = f_{2S}$. Thus, f_S for $\forall\, f \in F$, represents the unique function $F_S \in W^1_{q,\lambda}(S)$.

Consider the mapping $\Gamma_S : \mathscr{F}^1_{q,\lambda}(S) \to W^1_{q,\lambda}(S)$; $\Gamma_S F = F_S$, $\forall F \in \mathscr{F}^1_{q,\lambda}(S)$. It is clear that Γ_S is a linear mapping. Let us show that $Ker\,\Gamma_S = 0$. Let $F \in Ker\,\Gamma_S \Rightarrow F_S = 0$. This immediately implies that the class F contains a zero function in Ω, and thus F is a zero element of the space $\mathscr{F}^1_{q,\lambda}(S) \Rightarrow Ker\,\Gamma_S = 0$. Let us show that $R_{\Gamma_S} = W^1_{q,\lambda}(S)$. Let $f \in W^1_{q,\lambda}(S)$ be an arbitrary function. Then it is clear that $\exists\, u \in W^1_{q,\lambda}(\Omega) : u_S = f$. Let $F_f \in \mathscr{F}^1_{q,\lambda}(S)$ be a class containing u. It is obvious that $\Gamma_S F_f = f$.

Lemma is proved. ◄

The operator $\Gamma_S : \mathscr{F}^1_{q,\lambda}(S) \to W^1_{q,\lambda}(S)$ *defined in Lemma 7 is called the trace operator on* S.

Based on Lemma 7, we define the norm in $W^1_{q,\lambda}(S)$ by the following expression

$$\|f\|_{W^1_{q,\lambda}(S)} = \left\|\Gamma_S^{-1} f\right\|_{\mathscr{F}^1_{q,\lambda}(S)}, \forall\, f \in W^1_{q,\lambda}(S)\,. \tag{12}$$

Since $\mathscr{F}^1_{q,\lambda}(S)$ is a Banach space with respect to the factor norm $\|F\|_{\mathscr{F}^1_{q,\lambda}(S)}$, it immediately follows from this lemma that $W^1_{q,\lambda}(S)$ is also a Banach space with respect to the norm (12).

The space $W^m_{q,\lambda}(S)$ is defined similarly for $m \geq 2$ and the corresponding lemma is true with respect to the space $\mathscr{F}^m_{q,\lambda}(S)$, where

$$W^m_{q,\lambda}(S) = W^m_{q,\lambda}(\Omega)\,/_S = \Gamma_S\left(W^m_{q,\lambda}(\Omega)\right) =$$

$$= \left\{f \in L_q(S) : \exists\, u \in W^m_{q,\lambda}(S) \Rightarrow f = \Gamma_S u\right\}$$

is the image of the space $W^m_{q,\lambda}(S)$ under the mapping Γ_S, and the space $\mathscr{F}^m_{q,\lambda}(S)$ is defined in a similar way, i.e. it is a factor space

$$\mathscr{F}^m_{q,\lambda}(S) = W^m_{q,\lambda}(\Omega) \Big/_{\overset{0}{W}{}^m_{q,\lambda}(\Omega;S)},$$

where

$$\overset{0}{W}{}^m_{q,\lambda}(\Omega;S) = \left\{f \in W^m_{q,\lambda}(\Omega) : f_S = \Gamma_S f = 0 \;\; a.e. \;\, on \;\, S \right.$$

$$\left. with \;\, respect \;\, to \;\, the \;\, measure \;\, d\sigma\right\}.$$

These results allow us to prove the following

Main Lemma. *Let* L *be an* m*-th order elliptic operator with the coefficients* $a_p \in L_\infty(\Omega)$, $\forall p : |p| \leq m$, *and* L_{x_0} *be the corresponding tangential operator at the point* $x_0 \in \Omega$, *in which the ellipticity condition is satisfied and* $|a_p(x_0)| \leq$

$\|a_p\|_{L_\infty(\Omega)}$, $\forall p : |p| \le m$. *Let* $J_{x_0}(\cdot)$ *be the fundamental solution of the equation* $L_{x_0} = 0$ *from Theorem 2.1 [4] and* $S_{x_0}\varphi = J_{x_0} * \varphi$; $T_{x_0} = S_{x_0}(L_{x_0} - L)$. *If*

$$\varphi \in W^m_{q,\lambda}(B_{x_0}(r)) \ (B_{x_0}(r) \subset \Omega, \ r > 0), 1 < q < +\infty, \ 0 < \lambda < n,$$

then $\varphi = T_{x_0}\varphi + S_{x_0}L\varphi$. *Moreover, if for some* $f \in L_{q,\lambda}(B_{x_0}(r))$ *the equality* $\varphi = T_{x_0}\varphi + S_{x_0}f$ *holds a.e. on* $B_{x_0}(r)$, *then* φ *is a strong solution of the equation* $L\varphi = f$ *in* $W^m_{q,\lambda}(B_{x_0}(r))$.

Proof of Main Lemma. Without loss of generality, we can assume that $x_0 = 0 \in \Omega$. It is easy to see that the operator L acts boundedly from $W^m_{q,\lambda}(r)$ to $L_{q,\lambda}(r)$, i.e. $L \in \left[W^m_{q,\lambda}(r) \ ; \ L_{q,\lambda}(r)\right]$. Take $\forall\varphi \in W^m_{q,\lambda}(r)$. It is clear that $\varphi \in W^m_q(r)$. Following Theorem 2 (on extension), we will assume that $\varphi \in \overset{0}{W}{}^m_{q,\lambda}(r+\delta) \subset \overset{0}{W}{}^m_q(r+\delta)$, for some $\delta > 0$. Thus, φ has a compact support in R^n. Then, following Lemma A from the monograph [4, p. 225] we get the relation $\varphi = T_0\varphi + S_0L\varphi$ in $W^m_q(r)$, i.e. $I = T_0 + S_0L$ in $W^m_q(r)$. Assume $\psi = (L_0 - L)\varphi$ and let $\chi(x) = (T_0\varphi)(x)$. We have

$$\chi(x) = T_0\varphi = S_0(L_0 - L)\ \varphi = S_0\psi = \int_{B(r+\delta)} J_0(x-y)\ \psi(y)\,dy, x \in B(r+\delta).$$

It is obvious that $\psi \in L_q(r+\delta)$. For smooth functions ψ the differentiation formula holds for $|p| = m$:

$$\partial^p\chi(x) = \int_{B(r+\delta)} \partial^p_x J_0(x-y)\ \psi(y)\,dy + const\psi(x), \tag{13}$$

where *const* is independent of ψ. For this formula we refer the reader to [4, p. 235] (formula (5.26)). $\partial^p_x J_0(x-y)$ is a singular kernel, and the boundedness of the singular operator in $L_q(r+\delta)$ for $1 < q < +\infty$ implies the validity of formula (13) (a.e. $x \in B\ (r+\delta)$) $\forall\psi \in L_q(r+\delta)$. Indeed, let $\{\psi_k\}$ be sufficiently smooth functions converging to $\psi \in L_q(r+\delta)$ in $L_q(r+\delta)$. Put

$$\chi_k(x) = \int_{B(r+\delta)} J_0(x-y)\ \psi_k(y)\,dy, x \in B(r+\delta).$$

It is clear that

$$\partial^p\chi_k(x) = \int_{B(r+\delta)} J^p_x(x-y)\ \psi_k(y)\,dy + const\ \psi_k(x)\ ,\ x \in B(r).$$

It immediately follows from the boundedness of the singular operator in $L_q(r+\delta)$ that

$$\int_{B(r+\delta)} J_x^p (x-y)\ \psi_k (y)\, dy \to \int_{B(r+\delta)} J_x^p (x-y)\ \psi (y)\, dy, \text{ in } L_q (r+\delta).$$

So, $L_{q,\lambda}(r+\delta) \subset L_q$, it is clear that the formula (13) is also true for $\forall\, \psi \in L_{q,\lambda}(r+\delta)$. From this formula it follows that

$$\|\partial^p \chi\|_{L_{q,\lambda}(r)} \le \|\partial^p \chi\|_{L_{q,\lambda}(r+\delta)} \le c\,\|\psi\|_{L_{q,\lambda}(r+\delta)} \le$$

$$\le c\left(\|L_0\varphi\|_{L_{q,\lambda}(r+\delta)} + \|L\varphi\|_{L_{q,\lambda}(r+\delta)}\right),$$

where the constant $c > 0$ is independent of φ. Hence we directly obtain the following estimate

$$\|\partial^p \chi\|_{L_{q,\lambda}(r)} \le c\ \|\varphi\|_{W^m_{q,\lambda}(r+\delta)}\ ,\quad \forall p:\ |p| = m.$$

Paying attention to Theorem 2 (extension), we finally have the estimate:

$$\|\partial^p \chi\|_{L_{q,\lambda}(r)} \le c\ \|\varphi\|_{W^m_{q,\lambda}(r)}\ ,\quad \forall p:\ |p| = m\,.$$

Since for $|p| < m$ the derivatives $\partial^p J_0$ are not singular kernels, completely similar to the classical case (i.e., the case $W^m_q(\Omega)$), the validity of the estimate

$$\|\partial^p \chi\|_{L_{q,\lambda}(r)} \le c\ \|\varphi\|_{W^m_{q,\lambda}(r)}\ ,\quad \forall p:\ |p| < m,$$

is established. From the obtained estimates it follows

$$\|T_0\varphi\|_{W^m_{q,\lambda}(r)} = \ \|\chi\|_{W^m_{q,\lambda}(r)}\ \le c\ \|\varphi\|_{W^m_{q,\lambda}(r)}\,,$$

where the constant $c > 0$ is independent of φ. Consequently, $T_0 \in \left[W^m_{q,\lambda}(r)\right]$. In exactly the same way, we establish $S_0 \in \left[L_{q,\lambda}(r)\ ;\ W^m_{q,\lambda}(r)\right]$ and as a result, we have $S_0 L \in \left[W^m_{q,\lambda}(r)\right]$. Since the relation $\varphi = T_0\varphi + S_0 L\varphi$ holds in $W^m_q(r)$, it is clear that it also holds in $W^m_{q,\lambda}(r)$, i.e. $I = T_0 + S_0 L$ in $W^m_{q,\lambda}(r)$. The first part of the lemma is proved.

Let for some $f \in L_{q,\lambda}(r)$ the relation $\varphi = T_0\varphi + S_0 f$ hold, where $\varphi \in W^m_{q,\lambda}(r)$. Taking into account $\varphi = T_0\varphi + S_0 L\varphi$, we have $S_0 L\varphi = S_0 f$. Consequently

$$L\varphi = L_0 S_0 L\varphi = L_0 S_0 \varphi.$$

Since $L_0 S_0 f = f$ a.e. in $B(r)$ (it follows from Theorem 2.1 [4] (on a fundamental solution)) and $f \in L_q(r)$, from $S_0 \in \left[L_{q,\lambda}(r)\ ;\ W^m_{q,\lambda}(r)\right]$ and $L_0 \in$

$\left[W^m_{q,\lambda}(r); L_{q,\lambda}(r)\right]$ it follows that $L_0 S_0 = I$ in $L_{q,\lambda}(r)$. As a result, we obtain $L\varphi = f$.

Lemma is proved. ◀

To establish interior estimates we need the following

Lemma 8. *Let the m-th order elliptic operator L have property P_{x_0}) at the point $x_0 \in \Omega$ and $\varphi \in \overset{0}{N}{}^m_{q,\lambda}(B_r(x_0))$, $r > 0$. Then the inequality*

$$\|T_{x_0}\varphi\|_{N^m_{q,\lambda}(B_r(x_0))} \leq \sigma(r)\ \|\varphi\|_{N^m_{q,\lambda}(B_r(x_0))}$$

holds, where the function $\sigma(r) \to 0$, $r \to 0$, depends only on ellipticity constant of the operator L_{x_0} and on the coefficients of the operator L.

Proof. Without loss of generality, we will assume that $x_0 = 0 \in \Omega$. By the value of the function $a_p(\cdot)$ at the point $x_0 = 0$ for $|p| = m$ we will mean the value of the corresponding function from the property P_{x_0}), continuous at this point. For simplicity, we assume that $m \geq 3$ is odd and $0 < r < 1$. Assume

$$\psi = (L_0 - L)\varphi = \psi_1 + \psi_2,$$

$$\psi_1(x) = \sum_{|p|=m} (a_p(0) - a_p(x))\ \partial^p\varphi(x) = \sum_{|p|=m} b_p(x)\ \partial^p\varphi(x),$$

where $b_p(x) = a_p(0) - a_p(x) \ \Rightarrow\ b_p(0) = 0 \ \Rightarrow\ \sup\limits_{|x|<r} |b_p(x)| = \bar{\bar{o}}(1)$, $r \to 0$;

$$\psi_2(x) = -\sum_{|p|<m} a_p(x)\ \partial^p\varphi(x).$$

It is quite obvious that

$$\|\psi_1\|_{L_{q,\lambda}(r)} \leq \bar{\bar{o}}(1) \sum_{|p|=m} \|\partial^p\varphi\|_{L_{q,\lambda}(r)}, \quad r \to 0.$$

Let $\chi = T_0\varphi$. Paying attention to the expression for T_0 we obtain

$$\chi = S_0(L_0 - L)\varphi = S_0\psi = \int_{B_r} J_0(x-y)\ \psi(y)\ dy.$$

For $|p| < m$ we have

$$\partial^p\chi(x) = \int_{B_r} \partial^p_x J_0(x-y)\ \psi(y)\ dy.$$

Taking into account the estimate

$$|\partial^p J_0(x)| \le c\,|x|^{m-n-|p|},$$

we have

$$|\,\partial^p \chi(x)| \le c \int_{B_r} |x-y|^{m-n-|p|}\,|\,\psi(y)|\;dy.$$

Assume

$$I(x) = \int_{B_r} |x-y|^{m-n-|p|}\,|\,\psi(y)|\;dy,$$

and let

$$f(x) = |x|^{m-n-|p|}\,\chi_{B_r}(x),$$

$$g(x) = \psi(x)\;\chi_{B_r(x)}\;,\quad \forall\, x \in R^n.$$

It is quite obvious that $\sup p\,(f*g) \subset B_{2r}$. Thus

$$I(x) = \int_{R^n} f(x-y)\;g(y)\,dy = \int_{R^n} f(y)\;g(x-y)\;dy.$$

Let $\tilde{B}_\rho \subset R^n$ be an arbitrary ball of the radius $\rho > 0$. The intersection $B_r \bigcap \tilde{B}_\rho$ is denoted by $B_{r;\rho}$. Applying the Minkowski's inequality for the integrals, we have

$$\left(\int_{B_{r;\rho}} |I(x)|^p\,dx\right)^{1/p} \le \left(\int_{B_{r;\rho}} \left(\int_{R^n} |f(y)|\;|g(x-y)|\,dy\right)^p dx\right)^{1/p} \le$$

$$\le \int_{R^n} \left(\int_{B_{r;\rho}} |f(y)^p|\;|g(x-y)|^p\,dx\right)^{1/p} dy =$$

$$= \int_{R^n} |f(y)| \left(\int_{B_{r;\rho}} |g(x-y)|^p\,dx\right)^{1/p} dy. \tag{14}$$

Suppose $B^y_{r;\rho} = \{x-y:\; x \in B_{r;\rho}\}$. It is quite obvious that $B^y_{r;\rho}$ is the intersection of some ball of radius ρ with the ball B_r. Then from (14) we get

$$\frac{1}{\rho^\lambda}\left(\int_{B_{r;\rho}} |I(x)|^p\,dx\right)^{1/p} \le \int_{R^n} |f(y)| \left(\frac{1}{\rho^\lambda}\int_{B^y_{r;\rho}} |g(x)|^p\,dx\right)^{1/p} dy \le$$

$$\le \|g\|_{L_{p,\lambda}(B_r)} \int_{R^n} |f(y)|\;dy.$$

Taking into account

$$\int_{R^n} f(y)\,dy = \int_{B_r} \frac{dx}{|x|^{n-m+|p|}} = cr^{m-|p|},$$

where the constant $c > 0$ is independent of r, we finally get

$$\|\partial^p \chi\|_{L_{q,\lambda}(r)} \leq cr^{m-|p|} \|\psi\|_{L_{q,\lambda}(r)},$$

where the constant $c > 0$ is independent of ψ. Consequently

$$r^{|p|} \|\partial^p \chi\|_{L_{q,\lambda}(r)} \leq cr^m \|\psi\|_{L_{q,\lambda}(r)} \ , \ \forall p : \ |p| < m.$$

Using these estimates, completely similar to [6], we establish

$$r^{|p|-\frac{n}{q}} \|\partial^p \chi\|_{L_{q,\lambda}(r)} \leq \overline{\overline{o}}\,(1) \ \|\psi\|_{N_{q,\lambda}(r)} \ , \ r \to 0, \ \ \forall p : \ |p| < m.$$

Estimate $\|\partial^p \chi\|_{L_{q,\lambda}(r)}$ for $|p| = m$. In this case $\partial^p J_0(\cdot)$ is a singular kernel

$$\partial^p \chi(x) = \int_{B_r} \partial^p J_0(x-y) \ \psi(y)\,dy + c\psi(x).$$

It directly follows that

$$\|\partial^p \chi\|_{L_{q,\lambda}(r)} \leq c \ \|\psi\|_{L_{q,\lambda}(r)}.$$

Consequently

$$r^{m-\frac{n}{q}} \|\partial^p \chi\|_{L_{q,\lambda}(r)} \leq c \left(\overline{\overline{o}} \ (1) \sum_{|p|=m} r^{m-\frac{n}{q}} \|\partial^p \varphi\|_{L_{q,\lambda}(r)} + \right.$$

$$\left. + r^{m-1} \sum_{|p|<m} r^{|p|-\frac{n}{q}} \|\partial^p \varphi\|_{L_{q,\lambda}(r)} \right) =$$

$$= \overline{\overline{o}} \ (1) \ \|\varphi\|_{N_{q,\lambda}(r)} \ , \quad r \to 0.$$

Combining these estimates, we finally get

$$\|\chi\|_{N_{q,\lambda}(r)} = \|T_0 \varphi\|_{N_{q,\lambda}(r)} \leq \overline{\overline{o}} \ (1) \ \|\varphi\|_{N_{q,\lambda}(r)} \ , \ r \to 0.$$

Lemma is proved. ◄

This lemma also makes it possible, quite similar to [6], to establish the validity of the following theorem on the existence of a strong solution in the small.

Theorem 3. *Let the m-th order elliptic operator L have the property* P_{x_0}) *at the point* $x_0 \in \Omega$ *and* $f \in L_{q,\lambda}(\Omega)$, $1 < q < +\infty$, $0 < \lambda < n$. *Then, for a sufficiently small* $r > 0$, *there exists a strong solution of the equation* $Lu = f$ *in* $N^m_{q,\lambda}(B_r(x_0))$.

5. On characterization of trace space and boundedness of trace operator

Let $\Omega \subset R^n$ be a bounded domain with the boundary $\partial\Omega \in C^{(1)}$ and $S \in C^{(1)} \bigcap \overline{\Omega}$ be an $(n-1)$ dimensional surface. As already established, each function $f \in W^1_{q,\lambda}(\Omega)$ has a trace f_S, moreover, $f_S \in L_q(S)$. Let us define the Morrey space $L_{q,\mu}(S)$, $1 \le q < +\infty$, $0 < \mu < n-1$, on the surface S:

$$L_{q,\mu}(S) = \left\{ g \in L_q(S) : \sup_{0<r<R_0} \left(\frac{1}{r^\mu} \int_{S_r} |f|^q \, d\sigma \right)^{\frac{1}{q}} < +\infty \right\},$$

where $S_r = S \bigcap B_r$ and $R_0 = diam\,\Omega$. Let us show that for $0 < \lambda < n$ the relation $f_S \in L_{q,\lambda-1}(S)$ holds, i.e. the following theorem is true.

Theorem 4. *Let $\Omega \subset R^n$ be a bounded domain with the boundary $\partial\Omega \in C^{(1)}$ and $S \in C^{(1)} \bigcap \overline{\Omega}$ be an $(n-1)$ dimensional surface. If $f \in W_{q,\lambda}(\Omega)$, $1 \le q < +\infty$, $0 < \lambda < n$, then its trace f_S belongs to the space $L_{q,\lambda-1}(S)$ and, moreover, the estimate*

$$\|f_S\|_{L_{q,\lambda-1}(S)} \le c\, \|f\|_{W^1_{q,\lambda}(\Omega)},$$

is valid, where the constant $c > 0$ is independent of f.

Proof. Let $S_0 \subset S$ be a simple piece projected uniquely onto some domain D in the plane $\{x_n = 0\}$ and

$$x_n = \varphi(x'), \; x' = (x_1, ..., x_{n-1}) \in D, \varphi(x') \in C^{(1)}(\bar{D}).$$

Without loss of generality, we assume that the domain Ω is located in the cube $K_a = \left\{ 0 < x_i < a : \forall i = \overline{1,n} \right\}$ for some $a > 0$. Based on Theorem 2 (extension), we assume that $f \in \overset{0}{W}{}^1_{q,\lambda}(K_a)$. Since $f \in W^1_q(\Omega)$, it is well known (see, for example, [3, p. 137]) that a.e. on S_0 the following formula holds

$$f/_{S_0} = f(x'; \varphi(x')) = \int_0^{\varphi(x')} \frac{\partial f(x'; \xi_n)}{\partial \xi_n} d\xi_n. \tag{15}$$

Put $S_{or} = S_0 \bigcap B_r$, where $r > 0$ is an arbitrary fixed number. Denote by D_r the projection S_{0r} onto D. It is quite obvious that $diam D_r \le 2r$. Denote by K'_{2r} a cube in the plane $\{x_n = 0\}$ containing D_r and with sides of length $2r$ and let $K_{2r} = \{(x'; x_n) \in R^n : x' \in K'_{2r}\}$ be a cylinder in R^n with the base K'_{2r}. Put $\Omega_{K_{2r}} = \Omega \bigcap K_{2r}$ and $K^a_{2r} = K_a \bigcap K_{2r}$. It is clear that $\Omega_{K_{2r}} \subset K^a_{2r}$. Let us show that there exists an absolute constant $c > 0$ such that K^a_{2r} can be covered with balls of radius r, the number of which does not exceed $\frac{c}{r} + 1$.

For simplicity, we will show this for the two-dimensional case. So, let $K_{2r}^{a} = \left\{(x_1; x_2) \in R^2 : \left|x_1 - x_1^0\right| < r;\ 0 < x_2 < a\right\}$ for some $x_1^0 \in [0, a)$. It is clear that an arbitrary square with side $2r$ can be covered with 4 balls of radius r, and the rectangle K_{2r}^{a} can be covered with squares of sides $2r$, the number of which does not exceed $\left[\frac{a}{2r}\right] + 1$. Therefore, for $0 < r < R_0$, where $R_0 > 0$ is some number, we can assume that the number of such squares does not exceed $\frac{c}{r}$, where $c > 0$ is a constant independent of r.

This reasoning can be carried out without much difficulty in the general m -dimensional case. From (15) it immediately follows

$$|f/_{S_0}|^q \le a^{\frac{q}{q'}} \int_0^{\varphi(x')} \left|\frac{\partial f\,(x'; \xi_n)}{\partial \xi_n}\right|^q d\xi_n\,,\ \frac{1}{q} + \frac{1}{q'} = 1.$$

Multiplying by $\sqrt{1 + \varphi_{x_1}^2 + ... + \varphi_{x_{n-1}}^2}$ and integrating with respect to D_r , we have

$$\int_{S_{0r}} |f_S|^q\, d\sigma \le c \int_{\Omega_{K_{2r}}} \left|\frac{\partial f}{\partial \xi_n}\right|^q dx. \tag{16}$$

Let $K_{2r}^{a} \subset \bigcup_{k=\overline{1,N_0}} B_r\,(x_k)$, where $N_0 \le \frac{c}{r}$, $\forall r \in (0, R_0]$. It is clear that $\Omega_{K_{2r}} \subset \bigcup_{k=\overline{1,N_0}} \Omega_r\,(x_k)$. Then $\exists k_0 \in \{1; ...; N_0\}$:

$$\int_{\Omega_{K_{2r}}} \left|\frac{\partial f}{\partial \xi_n}\right|^q dx \le N_0 \int_{\Omega_r\left(x_{k_0}\right)} \left|\frac{\partial f}{\partial \xi_n}\right|^q dx. \tag{17}$$

Indeed, if

$$\int_{\Omega_{K_{2r}}} \left|\frac{\partial f}{\partial \xi_n}\right|^q dx > N_0 \int_{\Omega_r(x_k)} \left|\frac{\partial f}{\partial \xi_n}\right|^q dx\,,\ \forall k = \overline{1, N_0},$$

then summing up we have

$$\int_{\Omega_{K_{2r}}} \left|\frac{\partial f}{\partial \xi_n}\right|^q dx > \int_{\bigcup_{k=\overline{1,N_0}} \Omega_r(x_k)} \left|\frac{\partial f}{\partial \xi_n}\right|^q dx.$$

On the other hand, it is obvious that

$$\int_{\Omega_{K_{2r}}} \left|\frac{\partial f}{\partial \xi_n}\right|^q dx \le \int_{\bigcup_{k=\overline{1,N_0}} \Omega_r(x_k)} \left|\frac{\partial f}{\partial \xi_n}\right|^q dx.$$

Consequently, inequality (17) holds, and as a result, from (16) it follows that

$$\int_{S_{0r}} |f_S|^q\, d\sigma \le \frac{c}{r} \int_{\Omega_r(k_0)} \left|\frac{\partial f}{\partial \xi_n}\right|^q dx \Rightarrow \frac{1}{r^{\lambda-1}} \int_{S_{0r}} |f_S|^q\, d\sigma \le$$

$$\leq \frac{c}{r^{\lambda}} \int_{\Omega_r(k_0)} \left| \frac{\partial f}{\partial \xi_n} \right|^q dx \Rightarrow \left(\frac{1}{r^{\lambda-1}} \int_{S_{0r}} |f|^q \, d\sigma \right)^{\frac{1}{q}} \leq$$

$$\leq c \sum_{|p|=1} \|\partial^p f\|_{L_{q,\lambda}(\Omega)} \leq c \, \|f\|_{W^1_{q,\lambda}(\Omega)} ,$$

where the constant $c > 0$ is independent of $r > 0$. Hence we immediately obtain $f_S \in L_{q,\lambda-1}(S_0)$ and the estimate

$$\|f_S\|_{L_{q,\lambda-1}(S_0)} \leq c \, \|f\|_{W^1_{q,\lambda}(\Omega)} .$$

Since S can be covered by a finite number of pieces of type S_0, by standard technique we obtain $f_S \in L_{q,\lambda-1}(S)$ and

$$\|f_S\|_{L_{q,\lambda-1}(S)} \leq c \, \|f\|_{W^1_{q,\lambda}(\Omega)} . \tag{18}$$

Theorem is proved. ◄

This theorem immediately implies the following

Corollary 1. *Let all conditions of Theorem 4 be satisfied. Then the trace space $W^1_{q,\lambda}(S)$ is continuously embedded in $L_{q,\lambda-1}(S)$, i.e., the estimate*

$$\|f_S\|_{L_{q,\lambda-1}(S)} \leq c \, \|f_S\|_{W^1_{q,\lambda}(S)} , \ \forall f_S \in W^1_{q,\lambda}(S) ,$$

is true, where the constant $c > 0$ is independent of f_S.

Indeed, let $f_S \in W^1_{q,\lambda}(S)$ and $f \in W^1_{q,\lambda}(\Omega) : \Gamma_S f = f_S$. Let $F \in \mathscr{F}^1_{q,\lambda}(S)$ be a factor class containing f. Then from the estimate (18) it follows that

$$\|f_S\|_{L_{q,\lambda-1}(S)} \leq c \, \|f\|_{W^1_{q,\lambda}(\Omega)} .$$

Taking inf with respect to F on the right, we obtain the required assertion.

It should be noted that the characterization of traces of functions from Morrey-Sobolev spaces in a different context was previously considered by S. Campanato [42].

Remark 1. *It is not hard to see that for $0 < \lambda \leq 1$, the space $L_{q,\lambda-1}(S)$ coincides with the space $L_q(S)$.*

6. Interior Schauder-type Estimates

In this section we establish interior estimates and will use the following result from the work [7]. Let $\omega(\cdot)$ be an infinitely differentiable function on $[0,1]$ such that for $0 \le t < \frac{1}{3} : \omega(t) \equiv 1$ and for $\frac{2}{3} < t \le 1 : \omega(t) \equiv 0$. For $0 < R_1 < R_2$ we put

$$\xi(x) = \begin{cases} 1, & |x| \le R_1, \\ \omega\left(\frac{|x|-R_1}{R_2-R_1}\right), & R_1 < |x| \le R_2. \end{cases}$$

Then the following lemma is true.

Lemma 9. *[7] There is a constant $C > 0$ depending only on R_2 and $\omega(\cdot)$, such that for $\forall R_1 : 0 < R_1 < R_2$, the following relation holds*

$$\|\xi\|_{C^m(R_2)} \le C\left(1 - \frac{R_1}{R_2}\right)^{-m}.$$

Using the estimates obtained in the previous subsection, completely similar to [7], we can prove the following

Lemma 10. *Let the coefficients of m-th order elliptic operator satisfy the conditions: $a_p \in C\left(\overline{B}_{R_2}\right)$, $\forall p : |p| = m$; $a_p \in L_\infty(B_{R_2})$, $\forall p : |p| < m$, where $R_2 > 0$ is some number. Then the following estimate holds*

$$\|u\|_{N^m_{q,\lambda}(R_1)} \le c\left(1 - \frac{R_1}{R_2}\right)^{-m}\left(\|Lu\|_{L_{q,\lambda}(R_2)} + \|u\|_{N^{m-1}_{q,\lambda}(R_2)}\right),$$

$$\forall R_1 \in (0, R_2), \ \forall u \in N^m_{q,\lambda}(R_2),$$

where the constant $c > 0$ is independent of R_1 and may only depend on R_2.

Let us also prove the following lemma, which is used to establish interior estimates.

Lemma 11. *Let $\Omega \subset R^n$ be a bounded domain with the boundary $\partial\Omega \in C^{(m)}$. Then $\exists\, c > 0$ depending only on n, q, λ and on the constant from Theorem 2 (extension), for which*

$$\|\varphi\|_{N^k_{q,\lambda}(\Omega)} \le \varepsilon\, \|\varphi\|_{N^{k+1}_{q,\lambda}(\Omega)} + c\varepsilon^{-k}\, \|\varphi\|_{L_{q,\lambda}(\Omega)}, \forall k = \overline{1, m-1};$$

$$\forall \varepsilon > 0, \forall \varphi \in N^m_{q,\lambda}(\Omega). \tag{19}$$

Proof. Without loss of generality, we will assume that $d_\Omega = 1$ (see [7]). Let $\Omega' \subset R^n$ be a bounded domain with a sufficiently smooth boundary such that $\Omega \subset\subset \Omega'$ (i.e. $\bar{\Omega} \subset \Omega'$). Let $\varphi \in N^m_{q,\lambda}(\Omega)$. By Theorem 2 (extension) $\exists\ \Phi \in \overset{0}{N}{}^m_{q,\lambda}(\Omega') : \Phi/_\Omega = \varphi$, and

$$\|\Phi\|_{N^k_{q,\lambda}(\Omega')} \leq c\, \|\varphi\|_{N^k_{q,\lambda}(\Omega)}\ ,\ \ \forall\, k = \overline{0,\, m},$$

where the constant $c > 0$ is independent of φ. Let us extend Φ to all of R^n setting $\Phi = 0$ in $R^n \backslash \overline{\Omega'}$. It is clear that $\|\Phi\|_{N^k_{q,\lambda}(\infty)} = \|\Phi\|_{N^k_{q,\lambda}(\Omega')}$, $\forall\ k = \overline{0,\, m}$. Let the inequalities (19) be satisfied with respect to the spaces $N^k_{q,\lambda}(\infty)$, $\forall\ k = \overline{0,\, m}$, for the functions with compact supports. Then we have

$$\|\varphi\|_{N^k_{q,\lambda}(\Omega)} \leq \|\Phi\|_{N^k_{q,\lambda}(\Omega')} = \|\Phi\|_{N^k_{q,\lambda}(\infty)} \leq \varepsilon\, \|\Phi\|_{N^{k+1}_{q,\lambda}(\infty)} + c\varepsilon^{-k}\, \|\Phi\|_{L_{q,\lambda}(\infty)} \leq$$

$$\leq \varepsilon\, c\, \|\varphi\|_{N^{k+1}_{q,\lambda}(\Omega)} + c\varepsilon^{-k}\, \|\varphi\|_{L_{q,\lambda}(\Omega)}\,.$$

Choosing an appropriate δ, we obtain the inequality (19). Therefore, it suffices to prove the validity of inequalities (19) with respect to the spaces $N^k_{q,\lambda}(\infty)$, $k = \overline{0,\, m}$.

Following the classical scheme, we first assume that f is a sufficiently smooth function of one variable with compact support. Let $B_r \subset R$ be an arbitrary ball (interval of length $2r$) and $h > 0$ be a sufficiently small number. We have (see [4, p. 261])

$$\left|f'(t)\right| \leq \int_t^{t+h} \left|f''(\tau)\right|\, d\tau + \frac{1}{h}\left(\left|f(t+h) - f(t)\right|\right)\ ,\ \ \forall\, t \in R.$$

Put

$$\left(Mf''\right)(t) = \sup_{h>0} \frac{1}{h} \int_t^{t+h} \left|f''(\tau)\right|\, d\tau.$$

Thus

$$\left|f'(t)\right| \leq h\left(Mf''\right)(t) + \frac{1}{h}\left(\left|f(t+h)\right| + \left|f(t)\right|\right)\ ,\ \ \forall\, t\, \in R.$$

Taking into account the boundedness of the maximal operator in the grand-Lebesgue spaces (see, for instance, [8, 9]), we directly obtain

$$\left\|f'\right\|_{L_{p,\lambda}(\infty)} \leq c_1 h\, \left\|f''\right\|_{L_{p,\lambda}(\infty)} + \frac{c_2}{h}\, \|f\|_{L_{p,\lambda}(\infty)}\ , \tag{20}$$

where the constants c_1; $c_2 > 0$ are independent of f.

Consider the general multidimensional case. Let $p_k : |p_k - p| = 1$, $k = 1,\, 2$; $p_2 \geq p \geq p_1$ be arbitrary multi indices (we assume that the multindex p_1 is less than or equal to the multindex p_2 if each component of p_1 is less than or equal to the corresponding component of p_2). Proceeding from inequality (20), completely similar to the classical case (see [4, pp. 261-262]), we establish

$$\|\partial^p f\|_{L_{q,\lambda}(\infty)} \leq c_1 h \, \|\partial^{p_2} f\|_{L_{q,\lambda}(\infty)} + c_2 h^{-1} \, \|\partial^{p_1} f\|_{L_{q,\lambda}(\infty)} \, .$$

Summing over all $p_2 \, : \, |p_2| = k+1$, for $\forall \, h > 0$ we have

$$n \sum_{|p|=k} \|\partial^p f\|_{L_{q,\lambda}(\infty)} \leq c_1 h \sum_{|p|=k+1} \|\partial^p f\|_{L_{q,\lambda}(\infty)} + c_2 n^2 h^{-1} \sum_{|p|=k-1} \|\partial^p f\|_{L_{q,\lambda}(\infty)} \, .$$

Subsequently, similar to [7], we establish the estimate

$$\|f\|_{N^k_{q,\lambda}(\infty)} \leq \varepsilon \, \|f\|_{N^{k+1}_{q,\lambda}(\infty)} + c\varepsilon^{-k} \, \|f\|_{L_{q,\lambda}(\infty)} \ , \ \ \forall \, \varepsilon > 0, \forall \, k = \overline{1, \, m-1},$$

where the constant $A > 0$ is independent of f.

Lemma is proved. ◀

These lemmas allow us to establish the following interior Schauder-type estimate.

Theorem 5. *Let $\Omega \subset R^n$ be a bounded domain and the coefficients of the m -th order elliptic operator L satisfy the conditions i) $a_p(\cdot) \in C\left(\bar{\Omega}\right), \forall p : |p| = m$, ii) $a_p(\cdot) \in L_\infty(\Omega), \forall p : |p| < m$. Let $\Omega_0 \subset \Omega-$be an arbitrary compact. Then for $\forall u \in N^m_{q,\lambda}(\Omega)$, $1 < q < +\infty$, $0 < \lambda < n$, the following a priori estimate holds:*

$$\|u\|_{N^m_{q,\lambda}(\Omega_0)} \leq c \left(\|u\|_{L_{q,\lambda}(\Omega)} + \|u\|_{L_{q,\lambda}(\Omega)} \right),$$

where the constant c depends only on the ellipcity constant, on the coefficients of the operator L and on m, Ω_0, Ω.

Proof. The proof is carried out in exactly the same way as in [7] (see also [4, p. 243]). For the sake of completeness, we present it. Let a domain Ω and a compact $\Omega_0 \subset \Omega$ be given. It is clear that Ω_0 can cover a finite number of open balls whose closures are contained in Ω. Therefore, it suffices to prove the theorem for the case where Ω and Ω_0 are concentric balls of small radius centered at a point $x_0 \in \Omega$ and without loss of generality we assume that $x_0 = 0$.

So, let $R > 0$ be sufficiently small number. Let us prove that the following estimate holds for $\forall r : 0 < r < R$:

$$\|u\|_{N^m_{q)}(r)} \leq C \left(1 - \frac{r}{R}\right)^{-m^2} \left(\|Lu\|_{L_{q)}(R)} + \|u\|_{L_{q)}(R)} \right),$$

where $C > 0$ is a constant depending on R (independent of r and u). Assume

$$A = \sup_{0 \le r \le R} \left(1 - \frac{r}{R}\right)^{m^2} \|u\|_{N^m_{q)}(r)} \le \|u\|_{N^m_{q)}(R)} .$$

Suppose that $\|u\|_{N^m_{q)}(R)} \neq 0$. Otherwise, there is nothing to prove. Then it is easy to see that $\exists R_1 : 0 < R_1 < R$ such that

$$A \le 2\left(1 - \frac{R_1}{R}\right)^{m^2} \|u\|_{N^m_{q)}(R_1)} .$$

Then, for $R_2 : R_1 < R_2 < R$, by Lemma 5, the inequality (8) holds. Taking this inequality into account, we have

$$A \le 2\left(1 - \frac{R_1}{R}\right)^{m^2} C_1 \left(1 - \frac{R_1}{R_2}\right)^{-m} \left(\|Lu\|_{L_{q)}(R_2)} + \|u\|_{N^{m-1}_{q)}(R_2)}\right) \le$$

$$\le 2C_1 \left(1 - \frac{R_1}{R}\right)^{m^2} \left(1 - \frac{R_1}{R_2}\right)^{-m} \left(\|Lu\|_{L_{q)}(R)} + \|u\|_{N^{m-1}_{q)}(R_2)}\right).$$

Taking into account the inequality (19) for $k = m - 1$, we obtain

$$A \le 2C_1 \left(1 - \frac{R_1}{R}\right)^{m^2} \left(1 - \frac{R_1}{R_2}\right)^{-m} \left(\|Lu\|_{L_{q)}(R)} + \varepsilon \|u\|_{N^m_{q)}(R_2)} + \right.$$

$$\left. + C_2 \varepsilon^{-m+1} \|u\|_{L_{q)}(R_2)}\right).$$

Paying attention to the fact that

$$\left(1 - \frac{R_2}{R}\right)^{\tau} \|u\|_{N^m_{q)}(R_2)} \le A,$$

we have

$$A \le 2C_1 \left(1 - \frac{R_1}{R}\right)^{m^2} \left(1 - \frac{R_1}{R_2}\right)^{-m} \|Lu\|_{L_{q)}(R)} +$$

$$+ 2\varepsilon\, C_1 \left(1 - \frac{R_1}{R}\right)^{m^2} \left(1 - \frac{R_1}{R_2}\right)^{-m} \left(1 - \frac{R_2}{R}\right)^{-\tau} A +$$

$$+ 2C_1 C_2 \varepsilon^{-m+1} \left(1 - \frac{R_1}{R}\right)^{m^2} \left(1 - \frac{R_1}{R_2}\right)^{-m} \|u\|_{L_{q)}(R_2)} ,$$

where $\varepsilon > 0$ is an arbitrary number. Let $\delta = 1 - \frac{R_1}{R}$ and choose R_2 and ε from the relations $1 - \frac{R_2}{R_1} = \frac{\delta}{2}$, $\varepsilon = 2^{-2-m-\tau}\frac{\delta^m}{C_1}$. We have $0 < \delta < 1$, $\frac{\delta}{2} < 1 - \frac{R_1}{R_2} < \delta$. Consequently

$$2\varepsilon\, C_1 \left(1 - \frac{R_1}{R}\right)^m \left(1 - \frac{R_1}{R_2}\right)^{-m} \left(1 - \frac{R_2}{R}\right)^{-\tau} < \frac{1}{2},$$

and as a result,

$$\frac{1}{2}A \leq 2C_1 \left(1 - \frac{R_1}{R}\right)^{m^2} \left(1 - \frac{R_1}{R_2}\right)^{-m} \|Lu\|_{L_{q)}(R)} +$$

$$+2C_1C_2\varepsilon^{-m+1} \left(1 - \frac{R_1}{R}\right)^{m^2} \left(1 - \frac{R_1}{R_2}\right)^{-m} \|u\|_{L_{q)}(R_2)} \leq$$

$$\leq C \left(\|Lu\|_{L_{q)}(R)} + \|u\|_{L_{q)}(R)}\right).$$

Taking into account the expression for A, we have

$$\|u\|_{N^m_{q)}(r)} \leq C \left(1 - \frac{r}{R}\right)^{-m^2} \left(\|Lu\|_{L_{q)}(R)} + \|u\|_{L_{q)}(R)}\right),$$

for $\forall r : 0 < r < R$, where $C > 0$is a constant independent of r.

Theorem is proved. ◄

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