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Matrix Transforms in Weighted Variable Exponent Lebesgue Spaces

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Abstract. In this work, approximation properties of matrix transforms in the weighted variable exponent Lebesgue space of periodic functions are investigated.

Key Words and Phrases: matrix transform, rate of approximation, Fourier series, Muckenhoupt weight, variable exponent Lebesgue space.

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1. Introduction and Main Results

Let $\mathbb{T} := [0, 2\pi]$ and let $p(\cdot) : \mathbb{T} \to [1, \infty)$ be a Lebesgue measurable 2π periodic function. The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{T})$ is defined as the set of all Lebesgue measurable 2π periodic functions f such that

$$\rho_{p(\cdot)}(f) := \int_0^{2\pi} |f(x)|^{p(x)} \, dx < \infty.$$

We suppose that the exponent functions $p(\cdot)$ satisfy the conditions

$$1 \le p_- := ess \inf_{x \in \mathbb{T}} p(x) \le ess \sup_{x \in \mathbb{T}} p(x) := p^+ < \infty,$$
$$|p(x) - p(y)| \ln \left(\frac{1}{|x-y|} \right) \le c, \quad x, y \in \mathbb{T}, \quad 0 < |x-y| \le 1/2. \tag{1}$$

If the exponent $p(\cdot)$ satisfies the conditions (1), then we say that $p(\cdot) \in \mathcal{P}(\mathbb{T})$. We also denote $\mathcal{P}_0(\mathbb{T}) := \{p(\cdot) \in \mathcal{P}(\mathbb{T}) : p_- > 1\}$.

From now on we suppose that $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$. Then, equipped with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \le 1 \right\},$$

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$L^{p(\cdot)}(\mathbb{T})$ becomes a Banach space.

For a given weight ω we define the variable exponent weighted Lebesgue space $L_{\omega}^{p(\cdot)}(\mathbb{T})$ as the set of all measurable 2π periodic functions f such that $f\omega \in L^{p(\cdot)}(\mathbb{T})$. The norm of $L_{\omega}^{p(\cdot)}(\mathbb{T})$ can be defined by $\|f\|_{p(\cdot),\omega} := \|f\omega\|_{p(\cdot)}$.

Note that Lebesgue space with variable exponent is a generalization of classical Lebesgue space, by replacing the constant exponent p with variable exponent function $p(\cdot)$.

Definition 1. *For a given exponent $p(\cdot)$ we say that $\omega \in A_{p(\cdot)}(\mathbb{T})$, if*

$$\sup_{I_j} |I_j|^{-1} \left\|\omega \chi_{I_j}\right\|_{p(\cdot)} \left\|\omega^{-1}\chi_{I_j}\right\|_{q(\cdot)} < \infty \ , \qquad 1/p(\cdot) + 1/q(\cdot) = 1,$$

where supremum is taken over all open intervals $I_j \subset \mathbb{T}$ with the characteristic functions χ_{I_j}.

Let $f \in L^1(\mathbb{T})$ and

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} (a_k(f)\cos kx + b_k(f)\sin kx)$$

be its Fourier series representation with the Fourier coefficients

$$a_k(f) := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt \text{ and } b_k(f) := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt,$$

and let

$$S_n(f) := S_n(f,x) = \sum_{k=0}^{n} u_k(f)(x), \quad n = 1,2,...,$$

be the nth partial sum of Fourier series of f, where

$$u_0(f)(x) := a_0(f)/2 \text{ and } u_k(f)(x) := (a_k(f)\cos kx + b_k(f)\sin kx) \ , \ k = 1,2,..., \quad .$$

Let $A = (a_{n,k})$ be an infinite matrix of real numbers $a_{n,k}$, such that

$$a_{n,k} \geq 0 \text{ for } k,n = 0,1,2,..., \ \lim_{n\to\infty} a_{n,k} = 0 \text{ and } \sum_{k=0}^{\infty} a_{n,k} = 1.$$

Let also $A_0 = (a_{n,k})$, where $k,n = 0,1,2,...$, be a lower triangular matrix such that

$$a_{n,k} \geq 0 \text{ for } k \leq n \ , \ a_{n,k} = 0 \text{ for } k > n \text{ and } \sum_{k=0}^{n} a_{n,k} = 1.$$

The matrix transforms, produced by the matrices $A = (a_{n,k})$ and $A_0 = (a_{n,k})$, are defined as

$$T_{n,A}[f](x) := \sum_{k=0}^{\infty} a_{n,k} S_k(f,x) \ , \ n = 0,1,2,...$$

and

$$T_{n,A_0}[f](x) = \sum_{k=0}^{n} a_{n,k} S_k(f,x) \ , \ n = 0,1,2,...,$$

respectively.

Let (p_n) be a sequence of positive real numbers and $P_n = \sum_{k=0}^{n} p_k$. If $a_{n,k} = p_k/P_n$, then $T_{n,A_0}[f](x)$ coincides with the Riesz means $R_n[f]$, defined as

$$R_n[f](x) := \frac{1}{P_n} \sum_{k=0}^{n} p_k S_k(f,x), \quad n = 0,1,2,...,$$

which in the case of $p_n = 1$, for all $n = 0,1,2,...$, reduce to the Fejér means (Cesàro means of first order)

$$\sigma_n[f](x) := \frac{1}{n+1} \sum_{k=0}^{n} S_k(f,x).$$

If $a_{n,k} = p_k/P_n$ and

$$p_k = \begin{cases} 1, & \text{for } n \leq k \leq n+m \\ 0, & \text{for } 0 \leq k < n, \end{cases}$$

where $m,n = 0,1,2,...$, then $T_{n+m,A_0}[f]$ coincides with the De La Vallée Poussin means of f defined as

$$V_n^m[f](x) := \frac{1}{m+1} \sum_{k=n}^{n+m} S_k(f,x).$$

The variable exponent Lebesgue spaces have some advantages in the solution processes of different application problems of mathematics and mechanics. The corresponding results can be found in the monographs [7, 9]. The results concerning different problems of approximation and constructive approximation theory, and also basicity problems of differential systems of functions in these spaces can be found in [15, 13, 1, 2, 28, 29, 30, 16, 17, 18, 19, 32, 21, 23, 3, 4, 10, 27] .

In [13, 14, 24, 20, 31, 22, 8, 33], the approximation properties of more general aggregates such as matrix transforms in the variable exponent spaces have been investigated. The results obtained in these works are natural generalization of the results proved earlier for the classical Lebesgue spaces (see, for example, [5, 26, 11, 12]).

In this work, approximation properties of the matrix transforms $T_{n,A}[f]$ and $T_{n,A_0}[f]$ in the variable exponent weighted Lebesgue spaces are investigated and the errors

$\|f-T_{n,A}[f]\|_{p(\cdot),\omega}$ and $\|f-T_{n,A_0}[f]\|_{p(\cdot),\omega}$ in terms of the modulus of smoothness constructed via Steklov means of $f\in L_{\omega}^{p(\cdot)}(\mathbb{T})$ are estimated.

Definition 2. *Let $f\in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}_0(\mathbb{T})$, $\omega(\cdot)\in A_{p(\cdot)}(\mathbb{T})$. We define the modulus of smoothness as*

$$\Omega(f,\delta)_{p(\cdot),\omega}:=\sup_{0<h\leq\delta}\left\|\frac{1}{h}\int_0^h[f(\cdot+t)-f(\cdot)]\,dt\right\|_{p(\cdot),\omega},\quad \delta>0.$$

This definition is correct. Indeed, the maximal operator

$$M(f):f\to Mf(x):=\sup_{I\ni x}\frac{1}{|I|}\int\limits_I|f(t)|\,dt$$

is bounded in $L_{\omega}^{p(\cdot)}(\mathbb{T})$ if $\omega(\cdot)\in A_{p(\cdot)}(\mathbb{T})$ and $p(\cdot)\in\mathcal{P}_0(\mathbb{T})$ (see, [6]) and there exists a positive constant $c(p)$ such that the inequality

$$\|M(f)\|_{p(\cdot),\omega}\leq c(p)\,\|f\|_{p(\cdot),\omega}\tag{2}$$

holds for every $f\in L_{\omega}^{p(\cdot)}(\mathbb{T})$. Hence, by (2) we obtain

$$\left\|\frac{1}{h}\int_0^h[f(x+t)-f(x)]\,dt\right\|_{p(\cdot),\omega}\leq\left\|\frac{1}{h}\int_x^{x+h}f(t)\,dt\right\|_{p(\cdot),\omega}+\|f\|_{p(\cdot),\omega}\leq c\,\|f\|_{p(\cdot),\omega}$$

and then from the inequality

$$\Omega(f,\delta)_{p(\cdot),\omega}\leq c(p)\,\|f\|_{p(\cdot),\omega}\tag{3}$$

we obtain correctness of *Definition 2*.

If $f,g\in L_{\omega}^{p(\cdot)}(\mathbb{T})$, then applying techniques used in [18] we get

$$\Omega(f+g,\delta)_{p(\cdot),\omega}\leq\Omega(f,\delta)_{p(\cdot),\omega}+\Omega(g,\delta)_{p(\cdot),\omega}\ \text{ and }\ \lim_{\delta\to0}\Omega(f,\delta)_{p(\cdot),\omega}=0.$$

We use the notation $m=\mathcal{O}(n)$ if there exists a positive constant c such that $m\leq cn$ throughout this work.

Below we state our main results.

Theorem 1. *Let $f\in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}_0(\mathbb{T})$ and $\omega(\cdot)\in A_{p(\cdot)}(\mathbb{T})$. If the conditions*

$$\sum_{k=0}^{\infty}(k+1)^{\beta}\left|\frac{a_{n,k}}{(k+1)^{\beta}}-\frac{a_{n,k+1}}{(k+2)^{\beta}}\right|=\mathcal{O}\left(\frac{1}{n+1}\right)\tag{4}$$

for some $\beta \geq 0$ *and*

$$\sum_{k=0}^{\infty} (k+1)\, a_{n,k} = \mathcal{O}\,(n+1) \tag{5}$$

hold, then

$$\|f - T_{n,A}\,[f]\|_{p(\cdot),\omega} = \mathcal{O}\left(\Omega\left(f, \frac{1}{n}\right)_{p(\cdot),\omega} + \sum_{k=0}^{n} a_{n,k}\Omega\left(f, \frac{1}{k}\right)_{p(\cdot),\omega}\right).$$

Theorem 2. *Let* $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ *and* $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. *If the conditions*

$$\sum_{k=0}^{n-1} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| = \mathcal{O}\,(a_{n,n}) \tag{6}$$

for some $\beta \geq 0$ *and*

$$(n+1)\, a_{n,n} = \mathcal{O}\,(1) \tag{7}$$

hold, then

$$\|f - T_{n,A_0}\,[f]\|_{p(\cdot),\omega} = \mathcal{O}\left(\sum_{k=0}^{n} a_{n,k}\Omega\left(f, \frac{1}{k}\right)_{p(\cdot),\omega}\right).$$

In particular, Theorems 1 and 2 in the case of $\omega = 1$ were proved in [24].

Theorem 2 implies

Corollary 1. *Let* $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ *and* $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. *If the conditions*

$$\sum_{k=0}^{n-1} (k+1)^{\beta} \left| \frac{p_k}{(k+1)^{\beta}} - \frac{p_{k+1}}{(k+2)^{\beta}} \right| = \mathcal{O}\,(p_n) \tag{8}$$

for some $\beta \geq 0$ *and*

$$(n+1)\, p_n = \mathcal{O}\,(P_n) \tag{9}$$

hold, then

$$\|f - R_n\,[f]\|_{p(\cdot),\omega} = \mathcal{O}\left(\frac{1}{P_n}\sum_{k=0}^{n} p_k\Omega\left(f, \frac{1}{k}\right)_{p(\cdot),\omega}\right).$$

Since $n = \mathcal{O}\,(m)$ implies that $(n+m+1) = \mathcal{O}\,(m+1)$, applying Corollary 1 we obtain

Corollary 2. *Let* $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ *and* $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. *If the condition* $n = \mathcal{O}\,(m)$ *holds, then*

$$\|f - V_n^m\,[f]\|_{p(\cdot),\omega} = \mathcal{O}\left(\frac{1}{m+1}\sum_{k=n}^{n+m} \Omega\left(f, \frac{1}{k}\right)_{p(\cdot),\omega}\right).$$

Corollary 2 was proved for $\omega = 1$ in [30].

Corollary 3. *Let* $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ *and* $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. *Then*

$$\|f - \sigma_n[f]\|_{p(\cdot),\omega} = \mathcal{O}\left(\frac{1}{n+1}\sum_{k=0}^{n}\Omega\left(f, \frac{1}{k}\right)_{p(\cdot),\omega}\right).$$

In some special subclasses of $L_{\omega}^{p(\cdot)}(\mathbb{T})$, the estimations obtained in Theorems 2 and 3 can be further simplified.

Let $\omega^*(\cdot)$ be a nondecreasing continuous function defined on $\mathbb{T}$ and satisfying the conditions:

i) $\omega^*(0) = 0$,

ii) $\omega^*(\delta_1 + \delta_2) \leq \omega^*(\delta_1) + \omega^*(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 2\pi$.

Let also

$$Lip_{p(\cdot),\omega}(\omega^*, M) := \left\{f \in L_{\omega}^{p(\cdot)}(\mathbb{T}) : \Omega(f, \delta)_{p(\cdot),\omega} \leq M\omega^*(\delta), \quad \delta > 0\right\}$$

with some positive constant M. Then we have

Theorem 3. *Let* $Lip_{p(\cdot),\omega}(\omega^*, M)$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ *and* $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. *If the conditions (4) for some* $\beta \geq 0$ *and (5) hold, then*

$$\|f - T_{n,A}[f]\|_{p(\cdot),\omega} = \mathcal{O}(\omega^*(1/n)).$$

Theorem 4. *Let* $Lip_{p(\cdot),\omega}(\omega^*, M)$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ *and* $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. *If the conditions (6) for some* $\beta \geq 0$ *and (7) hold, then*

$$\|f - T_{n,A_0}[f]\|_{p(\cdot),\omega} = \mathcal{O}(\omega^*(1/n)).$$

When $\omega = 1$, Theorems 3 and 4 were proved in [24]. If $\omega^*(\delta) = \delta^{\alpha}$, $\delta > 0$ and $\alpha \in (0,1]$, then under different assumptions on the given matrix $A_0 = (a_{n,k})$ Theorem 4 in non weighted and weighted cases was proved in [14] and [20], respectively. The case $p(\cdot) = const$ and $\omega = 1$ was considered in [26] and [25].

Theorem 4 implies

Corollary 4. *Let* $Lip_{p(\cdot),\omega}(\omega^*, M)$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ *and* $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. *If the conditions (8) for some* $\beta \geq 0$ *and (9) hold, then*

$$\|f - R_n[f]\|_{p(\cdot),\omega} = \mathcal{O}(\omega^*(1/n)).$$

Corollary 5. *Let* $Lip_{p(\cdot),\omega}(\omega^*, M)$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ *and* $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. *If* $n = \mathcal{O}(m)$, *then*

$$\|f - V_n^m[f]\|_{p(\cdot),\omega} = \mathcal{O}(\omega^*(1/n)).$$

Corollary 6. *Let* $Lip_{p(\cdot),\omega}(\omega^*, M)$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ *and* $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. *Then*

$$\|f - \sigma_n[f]\|_{p(\cdot),\omega} = \mathcal{O}(\omega^*(1/n)).$$

2. Auxiliary Results

Let Π_n be the class of trigonometric polynomials of degree not exceeding n. The *best approximation number* of $f \in L_\omega^{p(\cdot)}(\mathbb{T})$ is defined as

$$E_n(f)_{p(\cdot),\omega} := \inf\left\{\|f - T_n\|_{p(\cdot),\omega} : T_n \in \Pi_n\right\}, \quad n = 0, 1, 2, .. \ .$$

Lemma 1. *([20]) Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. If $f \in L_\omega^{p(\cdot)}(\mathbb{T})$, then for every $n = 1, 2, ...,$ the inequality*

$$E_n(f)_{p(\cdot),\omega} = \mathcal{O}\left(\Omega(f, 1/n)_{p(\cdot),\omega}\right)$$

holds.

Lemma 2. *([24]) If (4) for some $\beta \geq 0$ and (5) hold,then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=0}^{\infty} a_{n,k}\left(1 + \sum_{\nu=1}^{k}\frac{\pi}{4\sin\frac{\pi}{8}}\cos\nu t\right)\right| dt = \mathcal{O}(1).$$

If the trigonometric polynomial $T_n^* := T_n^*(f) \in \Pi_n$ satisfies the inequality

$$\|f - T_n^*\|_{p(\cdot),\omega} \leq cE_n(f)_{p(\cdot),\omega} \ , n = 0, 1, 2, ..,$$

for some positive constant c independent of n, then T_n^* is called near-best approximating polynomial to $f \in L_\omega^{p(\cdot)}(\mathbb{T})$.

Lemma 3. *Let $f \in L_\omega^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$ and let T_n^* be near-best approximating polynomial to f. If the conditions (4) for some $\beta \geq 0$ and (5) hold, then*

$$\left\|\sum_{k=0}^{\infty} a_{n,k} S_n(f - T_n^*)\right\|_{p(\cdot),\omega} = \mathcal{O}\left(\Omega\left(f, \frac{1}{n}\right)_{p(\cdot),\omega}\right).$$

Proof. We define $f_h(t) = \frac{1}{2h}\int_{-h}^{h} f(y + t)\,dy$. Then $a_0(f_h) = a_0(f)$ and after simple calculations we have

$$a_\nu(f_h) = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2h}\int_{-h}^{h} f(y + t)\,dy\right)\cos(\nu t)\,dt$$

$$\begin{aligned}
&= \frac{1}{\pi}\frac{1}{2h}\int\limits_{-h}^{h}\left(\int\limits_{-\pi}^{\pi} f\left(y+t\right)\cos\nu t dt\right)dy = \frac{1}{\pi}\frac{1}{2h}\int\limits_{-h}^{h}\left(\int\limits_{-\pi}^{\pi} f\left(t\right)\cos\nu(t-y)dt\right)dy \\
&= \frac{1}{\pi}\frac{1}{2h}\int\limits_{-h}^{h}\left(\int\limits_{-\pi}^{\pi} f\left(t\right)\left(\cos\left(\nu t\right)\cos\left(\nu y\right)+\sin\left(\nu t\right)\sin\left(\nu y\right)\right)dt\right)dy \\
&= \frac{1}{2h}\int\limits_{-h}^{h}\left(\cos\left(\nu y\right)\frac{1}{\pi}\int\limits_{-\pi}^{\pi} f\left(t\right)\cos\left(\nu t\right)dt+\sin\left(\nu y\right)\frac{1}{\pi}\int\limits_{-\pi}^{\pi} f\left(t\right)\sin\left(\nu t\right)dt\right)dy \\
&= a_{\nu}\left(f\right)\frac{1}{2h}\int\limits_{-h}^{h}\cos\left(\nu y\right)dy+b_{\nu}\left(f\right)\frac{1}{2h}\int\limits_{-h}^{h}\sin\left(\nu y\right)dy = a_{\nu}\left(f\right)\frac{\sin\left(\nu h\right)}{\nu h}
\end{aligned}$$

for every $\nu = 1, 2, ...$, and similarly we can show that

$$b_{\nu}\left(f_h\right) = b_{\nu}\left(f\right)\frac{\sin\left(\nu h\right)}{\nu h}.$$

Hence,

$$S_k\left(f,x\right) = \frac{a_0\left(f_h\right)}{2}+\sum_{\nu=1}^{k}\frac{\nu h}{\sin\left(\nu h\right)}a_{\nu}\left(f_h\right)\cos\left(\nu x\right)+\sum_{\nu=1}^{k}\frac{\nu h}{\sin\left(\nu h\right)}b_{\nu}\left(f_h\right)\sin\left(\nu x\right)$$

$$\begin{aligned}
&= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f_h\left(t\right)dt+\sum_{\nu=1}^{k}\frac{\nu h}{\sin\left(\nu h\right)}a_{\nu}\left(f_h\right)\cos\left(\nu x\right)+\sum_{\nu=1}^{k}\frac{\nu h}{\sin\left(\nu h\right)}b_{\nu}\left(f_h\right)\sin\left(\nu x\right) \\
&= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f_h\left(t\right)dt+\sum_{\nu=1}^{k}\frac{\nu h}{\sin\left(\nu h\right)}\frac{1}{\pi}\int\limits_{-\pi}^{\pi} f_h\left(t\right)\cos\left(\nu t\right)dt\cos\left(\nu x\right) \\
&\quad +\sum_{\nu=1}^{k}\frac{\nu h}{\sin\left(\nu h\right)}\frac{1}{\pi}\int\limits_{-\pi}^{\pi} f_h\left(t\right)\sin\left(\nu t\right)dt\sin\left(\nu x\right) \\
&= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f_h\left(t\right)dt+\sum_{\nu=1}^{k}\frac{\nu h}{\sin\left(\nu h\right)}\frac{1}{\pi}\int\limits_{-\pi}^{\pi} f_h\left(t\right)\cos\left(\nu\left(t-x\right)\right)dt \\
&= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f_h\left(t+x\right)dt+\frac{1}{\pi}\int\limits_{-\pi}^{\pi} f_h\left(t+x\right)\sum_{\nu=1}^{k}\frac{\nu h}{\sin\left(\nu h\right)}\cos\left(\nu t\right)dt
\end{aligned}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f_h(t+x)\left(\frac{1}{2}+\sum_{\nu=1}^{k}\frac{\nu h}{\sin(\nu h)}\cos(\nu t)\right)dt.$$

Let T_n^* be near-best approximating polynomial to $f \in L_\omega^{p(\cdot)}(\mathbb{T})$. Denoting

$$(f-T_n^*)_h(t) := \frac{1}{2h}\int_{-h}^{h}(f-T_n^*)(y+t)\,dy,$$

we obtain

$$S_k(f-T_n^*)(\cdot) = \frac{1}{\pi}\int_{-\pi}^{\pi}(f-T_n^*)_h(t+\cdot)\left(\frac{1}{2}+\sum_{\nu=1}^{k}\frac{\nu h}{\sin(\nu h)}\cos(\nu t)\right)dt$$

and

$$\begin{aligned}\sum_{k=0}^{\infty}a_{n,k}S_k(f-T_n^*)(\cdot) &= \sum_{k=0}^{\infty}a_{n,k}\frac{1}{\pi}\int_{-\pi}^{\pi}(f-T_n^*)_h(t+\cdot)\left(\frac{1}{2}+\sum_{\nu=1}^{k}\frac{\nu h}{\sin(\nu h)}\cos(\nu t)\right)dt\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}(f-T_n^*)_h(t+\cdot)\sum_{k=0}^{\infty}a_{n,k}\left(\frac{1}{2}+\sum_{\nu=1}^{k}\frac{\nu h}{\sin(\nu h)}\cos(\nu t)\right)dt.\end{aligned}$$

Now choosing $h=\frac{\pi}{8\nu}$ for $\nu=1,2,...$, without lost of generality, by (2), Lemmas 1 and 2 we have

$$\begin{aligned}&\left\|\sum_{k=0}^{\infty}a_{n,k}S_k(f-T_n^*)\right\|_{p(\cdot),\omega}\\ &\le \frac{1}{\pi}\int_{-\pi}^{\pi}\left\|(f-T_n^*)_{\frac{\pi}{8\nu}}(t+\cdot)\right\|_{p(\cdot),\omega}\left|\sum_{k=0}^{\infty}a_{n,k}\left(\frac{1}{2}+\frac{\pi}{8\sin\frac{\pi}{8}}\sum_{\nu=1}^{k}\cos(\nu t)\right)\right|dt\end{aligned}$$

$$\begin{aligned}&= \frac{1}{\pi}\int_{-\pi}^{\pi}\left\|\frac{4\nu}{\pi}\int_{-\frac{\pi}{8\nu}}^{\frac{\pi}{8\nu}}(f-T_n^*)(y+t+\cdot)\,dy\right\|_{p(\cdot),\omega}\left|\sum_{k=0}^{\infty}a_{n,k}\left(\frac{1}{2}+\frac{\pi}{8\sin\frac{\pi}{8}}\sum_{\nu=1}^{k}\cos(\nu t)\right)\right|dt\\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\|\frac{4\nu}{\pi}\int_{-\frac{\pi}{8\nu}+t+\cdot}^{\frac{\pi}{8\nu}+t-\cdot}\left|(f-T_n^*)(u)\right|du\right\|_{p(\cdot),\omega}\left|\sum_{k=0}^{\infty}a_{n,k}\left(1+\frac{\pi}{4\sin\frac{\pi}{8}}\sum_{\nu=1}^{k}\cos(\nu t)\right)\right|dt\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\|f-T_n^*\|_{p(\cdot),\omega}\left|\sum_{k=0}^{\infty}a_{n,k}\left(1+\frac{\pi}{4\sin\frac{\pi}{8}}\sum_{\nu=1}^{k}\cos(\nu t)\right)\right|dt \\
&\leq cE_n(f)_{p(\cdot),\omega}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=0}^{\infty}a_{n,k}\left(1+\frac{\pi}{4\sin\frac{\pi}{8}}\sum_{\nu=1}^{k}\cos(\nu t)\right)\right|dt \\
&= \mathcal{O}(1)E_n(f)_{p(\cdot),\omega}=\mathcal{O}\left(\Omega\left(f,\frac{1}{n}\right)_{p(\cdot),\omega}\right). \qquad ◀
\end{aligned}$$

3. Proof of Main Results

Proof of Theorem 1 Let T_n^* be a near-best approximating polynomial to $f \in L_\omega^{p(\cdot)}(\mathbb{T})$. Using the equality $\sum_{k=0}^{\infty} a_{n,k}=1$ we have

$$\begin{aligned}
\|f-T_{n,A}[f]\|_{p(\cdot),\omega} &\leq \left\|f-\sum_{k=0}^{n}a_{n,k}T_k^*-\sum_{k=n+1}^{\infty}a_{n,k}T_n^*\right\|_{p(\cdot),\omega} \\
&\quad+\left\|\sum_{k=0}^{n}a_{n,k}T_k^*+\sum_{k=n+1}^{\infty}a_{n,k}T_n^*-T_{n,A}[f]\right\|_{p(\cdot),\omega} \\
&= \left\|\sum_{k=0}^{n}a_{n,k}(f-T_k^*)+\sum_{k=n+1}^{\infty}a_{n,k}(f-T_n^*)\right\|_{p(\cdot),\omega} \\
&\quad+\left\|T_{n,A}[f]-\sum_{k=0}^{n}a_{n,k}T_k^*-\sum_{k=n+1}^{\infty}a_{n,k}T_n^*\right\|_{p(\cdot),\omega} \\
&= \left\|\sum_{k=0}^{n}a_{n,k}(f-T_k^*)+\sum_{k=n+1}^{\infty}a_{n,k}(f-T_n^*)\right\|_{p(\cdot),\omega} \\
&\quad+\left\|\sum_{k=0}^{n}a_{n,k}(S_k(f)-T_k^*)+\sum_{k=n+1}^{\infty}a_{n,k}(S_k(f)-T_n^*)\right\|_{p(\cdot),\omega}.
\end{aligned}$$

Since

$$S_k(f-T_n^*,x)=\begin{cases} S_k(f,x)-T_k^*(x), \text{ for } k\leq n \\ S_k(f,x)-T_n^*(x), \text{ for } k\geq n,\end{cases}$$

we get

$$\|f-T_{n,A}[f]\|_{p(\cdot),\omega}\leq\left\|\sum_{k=0}^{n}a_{n,k}(f-T_k^*)\right\|_{p(\cdot),\omega}+\left\|\sum_{k=n+1}^{\infty}a_{n,k}(f-T_n^*)\right\|_{p(\cdot),\omega}$$

$$+\left\|\sum_{k=0}^{n} a_{n,k}\left(S_k\left(f-T_n^*\right)\right)+\sum_{k=n+1}^{\infty} a_{n,k}\left(S_k\left(f-T_n^*\right)\right)\right\|_{p(\cdot),\omega}$$

$$\leq \sum_{k=0}^{n} a_{n,k}\left\|\left(f-T_k^*\right)\right\|_{p(\cdot),\omega}+\sum_{k=n+1}^{\infty} a_{n,k}\left\|\left(f-T_n^*\right)\right\|_{p(\cdot),\omega}+\left\|\sum_{k=0}^{\infty} a_{n,k}\left(S_k\left(f-T_n^*\right)\right)\right\|_{p(\cdot),\omega}$$

$$=\sum_{k=0}^{n} a_{n,k} E_k(f)_{p(\cdot),\omega}+\sum_{k=n+1}^{\infty} a_{n,k} E_n(f)_{p(\cdot),\omega}$$
$$+\left\|\sum_{k=0}^{\infty} a_{n,k}\left(S_k\left(f-T_n^*\right)\right)\right\|_{p(\cdot),\omega}. \tag{10}$$

By (10), Lemmas 1 and 3 we obtain

$$\left\|f-T_{n,A}[f]\right\|_{p(\cdot),\omega}=\mathcal{O}(1)\sum_{k=0}^{n} a_{n,k}\Omega\left(f,\frac{1}{k}\right)_{p(\cdot),\omega}$$

$$+\mathcal{O}(1)\,\Omega\left(f,\frac{1}{n}\right)_{p(\cdot),\omega}\sum_{k=n+1}^{\infty} a_{n,k}+\mathcal{O}\left(\Omega\left(f,\frac{1}{n}\right)_{p(\cdot),\omega}\right)$$

$$=\mathcal{O}(1)\,\Omega(f,1/n)_{p(\cdot),\omega}+\mathcal{O}(1)\sum_{k=0}^{n} a_{n,k}\Omega(f,1/k)_{p(\cdot),\omega}$$

$$=\mathcal{O}\left(\Omega(f,1/n)_{p(\cdot),\omega}+\sum_{k=0}^{n} a_{n,k}\Omega(f,1/k)_{p(\cdot),\omega}\right). \qquad \blacktriangleleft$$

Proof of Theorem 2 Since $\lim_{n\to\infty}\Omega(f,1/n)_{p(\cdot),\omega}=0$, we have

$$\Omega(f,1/n)_{p(\cdot),\omega}=\sum_{k=0}^{n} a_{n,k}\Omega(f,1/n)_{p(\cdot),\omega}\leq\sum_{k=0}^{n} a_{n,k}\Omega(f,1/k)_{p(\cdot),\omega}. \tag{11}$$

Using the conditions (6), (7) and Abel transform, we have

$$\sum_{k=0}^{\infty}(k+1)^{\beta}\left|\frac{a_{n,k}}{(k+1)^{\beta}}-\frac{a_{n,k+1}}{(k+2)^{\beta}}\right|$$

$$=\sum_{k=0}^{n-1}(k+1)^{\beta}\left|\frac{a_{n,k}}{(k+1)^{\beta}}-\frac{a_{n,k+1}}{(k+2)^{\beta}}\right|+a_{n,n}$$

$$=\mathcal{O}(a_{n,n})+a_{n,n}=\mathcal{O}(a_{n,n})=\mathcal{O}\left(\frac{1}{n+1}\right)$$

and
$$\sum_{k=0}^{\infty}(k+1)\,a_{n,k}=\sum_{k=0}^{n}(k+1)\,a_{n,k}\leq(n+1)\sum_{k=0}^{n}a_{n,k}=n+1.$$
Hence, the entries of $A_0=(a_{n,k})$ satisfy the conditions (4) and (5). Applying Theorem 1 and (11) we obtain

$$\begin{aligned}\|f-T_{n,A_0}[f]\|_{p(\cdot),\omega} &= \mathcal{O}\left(\Omega\left(f,\frac{1}{n}\right)_{p(\cdot),\omega}+\sum_{k=0}^{n}a_{n,k}\Omega\left(f,\frac{1}{k}\right)_{p(\cdot),\omega}\right)\\ &= \mathcal{O}\left(\sum_{k=0}^{n}a_{n,k}\Omega\left(f,\frac{1}{k}\right)_{p(\cdot),\omega}\right). \quad ◀\end{aligned}$$

Proof of Theorem 3 *Let* $f\in Lip_{p(\cdot),\omega}(\omega^*,M)$, $p(\cdot)\in\mathcal{P}_0(\mathbb{T})$ and $\omega(\cdot)\in A_{p(\cdot)}(\mathbb{T})$. Since $\omega^*(n\delta)\leq n\omega^*(\delta)$ and hence $\omega^*(\lambda\delta)\leq(\lambda+1)\,\omega^*(\delta)$, we have

$$\omega^*(\delta_2)=\omega^*\left(\frac{\delta_2}{\delta_1}\delta_1\right)\leq\left(\frac{\delta_2}{\delta_1}+1\right)\omega^*(\delta_1)\leq 2\frac{\delta_2}{\delta_1}\omega^*(\delta_1),$$

where $n=1,2,...,$ $\lambda\geq 0$ and $0\leq\delta_1\leq\delta_2$. Then

$$\frac{\omega^*(\delta_2)}{\delta_2}\leq 2\frac{\omega^*(\delta_1)}{\delta_1}. \tag{12}$$

Applying (12) and (4) for $\beta>0$ and Abel transform, we have

$$\begin{aligned}\sum_{k=0}^{n}a_{n,k}\Omega(f,1/k)_{p(\cdot),\omega} &= \mathcal{O}(1)\sum_{k=0}^{n}a_{n,k}\omega^*(1/k)=\mathcal{O}(1)\sum_{k=0}^{n}\frac{a_{n,k}}{k}\frac{\omega^*(1/k)}{1/k}\\ &= \mathcal{O}(1)\,2n\omega^*\left(\frac{1}{n}\right)\sum_{k=0}^{n}\frac{a_{n,k}}{k}=\mathcal{O}(1)\,2n\omega^*(1/n)\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}(1+1/k)\\ &= \mathcal{O}(1)\,4(n+1)\,\omega^*(1/n)\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\\ &= \mathcal{O}(1)\,4(n+1)\,\omega^*(1/n)\sum_{k=0}^{\infty}\frac{a_{n,k}}{(k+1)^{\beta}}(k+1)^{\beta-1}\\ &= \mathcal{O}(1)\,(n+1)\,\omega^*\left(\frac{1}{n}\right)\sum_{k=0}^{\infty}\left|\frac{a_{n,k}}{(k+1)^{\beta}}-\frac{a_{n,k+1}}{(k+2)^{\beta}}\right|\sum_{j=0}^{k}(j+1)^{\beta-1}\\ &= \mathcal{O}(1)\,(n+1)\,\omega^*\left(\frac{1}{n}\right)\sum_{k=0}^{\infty}(k+1)^{\beta}\left|\frac{a_{n,k}}{(k+1)^{\beta}}-\frac{a_{n,k+1}}{(k+2)^{\beta}}\right|\\ &= (n+1)\,\omega^*\left(\frac{1}{n}\right)\mathcal{O}\left(\frac{1}{n+1}\right)=\mathcal{O}\left(\omega^*\left(\frac{1}{n}\right)\right).\end{aligned} \tag{13}$$

Since $\Omega\left(f,1/n\right)_{p(\cdot),\omega}=\mathcal{O}(\omega^{*}\left(1/n\right))$, by Theorem 1 and (13) we have

$$\left\|f-T_{n,A}\left[f\right]\right\|_{p(\cdot),\omega}=\mathcal{O}\left(\omega^{*}\left(1/n\right)\right). \qquad \blacktriangleleft$$

Proof of Theorem 4 *Let* $Lip_{p(\cdot),\omega}\left(\omega^{*},M\right)$, $p\left(\cdot\right)\in\mathcal{P}_{0}\left(\mathbb{T}\right)$, $\omega\left(\cdot\right)\in A_{p(\cdot)}\left(\mathbb{T}\right)$. It follows from the proof of Theorem 2 that the conditions (6) and (7) imply that the entries of $A_{0}=\left(a_{n,k}\right)$ satisfy the conditions (4) and (5). Then, applying Theorem 3 we have

$$\left\|f-T_{n,A_{0}}\left[f\right]\right\|_{p(\cdot),\omega}=\mathcal{O}\left(\omega^{*}\left(1/n\right)\right). \qquad \blacktriangleleft$$

References

[1] R. Akgun, *Polynomial approximation of functions in weighted Lebesgue and Smirnov spaces with nonstandard growth*, Georgian Math. Journal, **18**, 2011, 203-235.

[2] R. Akgun, V.M. Kokilashvili, *The refined direct and converse inequalities of trigonometric approximation weighted variable exponent Lebesgue spaces*, Georgian Math. Journal, **18**, 2011, 399-423.

[3] B.T. Bilalov, Z.G. Guseynov, *On the basicity from exponents in Lebesgue spaces with variable exponent*, TWMS J. Pure Appl. Math., **1(1)**, 2010, 14-23.

[4] B.T. Bilalov, Z.G. Guseynov, *Basicity of a system of exponents with a piece-wise linear phase in variable spaces*, Mediterr. J. Math., **9(3)**, 2012, 487-498.

[5] P. Chandra, *Trigonometric approximation of functions in L_p-norm*, J. Math. Anal. Appl., **275**, 2002, 13–26.

[6] D.V. Cruz-Uribe, L. Diening, P. Hästö, *The maximal operator on weighted variable Lebesgue spaces*, Fractional Calculus and Applied Analysis, **14(3)**, 2011, 361-374.

[7] D.V. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue Spaces Foundation and Harmonic Analysis*, Birkhäsuser, 2013.

[8] U. Deger, *On approximation by Nörlund and Riesz submethods in variable exponent Lebesgue spaces*, Commun. Fac. Sci. Univ. Ank. Series A1, **67(1)**, 2018, 46–59.

[9] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, 2011.

[10] F.A. Guliyeva, S.R. Sadigova, *Bases of the pertrubed system of exponents in generalized weighted Lebesgue space with a general weight*, Afr. Math, **28(6)**, 2017, 781-791

[11] A. Guven, *Trigonometric approximation of functions in weighted L_p spaces*, Sarajevo Journal of Mathematics, **5(17)**, 2009, 99-108.

[12] A. Guven, *Approximation in weighted L^p spaces*, Revista De La Unión Matemática Argentina, 53(1), 2012, 11-23.

[13] A. Guven, D.M. Israfilov, *Trigonometric Approximation in Generalized Lebesgue Spaces* $L^{p(x)}$, Journal of Math. Inequalities, **4(2)**, 2010, 285-299.

[14] A. Guven, *Trigonometric Approximation By Matrix Transforms in* $L^{p(x)}$ *Space*, Analysis and Applications, **10(1)**, 2012, 47-65.

[15] D.M. Israfilov, V. Kokilashvili, S. Samko, *Approximation In Weighted Lebesgue and Smirnov Spaces With Variable Exponents*, Proceed. of A. Razmadze Math. Institute, **143**, 2007, 25-35.

[16] D.M. Israfilov, A. Testici, *Approximation in Smirnov Classes with Variable Exponent*, Complex Variables and Elliptic Equations, **60(9)**, 2015, 1243-1253.

[17] D.M. Israfilov, A. Testici, *Some Inverse Theorem of Approximation Theory in Weighted Lebesgue Space With Variable Exponent*, Analysis Mathematica, **44(4)**, 2018, 475-492.

[18] D.M. Israfilov, A. Testici, *Approximation Problems In Lebesgue Spaces With Variable Exponent*, Journal of Mathematical Analysis and Applications, **459(1)**, 2018, 112-113.

[19] D.M. Israfilov, A. Testici, *Simultaneous Approximation in Lebesgue Space with Variable Exponent*, Proceeding of the Institute of Mathematics and Mechanics, National Academy of Science of Azerbaijan, **44(1)**, 2018, 3-18.

[20] D.M. Israfilov, A. Testici, *Approximation by matrix transforms in weighted Lebesgue spaces with variable exponent*, Results Math., **73(8)**, 2018.

[21] S.Z. Jafarov,*Linear methods for summing Fourier series and approximation in weighted Lebesgue spaces with variable exponents*, Ukrainian Mathematical Journal, **66(10)**, 2015, 1509-1518.

[22] S.Z. Jafarov, *Linear methods of summing Fourier series and approximation in weighted Orlicz spaces*, Turkish Journal of Mathematics, **42**, 2018, 2916-2925.

[23] S.Z. Jafarov, *Approximation by matrix transforms in weighted Orlicz spaces*, Turkish Journal of Mathematics, **44**, 2020, 179–193.

[24] W. Lenski, B. Szal, *Trigonometric Approximation of Functions from* $L^{p(x)}_{2\pi}$, Results Math, 2020, **75:56**.

[25] B.E. Mittal, B.E. Rhoades, V.N. Mishra, Uaday Singh, *Using infinite matrices to approximate functions of class image using trigonometric polynomials*, Journal of Mathematical Analysis and Applications, **326(1)**, 2007, 667-676.

[26] R.N. Mohapatra, B. Szal, *On trigonometric approximation of functions in the* L^q *norm*, Demonstr. Math., **51**, 2018, 17–26.

[27] T.I. Najafov, N.P. Nasibova, *On the Noetherness of the Riemann problem in generalized weighted Hardy classes*, Azerb. J. of Math., **5(2)**, 2015, 109-124.

[28] I.I. Sharapudinov, *Approximation of functions in* $L_{2\pi}^{p(x)}$ *by trigonometric polynomials*, Izvestiya RAN : Ser. Math., **77(2)**, 2013, 197-224; English transl., Izvestiya : Mathematics, **77(2)**, 2013, 407-434.

[29] I.I. Sharapudinov, *On direct and inverse theorems of approximation theory in variable Lebesgue and Sobolev spaces*, Azerb. J. of Math., **4(1)**, 2014, 55-72.

[30] I.I. Sharapudinov, *Approximation of functions by De Vallée Poussin means in the Lebesgue and Sobolev spaces with variable exponent*, Math. Sb., **207(7)**, 2016, 131–158.

[31] A. Testici, *Approximation By Nörlund and Riesz Means In Weighted Lebesgue Space With Variable Exponent*, Commun. Fac. Sci. Univ. Ank. Series A1, **68**, 2019, 2014-2025.

[32] A. Testici, *On Derivative of Trigonometric Polynomials and Characterizations of Modulus of Smoothness In Weighted Lebesgue Space With Variable Exponent*, Periodica Mathematica Hungarica, **80(1)**, 2020, 59-73.

[33] S.S. Volosivets, *Approximation of functions and their conjugates in variable Lebesgue spaces*, Mat. Sb., **208(1)**, 2017, 48–64.

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